Lecture 27

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1. (a) $\displaystyle\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$

   Right hand side counts the number of combinations of n objects from $2n$ objects.

   Left hand side counts the same thing. Paint n of the objects red, and n of the objects blue. To pick n objects, there are cases for each $0 \le k \le n$, where Case k represents picking i red objects and $n-k$ blue objects. There are $\binom{n}{k}$ ways to pick red objects and $\binom{n}{n-k} = \binom{n}{k}$ ways to pick blue objects. Therefore, for case k, there are $\binom{n}{k}^2$ choices. Therefore, the total number of choices are $\sum_{k=0}^{n}\binom{n}{k}^2$.

   (b) $\displaystyle\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$

   Given $2n$ people, n male and n female. We are counting the numbers of ways to pick a male team leader, and $n-1$ more team leaders to make a team of n people.

   Left hand side: we are considering cases for $1 \le k \le n$, where Case k is the case where there are k male, and $n-k$ females, where one of the k males is the leader, so for Case k there are k ways to pick the leader from the k male chosen, $\binom{n}{k}$ ways to choose the k male team members, and $\binom{n}{k} = \binom{n}{n-k}$ ways to pick the $n-k$ female team members. Therefore, $k\binom{n}{k}^2$ ways in each Case k, so $\sum_{k=1}^{n} k\binom{n}{k}^2$ total.

   Right hand side: pick one of the n male leaders, then from the $2n-1$ remaining people pick $n-1$ more team members.

   (c) $\displaystyle\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}.$

   Given n people, and we have to pick a team with a leader (can be any size ≥ 1).

   Left hand side: cases for each $1 \le k \le n$, where Case k is the case where the team has k people, so there are $\binom{n}{k}$ ways to pick the team, and in the team of k there are k choices of leaders.

   Right hand side: we first pick a leader, of which there are n choices. And for each of the remaining $n-1$ people, we decide if they are in or out of the team (2 choices per person), so there are 2^{n-1} choices. Thus, total, there are $n \cdot 2^{n-1}$ choices.

   (d) $\displaystyle\sum_{k=0}^{n}\binom{2n}{2k} = 2^{2n-1}.$

   Given $2n$ people, we have to pick a team of even size.

   Left hand side: there are cases for $0 \le k \le 2n$, where Case k is the case where the team has $2k$ people, so there are $\binom{2n}{2k}$ choices for Case k. Thus, there are $\sum_{k=0}^{n}\binom{2n}{2k}$ total choices.

   For the right hand side: order the $2n$ people in some way. For the first $2n-1$ people, we decide whether they are in or out of the team (2 choices per person). For the last person, if our team is already even, the the last person is out, otherwise the team is odd, and the last person is in. Thus, there are 2^{2n-1} choices.

2. $\forall m \in \mathbb{N}.\ m^3 = 3!\binom{m}{3} + 3!\binom{m}{2} + \binom{m}{1}$

   Consider arbitrary $m \in \mathbb{N}$. We are counting 3 lettered words from an alphabet of size m.

   Left hand side: we pick 3 letters one by one to form the word, so m^3 ways.

   Right hand side: there are 3 cases.

   Case 1 is the case where all 3 letters of the word are distinct, so we pick a subset of 3 letters from m so there are $\binom{m}{3}$ ways to do this, then we count permutations of the 3 chosen letters to make words, and there are 3! permutations. Thus, Case 1 has $3!\binom{m}{3}$ ways. Note that if $m < 3$ then this number is 0 which makes sense.

Case 2 is the case where the word has 2 distinct letters and one of which is repeated. There are $\binom{m}{2}$ ways to pick the subset of letters that are used. There are 2 ways to pick which letter is unique and which is repeated, and 3 choices of positions for the unique letter. Therefore, case 2 has $2 \cdot 3 \cdot \binom{n}{2} = 3!\binom{n}{2}$ ways.

Case 3 is the case where all 3 letters in the word are the same. There are $\binom{m}{1}$ choices of the repeated letter.

Therefore, the total number of ways are $3!\binom{m}{3} + 3!\binom{m}{2} + \binom{m}{1}$.