Open Problems

Problem 1. Determine the consistency strength of the statement "$u_2 = \omega_2$", where u_2 is the second uniform indiscernible.

Best known bounds: Con(there is a strong cardinal) $\leq$ Con($u_2 = \omega_2$) $\leq$

Con(there is a Woodin cardinal with a measurable above it)

Remark: The difficulty is that we don't know how to use the hypothesis $u_2 = \omega_2$ to build larger models, even if we were given a measurable to make sense of the core model K.

Related Problem: Determine the consistency strength of the statement "every real has a sharp + every subset of $\omega_1^{L(\mathbb{R})}$ in $L(\mathbb{R})$ is constructible from a real".

Problem 2. Assume PD. C_3 is the largest

countable Π^1_3-set of reals. Is it true that $C_3 = \{\mathrm{x} \in M_2 \cap \mathbb{R} \mid x$ is

$\Delta^1_3 -$ definable from a mastercode of $M_2\}$?

Known:

- $C_1 = \{x \mid x$ is $\Delta^1_1 -$ equivalent to a mastercode of $L\}$
- $C_2 = \mathbb{R} \cap L$
- The reals in C_3 are Turing cofinal in $C_4 = M_2 \cap \mathbb{R}$.

Problem 3. Working in ZFC, how large can $\Theta^{L(\mathbb{R})}$ be?

Known:

- $\omega_1^V < \Theta^{L(\mathbb{R})}$
- Con($\omega_2^V < \Theta^{L(\mathbb{R})}$) if e.g. $u_2 = \omega_2$

But what about $\omega_3^V < \Theta^{L(\mathbb{R})}$?

Variant: Assume there are arbitrarily large Woodin cardinals. Is it possible that there is a universally Baire well-ordering of ordertype ω_3^V?

Problem 4. Assume AD^+ and assume that there is no iteration strategy for a countable mouse with a superstrong. Let $a \in \mathbb{R} \cap \mathrm{OD}$. Does there exist a countable iterable mouse M such that $a \in M$?

Remark: If there *is* an iteration strategy Σ for a countable mouse with a superstrong then we work in an initial segment of the Wadge hierarchy $<_W \Sigma$.

Known: (Woodin) If we replace the hypothesis with "no iteration strategy for a countable mouse satisfying the $\mathrm{AD}_{\mathbb{R}}-$ hypothesis" then the conclusion follows. This is the best result known.

Problem 5. Suppose $M_1^\sharp(x)$ exists for all sets x. K will be closed under the operation $x \mapsto M_1^\sharp(x)$ and any model closed under this operation will be Σ_3^1– correct. Is K Σ_4^1– correct under the hypotheses that $M_1^\sharp(x)$ exists for all x and that there is no inner model with 2 Woodins?

Remark: (Steel) Assume that K exists below a Woodin cardinal (e.g. ORD is measurable) and assume that there is a measurable and that there is no inner model with 1 Woodin. Then K is Σ_3^1– correct.

Conjectured improvement to Steel's Theorem: Assume that $\forall x(x^\sharp$ exists) and that there is no inner model with a Woodin. Is K Σ_3^1– correct? There are partial results in this direction (Woodin and others).

Problem 6. Assume that $\forall x(M_1^\sharp(x)$ exists) and that there is a least Π_3^1– singleton, z, that is not in the least inner model, N, closed under the operation $x \mapsto M_1^\sharp(x)$. Does $N^\sharp$ exist and is z Δ_3^1– isomorphic to $N^\sharp$?

Remark: This would give Σ_4^1– correctness for K in the case that K doesn't go beyond N. The second clause in the above conclusion is an instance of problem 2.

Problem 7. a) Let M be a countable, transitive structure that is elementarily embeddable into some V_α. Is M (ω_1+1)– iterable?

b) (An instance of CBH) For every countable iteration tree on V of limit length such that every extender used is countably closed from the model from which it was taken, is there a cofinal well-founded branch? Note: Countably closed means that ${}^\omega\mathrm{Ult}(V,E) \subseteq \mathrm{Ult}(V,E)$.

Known:

- In any $L[\vec{E}]$ model UBH is true.
- (Woodin) If you drop countable closure then CBH (i.e. full–CBH) is false.

Problem 8. Let $L[\vec{E}]$ be an extender model such that every countable structure elementarily embeddable into a level of the model is (ω_1+1)– iterable (so that many forms of condensation hold). Characterize (in terms of large cardinal axioms) all successor cardinals $(\kappa^+)^{L[\vec{E}]}$ of $L[\vec{E}]$ such that

$L[\vec{E}] \models$ (every stationary subset $S \subseteq \kappa^+ \cap \mathrm{Cof}(\omega)$ reflects).

Variant: Characterize all $(\kappa^+)^{L[\vec{E}]}$ such that

$L[\vec{E}] \models$ (every stationary subset $S \subseteq \kappa^+ \cap \mathrm{Cof}(<\kappa)$ reflects).

