Lecture 23

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1. Let A, B, C be sets and $f : A \to B$ and $g : B \to C$ be functions. Prove or disprove the following:

   (a) If $g \circ f$ is injective then f is injective.

   This is true. Note $g \circ f : A \to C$. To show that f is injective, consider arbitrary $a, a' \in A$ such that $f(a) = f(a')$, then $g(f(a)) = g(f(a'))$ so by injectivity of $g \circ f$, $a = a'$.

   (b) If f is injective then $g \circ f$ is injective.

   This is false. Consider $f : [2] \to [2]$ defined as $f(n) = n$ for all $n \in [2]$. Clearly f is injective. Consider $g : [2] \to [1]$ where $g(n) = 1$ for all $n \in [2]$, then $g \circ f(n) = 1$ for all $n \in [2]$, so $g \circ f$ is not injective.

   (c) If $g \circ f$ is injective then g is injective.

   This is false. Consider $f : [1] \to [2]$ where $f(1) = 1$, and $g : [2] \to [1]$ where $g(1) = g(2) = 1$. Then $g \circ f : [1] \to [1]$ is injective, but g is not injective.

2. (Example 7.5.9) Let $h : \mathbb{R} - \{-1\} \to \mathbb{R} - \{1\}$ be defined by $h(x) = \frac{x}{1+x}$. Prove that h is a bijection by

   (a) Showing h is 1-to-1 and onto.

   To see that h is 1-to-1 (injective), consider arbitrary $x, x' \in \mathbb{R} - \{1\}$ such that $h(x) = h(x')$, then

$$\begin{aligned} \frac{x}{1+x} &= \frac{x'}{1+x'} \\ x(1+x') &= x'(1+x) \\ x + xx' &= x' + x'x \\ x &= x' \end{aligned}$$

   To see that h is onto (surjective), consider arbitrary $y \in \mathbb{R} - \{-1\}$, then note that $f(\frac{y}{1-y}) = y$ (see below), and $\frac{y}{1-y} \in \mathbb{R} - \{-1\}$ because $y \neq 1$, and $\frac{y}{1-y} \neq -1$.

   (b) Showing h is invertible.

   To find $h^{-1} : \mathbb{R} - \{1\} \to \mathbb{R} - \{-1\}$, note that $\forall x \in \mathbb{R} - \{1\}$ and $\forall y \in \mathbb{R} - \{-1\}$,

$$y = \frac{x}{1+x} \iff y(1+x) = x \iff y + xy - x = 0 \iff y + x(y-1) = 0 \iff x = -\frac{y}{y-1} \iff x = \frac{y}{1-y}$$

   note that we are relying on the fact that $x \neq -1$ and $y \neq 1$.

   Therefore, $h^{-1} : \mathbb{R} - \{1\} \to \mathbb{R} - \{-1\}$ defined $h^{-1}(y) = \frac{y}{1-y}$ is the inverse of h (check that $\frac{y}{1-y} \neq -1$ for $y \neq 1$).

3. (Try it #3) Let $U = \{x \in \mathbb{R} \mid -1 < x < 1\}$ and $I = \{x \in \mathbb{R} \mid -6 < x < 12\}$. Let $g : U \to I$ be defined by $g(x) = 9x + 3$ for all $x \in U$. Prove that g is a bijection by finding g^{-1}.

   Given $x \in U$ and $y \in I$,

$$y = 9x + 3 \iff \frac{y-3}{9} = x$$

   therefore, consider $g^{-1} : I \to U$ defined $g^{-1}(y) = \frac{y-3}{9}$. For any $y \in I$, $-6 < y < 12$ so

$$-1 = \frac{-6-3}{9} < \frac{y-3}{9} < \frac{12-3}{9} = 1$$

   so $\frac{y-3}{9} \in U$ as desired.