Lecture 13

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1. Suppose I want to modify PMI to conclude $\forall n \in S.\ P(n)$ holds for the following sets S. how should I modify the two conditions from PMI to prove this?

   (a) $S = \{\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \dots\}$.
   Base case: Show $P(\frac{1}{3})$. Inductive step: Show $\forall x \in S.\ P(x) \implies P(\frac{x}{2})$.

   (b) Assume $a \in \mathbb{R}$ and $r \in \mathbb{R}^+$ with $r \neq 0$. Let $S = \{x \in \mathbb{R} \mid \exists n \in \mathbb{N} \cup \{0\}.\ x = ar^n\}$.
   Base case: Show $P(ar^0)$. Inductive step: Show $\forall x \in S.\ P(x) \implies P(rx)$.

   (c) $S = \{2, 2^2, 2^{2^2}, \dots\}$
   Base case: Show $P(2)$. Inductive step: Show $\forall x \in S.\ P(x) \implies P(2^x)$.

2. Induction practice

   (a) Claim: $\forall n \in \mathbb{N} - \{1, 2\}.\ n^2 \geq 2n + 3$.
   We induct on $n \geq 3$. Base case: For $n = 3$, $3^2 = 9 \geq 9 = 2 \cdot 3 + 3$.
   Inductive case: Assume for some $n \geq 3$ that $n^2 \geq 2n + 3$, then

   $$(n+1)^2 = n^2 + 2n + 1 \geq 2n + 3 + 2n + 1 = 2n + 3 + 2 \cdot 3 + 1 = 2n + 10 \geq 2n + 5$$

   by IH, and we make the substitution $n \geq 3$.

   (b) Claim: $13^n + 6$ is a multiple of 7 for all even number n.
   We induct on n even. Base case: For $n = 0$, $13^0 + 6 = 1 + 6 = 7$ is a multple of 7.
   Inductive case: Assume for some even n that $13^n + 6 = 7k$ for some $k \in \mathbb{Z}$. Then

   $$13^{n+2}+6 = 13^n \cdot 13^2 + 6 = (7k-6)\cdot 13^2 + 6 = 7\cdot 13^2 k - 6 \cdot 13^2 + 6 = 7 \cdot 13^2 k - 1008 = 7 \cdot 13^2 k - 7 \cdot 12^2 = 7(13^2 k - 12^2)$$

   is a multiple of 7.

   (c) Define a sequence $\langle a_i \rangle_{i \in \mathbb{N}}$ recursively as follows: $a_1 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$ for $n \in \mathbb{N}$.

      i. Claim: $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.
      Observe that for all $x \in \mathbb{R}$,

      $$x^2 \geq 1 + x \implies$$

      We induct on n to show $a_n < a_{n+1}$. Base case: $a_1 = 1 < \sqrt{2} = a_2$.
      Inductive case: Assume $a_n < a_{n+1}$ for some $n \in \mathbb{N}$. Then

      $$a_n < a_{n+1} \implies a_n + 1 < a_{n+1} + 1 \implies \sqrt{a_n + 1} < \sqrt{a_{n+1} + 1} \implies a_{n+1} < a_{n+2}$$

      ii. Claim: $a_n < 2$ for all $n \in \mathbb{N}$.
      We induct on n. Base case: $a_1 = 1 < 2$.
      Inductive case: Assume $a_n < 2$ for some $n \in \mathbb{N}$. Then

      $$a_n < 2 \implies a_n + 1 < 3 \implies \sqrt{a_n + 1} < \sqrt{3} \implies a_{n+1} < \sqrt{3} < 2$$

      Observe that $3 < 4 \implies \sqrt{3} < \sqrt{4} = 2$.

      iii. This shows that the sequence $\langle a_i \rangle_{i \in \mathbb{N}}$ converges to some $a \leq 2$.

   (d) Define a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ recursively by $x_1 = 1$ and $x_{n+1} = x_n + \frac{1}{x_n}$ for $n \in \mathbb{N}$.
   Claim: $x_n > \sqrt{n}$ for all $n \in \mathbb{N} - \{1\}$.
   Induct on $n \geq 2$ that $x_n > \sqrt{n}$. Base case: $x_2 = 2 > \sqrt{2}$.
   Inductive case: Assume for some $n \geq 2$ that $x_n > \sqrt{n}$. Then

   $$(x_{n+1})^2 = (x_n + \frac{1}{x_n})^2 = x_n^2 + 2 + \frac{1}{x_n^2} > n + 2 + \frac{1}{x_n^2} > n + 1$$

   by inductive hypothesis, and using $1 + \frac{1}{x_n^2}$ is positive. Therefore, $x_{n+1} > \sqrt{n+1}$.