Lecture 5

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1. A binary tree is a rooted tree in which every node has either 0 or 2 children. These are computer scientist trees, not graph-theorist trees, meaning that "left" and "right" matter. So there are two trees with 3 leaves, even though they are mirror images of each other, and isomorphic as graphs.

   Let $t(n)$ count the number of binary trees with n leaves (nodes with no children). Thus $t(1) = 1$, $t(2) = 1$, $t(3) = 2$. Convince yourself that $t(4) = 5$. Now given an inductive definition of $t(n)$

$$t(n) = \sum_{1 \le i < n} t(i) \cdot t(n-i)$$

2. Prove that the Fibonacci numbers satisfies $F(n) \ge (3/2)^{n-2}$ for all $n \in \mathbb{N}$.

   Base case: $F(1) = 1 \ge \frac{2}{3} = (3/2)^{1-2}$, $F(2) = 1 \ge 1 = (3/2)^{2-2}$

   Inductive case: Assume $F(n-2) \ge (3/2)^{n-4}$ and $F(n-1) \ge (3/2)^{n-3}$.

   By definition of the Fibonacci numbers

$$\begin{aligned} F(n) &= F(n-1) + F(n-2) \\ &\ge (3/2)^{n-3} + (3/2)^{n-4} = \frac{2}{3}(3/2)^{n-2} + \frac{4}{9}(3/2)^{n-2} = (\frac{2}{3} + \frac{4}{9})(3/2)^{n-2} \\ &\ge (3/2)^{n-2} \end{aligned}$$

3. Prove that the Fibonacci numbers $F(3n)$ is even for every $n \in \mathbb{N}$.

   Base case: $F(3) = 2$ is even.

   Inductive case: Assume $F(3(n-1)) = F(3n-3)$ is even.

$$F(3n) = F(3n-1) + F(3n-2) = F(3n-2) + F(3n-3) + F(3n-2) = 2 \cdot F(3n-2) + F(3n-3)$$

   so since $F(3n-3)$ is even by the inductive hypothesis, $F(3n)$ is even.

4. (Similar to the triomino homework question): If an equilateral triangle is cut into 4^n equilateral triangles and one corner is removed, the remaining area can be covered by trapezoidal tiles made up of 3 triangles.

   Base case: A triangle made of 4 triangles with the top corner removed is exactly a trapezoid made of 3 triangles.

   Inductive case: Assume that a tiling exists for triangle of size 4^{n-1}

   Observe that

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   is a trapezoid made up for 4 smaller trapezoids.

   For a triangle with 4^n equilateral triangles, consider it to be made of 4^{n-1} larger triangles, each 4 times the area (so each large triangle is made of 4 triangles). Using the inductive hypothesis, we can tile a triangle with 4^{n-1} larger triangles with larger trapezoids each 4 times the size. However, each larger trapezoid can be tiled by 4 standard sized trapezoids. The tiling from the inductive hypothesis leaves a large triangle empty, so we can tile the remaining large triangle using the base case, leaving a standard triangle empty, which is what we wanted.