Lecture 22

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November 11, 2013

1. Let A, B be sets and $f : A \to B$ be a function. Further suppose $X_1, X_2 \subseteq A$ and $Y \subseteq B$. Prove or disprove:

   Recall the definitions $\text{Im}_f(X) = \{b \in B \mid \exists a \in X \ f(a) = b\}$, and $\text{PreIm}_f(Y) = \{a \in A \mid f(a) \in Y\}$.

   (a) $\text{Im}_f(\text{PreIm}_f(Y)) \subseteq Y$

   This is true. Consider arbitrary $y \in \text{Im}_f(\text{PreIm}_f(Y))$, then there is some $x \in \text{PreIm}_f(Y)$ such that $f(x) = y$. Therefore, $y \in Y$.

   (b) $Y \subseteq \text{Im}_f(\text{PreIm}_f(Y))$

   This is false. Counter-example: Consider $f : [1] \to [2]$ where $f(1) = 1$, and $Y = [2]$. Therefore, $\text{PreIm}_f(Y) = [1]$ and $\text{Im}_f(\text{PreIm}_f(Y)) = [1]$, but $Y = [2]$.

   (c) $\text{Im}_f(X_1) - \text{Im}_f(X_2) \subseteq \text{Im}_f(X_1 - X_2)$

   This is true. Consider $y \in \text{Im}_f(X_1) - \text{Im}_f(X_2)$, then $y \in \text{Im}_f(X_1)$ and $y \notin \text{Im}_f(X_2)$, so there is some $x \in X_1$ such that $f(x) = y$, and note that $x \notin X_2$ since $y \notin \text{Im}_f(X_2)$. Therefore, $x \in X_1 - X_2$ so $\text{Im}_f(X_1 - X_2)$.

   (d) $\text{Im}_f(X_1 - X_2) \subseteq \text{Im}_f(X_1) - \text{Im}_f(X_2)$

   This is false. Counter-example: Consider $f : [2] \to [1]$ with $f(1) = f(2) = 1$, and $X_1 = [2]$ and $X_2 = [1]$. Therefore, $\text{Im}_f(X_1 - X_2) = \text{Im}_f(\{2\}) = \{1\}$. However, $\text{Im}_f(X_1) - \text{Im}_f(X_2) = [1] - [1] = \varnothing$.

2. Consider the following function $f : \mathbb{N} \to \mathbb{N}$.

$$f(n) = \begin{cases} n+1 & n \text{ odd} \\ n-1 & n \text{ even} \end{cases}$$

   f is injective. Consider $n, m \in \mathbb{N}$ such that $f(n) = f(m)$. If $f(n) = f(m)$ is odd, n, m must have been even, so $n + 1 = m + 1$ so $n = m$. If $f(n) = f(m)$ is even, n, m must have been odd, so $n - 1 = m - 1$ so $n = m$.

3. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $\forall (m, n) \in \mathbb{N} \times \mathbb{N}$, $f(m, n) = 2^{m-1}(2n - 1)$. Prove that f is a bijection.

   First, to see that f is injective, suppose $f(m, n) = f(m', n')$ so $2^{m-1}(2n - 1) = 2^{m'-1}(2n' - 1)$. Note that $2n - 1$ and $2n' - 1$ are both odd, so it does not have 2 in its prime factorization. Therefore, the coefficient of 2 in the prime factorization of $f(m, n)$ is $m - 1$ and the coefficient of 2 in the prime factorixation of $f(m', n')$ is $m' - 1$. By uniqueness of prime factorizations, $m - 1 = m' - 1$, so $m = m'$. Therefore, $2^{m-1}(2n - 1) = 2^{m-1}(2n' - 1)$, so $2n - 1 = 2n' - 1$, so $n = n'$ as well. Therefore, $(m, n) = (m', n')$. Therefore, f is injective.

   Now, we will check that f is surjective. Consider $y \in \mathbb{N}$, then y can be written as a product of a power of 2 and an odd number, so $y = 2^a(2b + 1)$ for some $a, b \in \mathbb{Z}_+$. Therefore, letting $m = a + 1 \in \mathbb{N}$ and $n = b + 1 \in \mathbb{N}$, so $y = 2^{m-1}(2n - 1) = f(m, n)$. Thus f is surjective.

   Therefore, f is a bijection.

4. Define a bijection $f : \mathbb{N} \to \mathbb{Z}$. Prove that your function is a bijection.

   Let

$$f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ -\frac{n-1}{2} & n \text{ odd} \end{cases}$$

   Hint: Note that $f(n) > 0 \iff n$ is even.