Lecture 9

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1. What is $s_n = 1^2 + 2^2 + 3^2 + \cdots + n^2$ in summation notation? Prove that

$$s_n = \frac{n(n+1)(2n+1)}{6}$$

In summation notation:

$$s_n = \sum_{i=1}^{n} i^2$$

Base case: $s_1 = 1 = \frac{1 \cdot 2 \cdot 3}{6} = 1$

Inductive case: Suppose $s_n = \frac{n(n+1)(2n+1)}{6}$, we want to show $s_{n+1} = \frac{(n+1)(n+2)(2n+3)}{6}$

$$\begin{aligned}
s_{n+1} &= \sum_{i=1}^{n+1} i^2 \\
&= \sum_{i=1}^{n} i^2 + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \qquad \text{by the inductive hypothesis} \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\
&= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6}
\end{aligned}$$

2. Let $t_1 = t_2 = t_3 = 1$ and $t_n = t_{n-1} + t_{n-2} + t_{n-3}$. Show that $t_n < 2^n$ for all $n \in \mathbb{N}$.

Base cases: $t_1 = 1 < 2$, $t_2 = 1 < 4 = 2^2$, $t_3 = 1 < 8 = 2^3$

Inductive case: Suppose $t_k < 2^k$ for all $k < n$. We wish to show $t_n < 2^n$.

$$\begin{aligned}
t_n &= t_{n-1} + t_{n-2} + t_{n-3} \\
&< 2^{n-1} + 2^{n-2} + 2^{n-3} \\
&= \frac{1}{2}2^n + \frac{1}{4}2^n + \frac{1}{8}2^n \\
&= (\frac{1}{2} + \frac{1}{4} + \frac{1}{8})2^n \\
&= \frac{7}{8}2^n \\
&< 2^n
\end{aligned}$$

3. Let $\{A_i\}_{i \in \mathbb{N}}$ and $\{B_i\}_{i \in \mathbb{N}}$ be two indexed families of sets such that for all $i \in \mathbb{N}$, $A_i \subsetneq B_i$. Show that

$$\bigcup_{i \in \mathbb{N}} A_i \subseteq \bigcup_{i \in \mathbb{N}} B_i$$

Is it true always true that

$$\bigcup_{i\in\mathbb{N}} A_i \subsetneq \bigcup_{i\in\mathbb{N}} B_i$$

Let $x \in \bigcup_{i\in\mathbb{N}} A_i$, then $x \in A_k$ for some k, so $x \in B_{k+1} \subseteq \bigcup_{i\in\mathbb{N}} B_i$.

It is not true. Conisder $A_i = [i]$ and $B_i = [i+1]$. Certainly $A_i \subsetneq B_i$. However, consider $x \in \bigcup_{i\in\mathbb{N}} B_i$, then $x \in B_k = [k+1]$ for some k so $x \in A_{k+1} = [k+1]$ so $x \in \bigcup_{i\in\mathbb{N}}$.

4. Prove or disprove $\mathcal{P}(A) \cup A \neq \mathcal{P}(A)$ for all sets $A \neq \varnothing$.

   This is false. Let $A = \{\varnothing\}$, then $\mathcal{P}(A) = \{\varnothing, \{\varnothing\}\}$.

5. Let $A \subseteq B$ and $C \subseteq D$. Prove or disprove $A \times C \subseteq B \times D$.

   Suppose $(x, y) \in A \times C$, then $x \in A \subseteq B$ and $y \in C \subseteq D$, so $(x, y) \in B \times D$.

6. Prove or disprove $A \times B - C \times D = (A - C) \times (B - D)$ for all A, B, C, D.

   Let $A = [2], B = [2], C = [1], D = [2]$, then $A \times B - C \times D = \{(1, 1), (1, 2), (2, 1), (2, 2)\} - \{(1, 1), (1, 2)\} = \{(2, 1), (2, 2)\}$. However, $(A - C) \times (B - D) = \{2\} \times \varnothing = \varnothing$.