Lecture 6

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1. 2 people sit facing each other, players I and II. A third person secretly writes 2 consecutive natural numbers on 2 slips of paper, and tapes each piece on the 2 players' foreheads.

   Player I can see the number on Player II's forehead, and vice versa, but neither can see their own (but they know that the numbers are consecutive natural numbers). On the first turn, player I asks II if she knows her number. If she says "yes", the game ends. If not, player I's turn ends and player II asks player I if he knows his number. If he says "yes", the game ends. If not, it's player I's turn, and the game proceeds exactly the same. Assuming player I and player II are both "perfect reasoners" and nobody says "yes" unless they are absolutely certain about their number, does the game ever end?

   Go over the question as to whether the game mentioned above ends or not. They will have an idea that it will always end and how, but it'll probably need to be fleshed out. After their conjecture ask them "How can we write this as an inductive argument?" and "What are we going to induct on?" The basic idea of the proof is:

   Let n denote the lower of the 2 numbers (this is what we are going to induct on) and then prove the stronger claim that the game will end in no more than $2n$ moves. They should be able to determine what is the base case, what the inductive hypothesis should be and how to prove the inductive step (how are we going to relate the situation in which the lower number is $k+1$ to the situation in which the lower number is k?).

   Base case: If player I sees 1, then player I's number is 2, so the game ends in 1 turn. If player I says no, and player II sees 1, then player II's number is 2 so the game ends in 2 turns.

   Inductive case: Assume the game ends in $2n$ turns if the lowest number is n. Consider the case where $n+1$ is the lowest number. Then if the first two turns neither player says yes, then we have eliminated 1 from the pool of possible numbers. After the first two turns, both players gain the information that 1 is eliminated from the pool of possible numbers. So for the remaining game, both players can subtract 1 from the number they see to obtain the original game as if neither player has information. After subtracting 1, the lowest number becomes n so the game will now end in $2n$ turns. Including the first 2 turns, the total number of turns is at most $2n+2$.

   Note that in this proof, we not only answered the question, but made an upper bound for the length of the game. Why was this included in the proof?

2. The Coin Removal Problem

   Let a string be a row of coins without gaps and without other coins beyond the ends. We write a string as a list of Hs and Ts. When we remove an H, we leave a gap (marked by a dot), and we flip all of the (at most two) coins next to it that remain. Thus HHT becomes $T.H$ when we remove the H in the middle. We then get $T..$ when we remove the new H. Removing a coin from a string leaves two strings except when we remove the end.

   We begin with a string of length n. Examination of examples suggests that we can empty a string (remove all its coins) if and only if it has an odd number of Hs. We prove this by strong induction on n.

   Base case: A string of length 1 with an odd number of Hs must be the string H, which is emptied in 1 turn. A string of length 1 with an even number of Hs must be the string T, which cannot be emptied.

   Inductive case: Suppose all strings of length $m < n$ can be emptied if and only if there are an odd number of Hs. Consider a string of length n with $x =$ the number of Hs. If $x = 0$, then the string cannot be emptied. Otherwise, remove the first H from the left. The dot it creates partitions the sequence into two strings of length $< n$, where the left sequence has exactly 1 H, and the right sequence has either x or $x - 2$ Hs depending on the what follows the H we remove

$$\underbrace{TTTTHT\ldots}_{xHs} \longrightarrow \underbrace{TTTH}_{1Hs}.\ \underbrace{H\ldots}_{x-2Hs}$$
$$\underbrace{TTTTHH\ldots}_{xHs} \longrightarrow \underbrace{TTTH}_{1Hs}.\ \underbrace{T\ldots}_{x-2Hs}$$

thus, if x is odd, both the left sequence and the right sequence can be emptied by the inductive hypothesis. So we empty the left one and then the right one without it affected each other since they are isolated by a dot, and we have emptied the entire sequence.

Otherwise, if x is even, then if the first H removed is the left most one, we get the right sequence from the above 2 cases, meaning that there are $x - 2$ Hs so there is an even number of Hs, so by inductive hypothesis the remaining sequence cannot be emptied. The case where the first H is the right-most is the same. If first H removed is in the middle, there are four possible cases

$$\ldots THT\ldots \to \ldots H.H\ldots$$
$$\ldots THH\ldots \to \ldots H.T\ldots$$
$$\ldots HHT\ldots \to \ldots T.H\ldots$$
$$\ldots HHH\ldots \to \ldots T.T\ldots$$

In the first case, the number of Hs is $x+1$, in the next two cases, the number of Hs is $x-1$, for the last case the number of Hs is $x-3$. Thus, if x is even, the total number of Hs is odd, so split between two sequences, one of the sequence must contain an even number of Hs, so it cannot be emptied. Thus, the whole sequence cannot be emptied.