Lecture 16

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1. Let $A = \{n \in \mathbb{Z} \mid n \geq 100\}$. Consider the relation R on A defined by $(a, b) \in R$ iff a and b agree in their first 3 decimal digits. For example, $(999, 9990) \in R$.

   Clearly, R is an equivalence relation, because R just looks at equality of the first three digits, so reflexivity, symmetry and transitivity follows from the properties of equality.

   The elements in the equivalence class of 2013 has the form $201\ldots$. In general, if x, y, z are decimal digits, $x > 0$, then
   $$[xyz]_R = \{n \in \mathbb{Z} \mid n = xyz\ldots\}$$

   Therefore, each equivalence class can be identified with the first three digits xyz. There are 9 choices for x, 10 choices for y and 10 choices for z. Therefore,
   $$|A/R| = 9 \cdot 10 \cdot 10 = 900$$
   there are 900 equivalence classes.

2. Let $X =$ the set of triangles in the cartesian plane. For $x, y \in X$ define $x \sim y \iff x$ and y are similar. Is $\sim$ an equivalence relation? Why or why not? If it is an equivalence relation, describe $X/\sim$.

   The answer to this question depends on whether or not we consider a degenerate triangle to be a triangle (i.e. a triangle with 0 area, with vertices $(0,0), (0,0), (0,0)$). If we do consider degenerate triangles, then depending on our notion of similarity, the degenerate triangle of a single point could be similar to every triangle, which would mean that $\sim$ would not be an equivalence relation.

   Therefore, we will consider X to contain only triangles with area > 0. In which case, $\sim$ would be an equivalence relation. One presentation of similarity of triangles is to look at their angles: two triangles are similar if and only if their angles are the same (up to permutation, so $(30°, 60°, 90°)$ would be the same as $(60°, 30°, 90°)$). We also know that for any triple $x, y, z \in \mathbb{R}$, $0 < x, y, z < 180$ and $x + y + z = 180$, there is a triangle with angles (x, y, z).

   Therefore, let T be a triangle with angles (x, y, z), then $[T]_R =$ the set of triangles with angles (x, y, z).

3. Define a relation R on $\mathbb{Z}$ by $(x, y) \in R \iff 4|x^2 - y^2$. Prove that R is an equivalence relation. Describe the equivalence class $[0]_R$. How many equivalence classes are in $\mathbb{Z}/R$?

   Note that
   $$4|(x+y)(x-y) = x^2 - y^2 \iff x, y \text{ both even or both odd}$$
   because x, y both even or both odd $\implies x + y, x - y$ both even $\implies 4|x^2 - y^2$. On the other hand, x, y not both even or both odd $\implies x + y, x - y$ both odd $\implies 4 \nmid x^2 - y^2$.

   Obviously $x \sim x$, so x, x are either both even or both odd. Suppose $x \sim y$, then x, y has the same parity, so $y \sim x$. If $x \sim y, y \sim z$ then x, y, z all have the same parity, so $x \sim z$.

   The equivalence class $[0]_R$ is the set of integers with the same parity as 0, so $[0]_R = \{x \in \mathbb{Z} \mid x \text{ is even}\}$.

   There are exactly two equivalence classes, $[0]_R$ the set of even integers and $[1]_R$ the set of odd integers, so $|\mathbb{Z}/R| = 2$.

4. Define a relation R on $\mathbb{R}$ by $(x, y) \in R \iff \exists k, l \in \mathbb{N}.\ x^k = y^l$.

   R is an equivalence relation. Reflexivity: clearly $x^1 = x^1$ so $(x, x) \in R$. Symmetry: If $(x, y) \in R$, then $\exists k, l \in \mathbb{N}\ x^k = y^l$ so $y^l = x^k$ so $(y, x) \in R$. Transitivity: if $(x, y), (y, z) \in R$, then $\exists k, l, m, n \in \mathbb{N}$ $x^k = y^l$ and $y^m = z^n$, so
   $$x^{km} = (x^k)^m = (y^l)^m = y^{lm} = (y^m)^l = (z^n)^l = z^{nl}$$

where $km, nl \in \mathbb{N}$, so $(x, z) \in R$.

$$
\begin{aligned}
[0]_R &= \{0\} \\
[1]_R &= \{1\} \\
[2]_R &= \{2^r \mid r \in \mathbb{Q}, r > 0\} \cup \{-2^r \mid r \in \mathbb{Q}, r > 0\}
\end{aligned}
$$

where in general $[x]_R = \{x^r \mid r \in \mathbb{Q}, r > 0\} \cup \{-x^r \mid r \in \mathbb{Q}, r > 0\}$ because for any $x \in \mathbb{R}$, $x^2 = |x|^2$ so $[x]_R = [|x|]_R$.

Now for any $y \in \mathbb{R}$, since $y \in [x]_R \iff |y| \in [x]_R$, we can consider the case where y is positive. For any $k, l \in \mathbb{N}$,

$$x^k = y^l \iff x^{\frac{k}{l}} = y$$

and we know that $\frac{k}{l}$ must be positive because $k, l \in \mathbb{N}$, and given any positive $r \in \mathbb{Q}$, $r = \frac{k}{l}$ for some positive integer k, l.