Lecture 4

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2.2.2 Lines On The Plane

Consider n lines on an infinite plane such that no two lines are parallel and no more than two lines intersect at one point. How many distinct regions do the lines create?

We can draw a few examples by hand for small n, and use it to guide our intuition into making a general argument for an arbitrary value of n.

For $n = 2$, a drawing shows that there are 4 regions, but how do we know that there are always 4? First, we start with 1 line, which splits it into two regions. For the next line, no matter how we draw it, it will split each region into two. Thus, we have 4 regions for $n = 2$.

What about $n = 3$? Again we don't want our answer to depend on particular arrangement of lines. Start with a 2 line diagram, see that after adding the third line, one of the regions remains unchanged, but the other three are split into two. Thus, each split increases our regions by one, so there are 3 new regions. Thus, we have 7 regions for $n = 3$.

For $n = 4$, we might want to be careful where we put the new line, but it turns out it doesn't matter. 4 new regions are introduced. Notice a pattern?

Let $R(n)$ be the number of regions from n lines. Assuming we know $R(n)$, maybe we can find $R(n+1)$ by adding the $(n+1)$-th line. How many new regions are created? A line intersects another line that is non-parallel exactly once. Thus, the $(n+1)$-th line must intersect the other lines n times. Note that each intersection must be a distinct point, because we said at most two lines join at a point. Thus, there are n distinct intersections on the new line.

n points on a line divides it up into $n+1$ segments. Each segment crosses a region, so $n+1$ regions are split into two. The other regions are untouched. Therefore, $n+1$ new regions are introduced. So

$$R(n+1) = R(n) + n + 1$$

Now note that $R(1) = 2$, so we can pull $R(n)$ all the way down to $R(1)$.

$$\begin{aligned} R(n) &= R(n-1) + n \\ &= R(n-2) + (n-1) + n \\ &= R(n-3) + (n-2) + (n-1) + n \\ &\vdots \\ &= R(1) + 2 + 3 + \cdots + (n-2) + (n-1) + n \end{aligned}$$

so since $R(1) = 2$, we have

$$R(n) = 2 + 2 + 3 + \cdots + (n-1) + n = 1 + \sum_{k=1}^{n} = 1 + \frac{n(n+1)}{2}$$

Induction

Using $R(n+1) = R(n) + n + 1$, we will prove $R(n) = 1 + \frac{n(n+1)}{2}$.

Base case: $R(1) = 2 = 1 + \frac{1(2)}{2}$.

Inductive case: Suppose we know that $R(n) = 1 + \frac{n(n+1)}{2}$, we wish to show that $R(n+1) = 1 + \frac{(n+1)(n+2)}{2}$.

$$R(n+1) = R(n) + n + 1 = 1 + \frac{n(n+1)}{2} + n + 1 = 1 + \frac{n(n+1) + 2n + 2}{2} = 1 + \frac{(n+2)(n+1)}{2}$$