Lecture 24

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1. Let A, B be countably infinite sets.

   By definition, there are bijections $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$. Therefore, there is a bijection

$$\begin{aligned} h : A \times B &\longrightarrow \mathbb{N} \times \mathbb{N} \\ (a, b) &\longmapsto (f(a), g(b)) \end{aligned}$$

   To see that h is an injection, consider $(a, b), (a', b') \in A \times B$, and suppose $h(a, b) = h(a', b')$, then by definition, $(f(a), g(b)) = (f(a'), g(b'))$. By injectivity of f and g, this implies that $a = a'$ and $b = b'$, so $(a, b) = (a', b')$.

   To see that h is surjective, consider $(n, m) \in \mathbb{N}$, then by surjectivity of f and g, there are $a \in A$ such that $f(a) = n$ and $b \in B$ such that $g(b) = B$. Therefore, $(a, b) \in A \times B$ such that $h(a, b) = (n, m)$.

   Therefore, $h : A \times B \to \mathbb{N} \times \mathbb{N}$ is a bijection.

   Recall from last time that $s : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined $s(m, n) = 2^{m-1}(2n - 1)$ is a bijection.

   We claim that $s \circ h : A \times B \to \mathbb{N}$ is a bijection.

   To see that $s \circ h$ is injective, consider $(a, b), (a', b') \in A \times B$, then if $s(h(a, b)) = s(h(a', b'))$, then by injectivity of s, $h(a, b) = h(a', b')$ and by injectivity of h we have $(a, b) = (a', b')$.

   To see that $s \circ h$ is surjective, for all $n \in \mathbb{N}$, since s is surjective there is some $(n', m') \in \mathbb{N}$ such that $s(n', m') = n$, and by surjectivity of h there is some $(a, b) \in A \times B$ such that $h(a, b) = (n', m')$. Therefore, $s \circ h(a, b) = s(n', m') = n$.

   Therefore, $s \circ h : A \times B \to \mathbb{N}$ is a bijection, which shows that $A \times B$ is countable.

2. Diagonalization:

   Show that there does not exists an injection $f : \mathcal{P}(A) \to A$ for any set A.

   Suppose such injection $f : \mathcal{P}(A) \to A$ exists. Consider $X = \{f(Y) \mid Y \subseteq A \wedge f(Y) \notin Y\}$. Note that $X \subseteq Y$, so let $a = f(X)$.

   To see that this is a contradiction, note that there are two cases $a \notin X$ or $a \in X$. Suppose $a \notin X$, then $a = f(X) \notin X$ so by definition $f(X) \in X$ which is a contradiction.

   On the other hand, if $a \in X$, then there is some $Y \subseteq A$ and $f(Y) \notin Y$ such that $f(Y) = a \in X$. However since f is injective, $f(Y) = a = f(X)$ implies that $X = Y$, which is a contradiction because $a = f(X) \in X$ but $f(Y) \notin Y$.

3. In class we proved that the open interval $|(0, 1)|$ was uncountable.

   (a) Prove that $|(0, 1)| = |[0, 1)|$ by showing that the function $f : (0, 1) \to [0, 1)$ below is injective.

$$f(x) = \begin{cases} 0 & x = \frac{1}{2} \\ \frac{1}{2^n} & x = \frac{1}{2^{n+1}} \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

   Injectivity: Suppose $x, y \in (0, 1)$ and $f(x) = f(y)$. If $f(x) = f(y) = 0$, then note that the only possibility is $x = y = \frac{1}{2}$. If $f(x) = f(y) = \frac{1}{2^n}$ for some $n \in \mathbb{N}$, then note that the only possibility is $x = y = \frac{1}{2^{n+1}}$. Otherwise, $f(x) = f(y)$ does not take the forms above, so $f(x) = f(y) = x = y$.

   Surjectivity: Consider any $y \in [0, 1)$. If $y = 0$, then $f(\frac{1}{2}) = 0$. If $y = \frac{1}{2^n}$ for some $n \in \mathbb{N}$, then $f(\frac{1}{2^n}) = \frac{1}{2^n}$. Otherwise, $y \neq 0$ and $y \in [0, 1)$ so $y \in (0, 1)$, so $f(y) = y$.

(b) Prove that $|(0,1)| = |\mathbb{R}|$ by showing that the function $f(x) = \ln(\frac{1-x}{x})$ is a bijection $f : (0,1) \to \mathbb{R}$.

We will provide an inverse to f. Note that for any $a \neq 0$ and $b \neq -1$,

$$\frac{1-a}{a} = b \iff \frac{1}{a} - 1 = b \iff \frac{1}{a} = b + 1 \iff a = \frac{1}{b+1}$$

Therefore, let $g : \mathbb{R} \to (0,1)$ be defined

$$g(y) = \frac{1}{e^y + 1}$$

to see that the fraction is well defined, note that for any $y \in \mathbb{R}$, $e^y > 0$. This also shows that $e^y + 1 > 1$ so $\frac{1}{e^y+1} < 1$, and $e^y + 1 > 0$ so $0 < \frac{1}{e^y+1} < 1$, so g has appropriate domain and codomain.

To check that g is the inverse of f, note that for any $x \in (0,1)$,

$$g(f(x)) = \frac{1}{e^{\ln(\frac{1-x}{x})} + 1} = \frac{1}{\frac{1-x}{x} + 1} = \frac{1}{\frac{1}{x}} = x$$

and for any $y \in \mathbb{R}$,

$$f(g(y)) = \ln(\frac{1 - \frac{1}{e^y+1}}{\frac{1}{e^y+1}}) = \ln(\frac{1}{\frac{1}{e^y+1}} - 1) = \ln(e^y + 1 - 1) = y$$

thus $g = f^{-1}$ is the inverse of f, so f is a bijection.