Lecture 14

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1. Claim: For all $n \in \mathbb{N} - \{1\}$, n has a prime factorization.

   We will prove this by strong induction on n. Base case: $n = 2$ is prime, so 2 is its prime factorization.

   Inductive case: Suppose $\forall k < n$ k has a prime factorization, then consider n. If n is prime, n is the prime factorization. Otherwise, n is not prime, so $n = ab$ where $1 < a, b < n$. By inductive hypothesis, a, b has prime factorizations $a = p_1 \cdot p_2 \cdots p_k$ and $b = q_1 \cdot q_2 \cdots q_l$. Then $n = ab = p_1 \cdot p_2 \cdots p_k \cdot q_1 \cdot q_2 \cdots q_l$ is a prime factorization.

2. Claim: $\forall n \in \mathbb{N}.\ (n \geq 12 \rightarrow (\exists a, b \in \mathbb{Z}^{\geq 0}.\ n = 4a + 5b))$.

   Base cases: Four cases $12 = 4 \cdot 3$, $13 = 4 \cdot 2 + 5$, $14 = 4 + 5 \cdot 2$, $15 = 5 \cdot 3$.

   Inductive hypothesis: For some $n \geq 12$ suppose $\forall 12 \leq k < n$ $\exists a, b \in \mathbb{Z}^{\geq 0}$. $k = 4a + 5b$.

   Then suppose $n > 15$ (so we are not in one of our base cases), then $12 \leq n - 4 < n$ so by inductive hypothesis $\exists a, b \in \mathbb{Z}^{\geq 0}$ such that $n - 4 = 4a + 5b$. Therefore, $n = 4(a + 1) + 5b$.

3. Suppose I have variable proposition $P(m, n)$ defined on $\mathbb{N} \times \mathbb{N}$ and I know: (1) $P(1, 1)$ holds, (2) $\forall m \in \mathbb{N}.\ P(m, 1) \rightarrow P(m + 1, 1)$, and (3) $\forall m, n \in \mathbb{N}.\ P(m, n) \rightarrow P(m, n + 1)$. For what values of $(m, n) \in \mathbb{N} \times \mathbb{N}$ can I conclude $P(m, n)$ holds?

   By induction on m, using (1) as base case and (2) as inductive step, we can conclude that $\forall m \in \mathbb{N}.\ P(m, 1)$ holds. Now for a chosen m, we can induct on n using what we just showed as a base case and (3) as our inductive step, we can show that $\forall n \in \mathbb{N}.\ P(m, n)$.

   Therefore, we have shown that $\forall m, n \in \mathbb{N}.\ P(m, n)$.

4. Define the sequence
   $$a_0 = 2,\ a_1 = 2,\ a_n = 2a_{n-1} + 8a_{n-2} \text{ for } n \geq 2$$

   Prove by induction that $a_n = 4^n + (-2)^n$ for all $n \in \mathbb{Z}^{\geq 0}$.

   Base cases: $a_0 = 2 = 1 + 1 = 4^0 + (-2)^0$, $a_1 = 2 = 4 - 2 = 4^1 + (-2)^1$.

   Inductive case: Suppose for some $n \geq 2$ that $\forall k < n$ $a_k = 4^k + (-2)^k$, then
   $$\begin{aligned} a_n &= 2a_{n-1} + 8a_{n-2} \\ &= 2(4^{n-1} + (-2)^{n-1}) + 8(4^{n-2} + (-2)^{n-2}) \\ &= 2 \cdot 4^{n-1} - (-2)^n + 2 \cdot 4^{n-1} + 2(-2)^n \\ &= (2 + 2)4^{n-1} + (2 - 1)(-2)^n \\ &= 4^n + (-2)^n \end{aligned}$$

5. Consider the following equation: $4x^4 + 2y^4 = z^4$. In this problem, you will prove that this equation has no solution $(x, y, z) \in \mathbb{N}^3$ by descent.

   (a) AFSOC that $(x, y, z) \in \mathbb{N}^3$ is such a solution, and suppose further that this solution has the smallest value of x amongst all solutions.

   (b) z is even because $z^4 = 2(2x^4 + y^4)$ is even, and z even $\iff$ z^2 even $\iff$ z^4 even.

   (c) y is even because z is even so let $z = 2k$ for $k \in \mathbb{Z}$, then $2y^4 = (2k)^4 - 4x^4$ so $y^4 = 2^3k^4 - 2x^4$ which is even, so y is even.

   (d) x is even because if y, z are even, let $y = 2l$ and $z = 2k$ for $k, l \in \mathbb{Z}$, then $4x^4 = z^4 - 2y^4 = (2k)^4 - 2(2l)^4 = 4 \cdot 2^2k^4 - 4 \cdot 2^3l^4$ so $x^4 = 2^2k^4 - 2^3l^4$ is even, so x is even.

   (e) Note therefore that $(a, b, c) = (\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) \in \mathbb{N}^3$ is also a solution because x, y, z are even and
   $$4x^4 + 2y^4 = z^4 \implies \frac{4x^4 + 2y^4}{2^4} = \frac{z^4}{2^4} \implies 4(\frac{x}{2})^4 + 2(\frac{y}{2})^4 = (\frac{z}{2})^4$$

(f) Therefore, since $\frac{x}{2} < x$ so (x, y, z) is not a solution with x being smallest amongst all solutions, which is a contradiction to (a), so there are no solutions in $\mathbb{N}^3$.

(g) Note that if $(x, y, z) \in \mathbb{Z}^3$ is a solution, then $(|x|, |y|, |z|) \in (\mathbb{Z}^{\leq 0})^3$ is a solution, since $|x|^4 = x^4$ etc. Therefore, we showed that $x \notin \mathbb{N}$, so $x \notin \mathbb{Z}^{<0}$ either, so $x = 0$. Therefore, the only possible solution is inside $\mathbb{Z}^3$, with $x = 0$. Thus, suppose $(0, y, z) \in (\mathbb{Z}^{\leq 0})^3$ is such a solution, then $2y^4 = z^4$, so taking the positive square roots we have $\sqrt{2}y^2 = z^2$ (because clearly both sides must be positive so we can take the square root, and $\sqrt{2}y^2$ and z^2 must be positive, so we are using the positive square root), so $\sqrt{2} = \frac{z^2}{y^2}$ is a contradiction because $\sqrt{2}$ is irrational. Therefore, $y = 0$, so $z^4 = 4 \cdot 0^4 + 2 \cdot 0^4 = 0$, which means that the only solution is $(x, y, z) = (0, 0, 0) \in \mathbb{Z}^3$ the trivial solution.