Lecture 11

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October 3, 2013

Universal Claims: Of the form $\forall x \in S.P(x)$. (*Note: Commonly $P(x)$ is a conditional statement. We'll revisit thse when we do the section on conditional claims.*)

- Direct Proofs: General idea - Let $x \in S$ be arbitrary. Show that $P(x)$ must hold

Ex 1) **Claim:** For all $x, y \in \mathbb{R}$. $x^2 + y^2 \geq 2xy$.

Let $x, y \in \mathbb{R}$ be arbitrary. Consider $(x-y)^2 \geq 0$, which we know to be non-negative because squares of real numbers are non-negative. Through algebraic manipulation

$$x^2 + y^2 - 2xy = (x-y)^2 \geq 0$$

so $x^2 + y^2 \geq 2xy$

Ex 2) **Claim:** The sum of any 2 rational numbers is a rational number.

We will write this out symbolically: $\forall a, b \in \mathbb{Q}$. $(a+b) \in \mathbb{Q}$.

Let $a, b \in \mathbb{Q}$ be arbitrary, then by definition of rational numbers there exists $s, t, u, v \in \mathbb{Z}$ where $t \neq 0$ and $v \neq 0$ such that

$$a = \frac{s}{t} \qquad b = \frac{u}{v}$$

Then it follows that

$$a + b = \frac{s}{t} + \frac{u}{v} = \frac{sv + tu}{tv} \in \mathbb{Q}$$

because $(sv + tu), (tv) \in \mathbb{Z}$ and $tv \neq 0$.

- Indirect Proofs: Observe that $\neg\forall x \in S.\ P(x) \iff \exists x \in S.\ \neg P(x)$.

  Proof strategy: Assume for sake of contradiction that there exists an $x \in S$ such that $\neg P(x)$ holds. Arrive at a contradiction.

Ex 3) **Claim:** $\forall m, n \in \mathbb{Z}$. $14m + 21n \neq 1$.

Assume for the sake of contradiction that $m, n \in \mathbb{Z}$ are such that $14m + 21n = 1$. Then

$$1 = 14m + 21n = 2 \cdot 7m + 3 \cdot 7n = (2m + 3n) \cdot 7$$

However, this implies that 7 divides 1, which is a contradiction.

Ex 4) **Claim:** $\sqrt{2}$ is irrational.

One way to write this out as a universal claim is $\forall x, y \in \mathbb{Z}$. $\frac{x}{y} \neq \sqrt{2}$.

Assume for the sake of contradiction that there are such $x, y \in \mathbb{Z}$ such that $\frac{x}{y} = \sqrt{2}$. Additionally, we will require that the fraction $\frac{x}{y}$ is irreducible, meaning that x, y are coprime, meaning that x and y are not both multiples of the same natural number ($\neg\exists a, b, c \in \mathbb{Z}.ac = x \ \wedge bc = y$). The reason why we can make this assumption is because given a fraction $\frac{x}{y}$, we can divide by whatever x and y are both multiples of to obtain smaller values x_1, y_1 such that $|x_1| < |x|$ and $|y_1| < |y|$ (simplify the fraction). We can iterate this process to eventually find x_n, y_n such that $\frac{x_n}{y_n}$ is irreducible, because if we cannot, then we have an infinite sequence $y, y_1, y_2, \ldots$ such that $|y| > |y_1| > |y_2| > \ldots$ where each $|y_i| \in \mathbb{N}$. This is impossible because there can be no infinte descending chain of natural numbers.

Therefore, it is sufficient to arrive at a contradiction from assuming that there exists $x, y \in \mathbb{Z}$ such that $\frac{x}{y} = \sqrt{2}$ and the fraction is irreducible.

By assumption, $\frac{x^2}{y^2} = 2$ or equivalently $x^2 = 2y^2$. This implies that x^2 is even, which means that x was even because the square of an odd number is odd, so let $x = 2z$ for some $x \in \mathbb{Z}$. Substitution into our original equation gives $4z^2 = (2z)^2 = 2y^2$ so $2z^2 = y^2$ so y^2 is even as well, which by the same argument means that y is also even. However, if x and y are both even, then $\frac{x}{y}$ is not irreducible (since we can divide both by 2). This gives a contradiction.

Use the intermediate value theorem to come up with a non-constructive direct proof of the following existential claim:

Claim: $\exists x \in \mathbb{R}.\ x^5 - 3x + 1 = 0$.

Consider the continuous function $f(x) = x^5 - 3x + 1$. Note that $f(0) = 1$ and $f(1) = -1$. Therefore, by the intermediate value theorem, since $f(1) = -1 < 0 < 1 = f(0)$, there exists some $c \in \mathbb{R}$ where $0 < c < 1$ such that $f(c) = 0$. It follows that $f(c) = c^5 - 3c + 1 = 0$, so c is a real number satisfying our requirement.