Lecture 12

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1. **Claim:**
$$\forall n \in \mathbb{N}. \sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$$

   We will prove this by induction.

   Base Case: for $n = 1$, $\sum_{i=1}^{1} (-1)^i i^2 = -1 = \frac{(-1)1(2)}{2}$.

   Inductive case: Assume for some $n \in \mathbb{N}$ we have $\sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$, then

$$\begin{aligned}
\sum_{i=1}^{n+1} (-1)^i i^2 &= \sum_{i=1}^{n} (-1)^i i^2 + (-1)^{n+1}(n+1)^2 \\
&= \frac{(-1)^n n(n+1)}{2} + (-1)^{n+1}(n+1)^2 \qquad \text{(by inductive hypothesis)} \\
&= \frac{-(-1)^{n+1} n(n+1) + 2(-1)^{n+1}(n+1)^2}{2} \\
&= \frac{(-1)^{n+1}(n+1)(-n+2n+2)}{2} \\
&= \frac{(-1)^{n+1}(n+1)(n+2)}{2}
\end{aligned}$$

   so we have shown by induction that $\forall n \in \mathbb{N}. \sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$

2. (a) Prove that $\forall x \in \mathbb{R}.\ x^2 \neq 1 \to x \neq 1$.

   We will prove this by contrapositive. Let $x \in \mathbb{R}$ be arbitrary, and $x = 1$, then $x^2 = 1$. Therefore, $\forall x \in \mathbb{R}.\ x = 1 \to x^2 = 1$, which is the contrapositive of the original statement.

   (b) Prove that $\forall n \in \mathbb{N}.\ n \geq 5 \to 2n^2 > (n+1)^2$.

   Suppose $n \in \mathbb{N}$ arbitrary, $n \geq 5$. Then

$$2n^2 = n^2 + n^2 > n^2 + 4n > n^2 + 2n + 1 = (n+1)^2$$

   where we made the substitution $n > 4$ and $2n > 1$.

   (c) Let $E(x)$ be the proposition "x is even." Prove that

$$\forall a, b \in \mathbb{Z}.\ E(a) \wedge E(b) \iff E(a+b) \wedge E(a \cdot b)$$

   We first prove $E(a) \wedge E(b) \implies E(a+b) \wedge E(a \cdot b)$. Consider $a, b \in \mathbb{Z}$ arbitrary, and $E(a) \wedge E(b)$ so let $a = 2h$ and $b = 2k$ for $h, k \in \mathbb{Z}$. Then $a + b = 2h + 2k = 2(h+k)$ is even and $a \cdot b = (2h)(2k) = 2(2hk)$ is even, so $E(a+b) \wedge E(a \cdot b)$.

   Now we prove $E(a) \wedge E(b) \impliedby E(a+b) \wedge E(a \cdot b)$ by contraposition. Consider $a, b \in \mathbb{Z}$ arbitrary such that $\neg(E(a) \wedge E(b))$. Equivalently, $\neg E(a) \vee \neg E(b)$. Consider the case where $\neg E(a)$, so a is odd. Then if $a + b$ is even then b is odd. However, this means that $a \cdot b$ is odd which gives a contradiction. Now consider the case where $\neg E(b)$, so b is odd. Then if $a + b$ is even then a is odd, so $a \cdot b$ is odd which gives a contradiction.

3. **Claim:** There are no positive integer solutions to the equation $x^2 - y^2 = 1$.

   Symbolically, this is $\forall x, y \in \mathbb{N}\ x^2 - y^2 \neq 1$.

   Suppose for sake of contradiction that $x, y \in \mathbb{N}$ such that $x^2 - y^2 = 1$. Then note that

$$(x+y)(x-y) = x^2 - y^2 = 1$$

   so in particular $(x+y)$ divides 1. However, since x, y are positive integers, $x + y \geq 2$, which is a contradiction because the only divisors of 1 are -1 and 1.

4. **Claim:** If a and b are both real numbers such that the product ab is irrational then either a or b must be irrational.

   To write this out symbolically:

$$\forall a, b \in \mathbb{R}.\ ab \notin \mathbb{Q} \to a \notin \mathbb{Q} \vee b \notin \mathbb{Q}$$

   Consider the contrapositive (with De Morgan's law): $\forall a, b \in \mathbb{R}.\ a \in \mathbb{Q} \wedge b \in \mathbb{Q} \to ab \in \mathbb{Q}$. Let $a, b \in \mathbb{R}$ be arbitrary, and suppose $a, b \in \mathbb{Q}$. Then let $a = \frac{s}{t}, b = \frac{u}{v}$ where $s, t, u, v \in \mathbb{Z}$ and $v, t \neq 0$. Then $ab = \frac{su}{tv} \in \mathbb{Q}$ since $uv \neq 0$. Therefore, we have shown the contrapositive.

5. **Claim:** For any distinct integers p and q it is the case that $p - 1$ is a multiple of $q - p$ if and only if $q - 1$ is a multiple of $q - p$.

   To write this out symbolically:

$$\forall p, q \in \mathbb{Z}.\ (p \neq q \implies ((q-p)|(p-1) \leftrightarrow (q-p)|(q-1)))$$

   Consider $p, q \in \mathbb{Z}$ arbitrary such that $p \neq q$.

   Suppose $(q-p)|(p-1)$, then there is some $n \in \mathbb{Z}$ such that $(q-p)n = p-1$. It follows that

$$\begin{aligned} & (q-p)n = p-1 \\ \implies \quad & (q-p)n - (p-q) = p - 1 - (p-q) \\ \implies \quad & (q-p)(n-1) = q-1 \end{aligned}$$

   so $(q-p)|(q-1)$ as desired.

   On the other hand, suppose $(q-p)|(q-1)$, then there is some $n \in \mathbb{Z}$ such that $(q-p)n = q-1$, then

$$\begin{aligned} & (q-p)n = q-1 \\ \implies \quad & (q-p)n + (p-q) = q - 1 + (p-q) \\ \implies \quad & (q-p)(n+1) = p-1 \end{aligned}$$

   so $(q-p)|(p-1)$ as desired.