Problem 9. What is the consistency strength of $\neg\Box_\lambda^*$ for some singular λ?

Best upper bound known: $\exists\kappa(\kappa$ is $\kappa^{+\omega}$– strongly compact)

Contrast this with the best upper bound known for $\neg\Box_{\aleph_\omega}^*$: there is a measurable subcompact.

Problem 10. Let $j : V \to M$ be elementary, assume ORD is measurable and assume there is no proper class inner model with a Woodin. Is $j \upharpoonright K$ an iteration map?

Known: (Schindler) This is true if ${}^{\omega}M \subseteq M$.

Problem 11. a) Rate the consistency strength of the following statement: Let $\mathcal{I}$ be a simply definable σ-ideal.Then the statement "Every Σ^1_2 (projective) $\mathcal{I}$-positive set has a Borel $\mathcal{I}$-positive subset" holds in every generic extension.

b) Assume $\neg 0^{\#}$. Is it possible to force over V a real x such that $\mathbb{R}^{V[G]} \cap L[x]$ is $\mathcal{I}$-positive?

Remark: Here definable means simple, i.e., countable sets, Lebegue null sets, meager sets, and etc.

Problem 12. Prove that if $V = W[r]$ for some real r, V and W have the same cofinalities , $W \models$ CH, and $V \models 2^{\aleph_0} = \aleph_2$, then there is an inner model with $\aleph_2$ many measurables.

Known: (Shelah) Under the above hypothesis, there is an inner model with a measurable.

Problem 13. Investigate the following ZFC-model: $\mathrm{HOD}^{V[G]}$ where G is generic over V for $Coll(\omega, < \mathrm{ORD})$. In particular, Does CH hold in this model?

Problem 14. Assume 0-Pistol doesn't exist.

Suppose κ is Mahlo and $\diamondsuit_\kappa(Sing)$ fails.

a) Must κ be a measurable in K?

b) Suppose, in addition, that GCH holds below κ.

Is there an inner model with a strong cardinal?

c) Can GCH hold?

Known: (Woodin)

$\mathrm{CON}(o(\kappa) = \kappa^{++} + \epsilon) \to$

CON("κ is Mahlo and $\diamondsuit_\kappa$ fails").

Known: (Zeman) If κ is Mahlo and $\diamondsuit_\kappa(Sing)$ fails then for all $\lambda < \kappa$ there is $\delta < \kappa$ such that $K \models o(\delta) > \lambda$.

Problem 15. a) Assume there is no proper class inner model with a Woodin cardinal. Must there exist a " set-iterable" extender model which satisfies weak covering?
b) Does CON("ZFC + NS_{ω_1} is $\aleph_2$-saturated") imply CON(ZFC + " there is a Woodin cardinal")?

Problem 16. Let M be the minimal fully iterable extender model which satisfies "there is a Woodin cardinal κ which is a limit of Woodin cardinals". Let D be the derived model of M bellow κ. Does $D \models$ "θ is regular"?

Problem 17. Determine the consistency strength of incompatible models of AD^+ (i.e. there are A and B such that $L(A, \mathbb{R})$ and $L(B, \mathbb{R})$ satisfy AD^+ but $L(A, B, \mathbb{R}) \not\models \mathrm{AD}$).
Known: (Neeman and Woodin) Upper Bound: Woodin limit of Woodin cardinals. (Woodin) Lower Bound: $\mathrm{AD}_{\mathbb{R}} + \mathrm{DC}$.

Problem 18. Is $\mathrm{HOD}^{L(\mathbb{R})} \restriction \theta$ a normal iterate of $M_\omega \restriction \delta_0$ where δ_0 is the least Woodin of M_ω? If not, is there a normal iterate Q of $\mathrm{HOD}^{L(\mathbb{R})}$ fixing θ such that $Q \restriction \theta$ is a normal iterate of every countable iterate of $M_\omega \restriction \delta_0$?
Known: (Neeman) The answer to the first question is "almost" no.

Problem 19. Assume $V = L(\mathbb{R}) + \mathrm{AD}$. Let $\boldsymbol{\Gamma}$ be a Π^1_1-like scaled pointclass (i.e., closed under $\forall^{\mathbb{R}}$ and non-self-dual). Let $\boldsymbol{\delta} = \sup\{|<| : < \text{ is a pwo in } \boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \boldsymbol{\Gamma}^{\smile}\}$. Then, is Γ closed under unions of length $< \boldsymbol{\delta}$?
Known: (Kechris-Martin) Known for $\boldsymbol{\Pi^1_3}$. (Jackson) Known for $\boldsymbol{\Pi^1_{2n+5}}$.

Problem 20. Is there an inner model M of $L[0^{\#}]$ such that $0^{\#} \notin M$, $0^{\#} \in M[G]$, and $(M[G], \in G) \models \mathrm{ZFC}$, where G is $\mathbb{P}$-generic over M for some M-definable class-forcing?