Lecture 25

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1. How many ways are there to pick two cards from a standard 52-card deck such that the first card is a spade and the second card is not an ace?

   Case 1: First card ace of spades, $52 - 4 = 48$ non-aces left. $1 \cdot 48$ ways this case can happen.

   Case 2: First card spades but not ace, $13 - 1 = 12$ choices of first card. $52 - 4 - 1 = 47$ non-aces left. $12 \cdot 47$ ways this case can happen.

   Therefore, there are $48 + 12 \cdot 47$ ways.

2. How many ways are there to deal a 5 card hand such that at least 3 cards have the same rank?

   There are 13 choices for the repeated rank. Case 1: The rank is repeated exactly 3 times, then there are $\binom{4}{3} = 4$ choices of the repeating cards, and $\binom{48}{2}$ choices for the remaining cards. Case 2: The rank is exactly 4 times, then there is 1 choice for the four cards, and $\binom{48}{1} = 48$ choices for the remaining card.

   Therefore, there are $13(4 \cdot \binom{48}{2} + 48)$ ways.

3. (a) How many 5-letter words can be made that have at least 1 vowel?

   It is easier to count the number of 5-letter words that has no vowels. There are $26 - 5 = 21$ consonants. Therefore, there are 21^5 words with no vowels. There are 26^5 total words. Therefore, there are $26^5 - 21^5$ words with at least one vowel.

   (b) How many 6-letter words only have 1 distinct vowel and that vowel occurs exactly twice.

   There are 5 choices of the vowel that appears. There are $\binom{6}{2}$ possible positions the vowel can appear (in the 6 letter string). There are 21^4 possible choices for the remaining letters. Therefore, the total number is $5 \cdot \binom{6}{2} \cdot 21^4$.

4. How many anagrams are there of the word BANANA are there?

   First, there are $\binom{6}{1}$ possible choices of locations of B. Next, there are $\binom{5}{3}$ possible choices of locations of the A's. The rest are N's. Therefore, the total number is $\binom{6}{1}\binom{5}{3} = \frac{6!}{1!3!2!}$.

5. Let $n \in \mathbb{N}$. How many lattice paths to $(2n, 2n)$ go through the point (n, n).

   Starting from $(0, 0)$, there are $\binom{2n}{n}$ lattice paths to (n, n), because each path is a distinct sequence of n UPs and n RIGHTs. From (n, n), there are $\binom{2n}{n}$ lattice paths to $(2n, 2n)$. Therefore, there are $\binom{2n}{n}^2$ total paths.

6. How many lattice paths to $(11, 7)$,

   (a) Pass though both $(2, 3)$ and $(7, 4)$?

   Starting from $(0, 0)$, there are $\binom{2+3}{2} = \binom{5}{2}$ lattice paths to $(2, 3)$, because each path is a distinct sequence of 2 RIGHTs and 3 UPs. Now from $(2, 3)$ there are $\binom{7-2+4-3}{7-2} = \binom{6}{5}$ lattice paths to $(7, 4)$. Finally, from $(7, 4)$ to $(11, 7)$ there are $\binom{11-7+7-4}{11-7} = \binom{7}{4}$ paths.

   Therefore, total there are $\binom{5}{2}\binom{6}{5}\binom{7}{4}$ paths.

   (b) Pass through neither $(2, 3)$ nor $(7, 4)$?

   The total number of paths from $(0, 0)$ to $(11, 7)$ are $\binom{11+7}{11} = \binom{18}{11}$. The number of paths that pass through $(2, 3)$ are $\binom{2+3}{2}\binom{11-2+7-3}{11-2} = \binom{5}{2}\binom{13}{9}$. The number of paths that pass through $(7, 4)$ are $\binom{7+4}{7}\binom{11-7+7+4}{11-7} = \binom{11}{7}\binom{7}{4}$. Note however that if a path goes through both $(2, 3)$ and $(7, 4)$, then it will be counted when we count paths that goes through $(2, 3)$ and also when we count paths that goes through $(7, 4)$. By (a), there are $\binom{5}{2}\binom{6}{5}\binom{7}{4}$ such paths.

   Therefore, the total number of paths that goes through neither $(2, 3)$ nor $(7, 4)$ are

$$\binom{18}{11} - \binom{5}{2}\binom{13}{9} - \binom{11}{7}\binom{7}{4} + \binom{5}{2}\binom{6}{5}\binom{7}{4}$$

7. How many ways are there to distribute 12 indistinguishable balls among 5 bins? "Stars and Bars" method: To divide up 12 balls (Stars) in a row into 5 bins, we need 4 dividers (Bars). There are $\binom{12+4}{4}$ possible positions of the Bars in a sequence of 12+4 objects. Therefore, there are $\binom{16}{4}$ ways.

   If we require at least one ball per bin, it is the same as asking how many ways to distribute 12-5 balls among 5 bins where they could be empty, then adding 1 ball to each of the 5 bins. Therefore, there are $\binom{12-5+4}{4} = \binom{11}{4}$ ways.