Lecture 3

Enoch Cheung

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1. n people, each person has at most k friends. Into how many clubs do we need to partition the n people to guarantee that nobody is in a club with a friend?

   Let $p_1, \ldots, p_n$ be the n people. At stage 0, nobody is in any club. At each stage i, we put p_i into some club $C_1, \ldots, C_c$ such that nobody is in a club with a friend. We need to show that we can always put p_i somewhere.

2. 17 rooks are placed on an 8×8 chessboard. Prove that there are at least 3 rooks that do not threaten each other.

   There are two ways to solve this using the pigeon hole principle.

   (a) There are 8 rows, thus at least one row contains at least 3 rooks, call it row A. There are at least $17 - 8 = 9$ rooks in the other 7 rows, so one row contains at least 2 rooks, call it B. Finally, there are at least $9 - 8 = 1$ rook in the remaining 6 rows, so one row contains at least 1 rook, call it C.

   Pick 1 rook in row C, then pick 1 rook in B that is not in the same column, since B has at least 2 rooks. Then pick 1 rook in A that is not in the same column as the previous 2, since A has at least 3 rooks.

   (b) Roll the chessboard into a cylinder. Observe that rooks on the same diagonal do not threaten each other. There are 8 diagonals. By the pigeon hole principle, at least 3 rooks must be on the same diagonal, thus do not threaten each other.

3. Suppose $f(x)$ is a polynomial with integer coefficients.

   (a) Prove that for any 2 different integers p and q, $p - q$ divides $f(p) - f(q)$.

   Let $f(p) - f(q) = \sum_{k=0}^{n} a_k(p^k - q^k)$ where $a_k \in \mathbb{Z}$. We claim that $p - q$ divides $p^k - q^k$ for all $k \in \mathbb{N}$.

   $$\frac{p^k - q^k}{p - q} = p^{k-1} + p^{k-2}q + p^{k-3}q^2 + \cdots + pq^{k-2} + q^{k-1}$$

   Thus $p - q$ divides $f(p) - f(q)$.

   (b) Now suppose $f(x) = 2$ for 3 distinct integers $x = a_1, a_2, a_3$. Show that if $f(b) = 3$ then b is not an integer.

   From (a) we know that $(b - a_i)$ divides $f(b) - f(a_i) = 1$ for each $i = 1, 2, 3$. The only integer divisors of 1 are $1, -1$, so by pigeon hole principle some $b - a_i = b - a_j$. Thus $a_i = a_j$ contradiction.

1.5.20 Let M_0 be the initial number of $M\&Ms$. Let $M_1 = \frac{2}{3}(M_0 - 1)$. Let $M_2 = \frac{2}{3}(M_1 - 1)$. Let $M_3 = \frac{2}{3}(M_2 - 1)$. Finally, 3 divides M_3.

Thus

$$M_0 = 1 + \frac{3}{2}(1 + \frac{3}{2}(1 + \frac{3}{2}M_3))$$

Note that M_3 has to be divisible by 3 and 2, so try $M_3 = 6$. Then $M_2 = 10$, and $M_1 = 16$, and $M_0 = 25$.