Lecture 17

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1. Find the Maclaurin series for $f(x)$ using the definition of a MacLaurin series. [Assuem that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

$$f(x) = \sin(\pi x)$$

Recall the definition of Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$$

so note that

$$\begin{aligned} f(x) &= \sin(\pi x) & f(0) &= 0 \\ f'(x) &= \pi \cos(\pi x) & f'(0) &= \pi \\ f''(x) &= -\pi^2 \sin(\pi x) & f''(0) &= 0 \\ f'''(x) &= -\pi^3 \cos(\pi x) & f'''(0) &= -\pi^3 \\ f^{(4)}(x) &= \pi^4 \sin(\pi x) & f^{(4)}(0) &= 0 \end{aligned}$$

and clearly the pattern continues, because $f^{(4)}(x) = \pi^4 f(x)$, so

$$f^{(2n+1)}(0) = (-1)^n \pi^{2n+1} \qquad f^{(2n)}(0) = 0$$

therefore

$$f(x) = \frac{\pi}{1!} x + \frac{-\pi^3}{3!} x^3 + \frac{\pi^5}{5!} x^5 + \frac{-\pi^7}{7!} x^7 + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} x^{2n+1}$$

Using the ratio test, we can find the radius of convergence

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1} \pi^{2n+3} x^{2n+3}}{(2n+3)!} \frac{(2n+1)!}{(-1)^n \pi^{2n+1} x^{2n+1}} \right| = \frac{\pi^2 x^2}{(2n+3)(2n+2)} \longrightarrow 0 \text{ as } n \to \infty$$

so converges for all x, so the radius of convergence is ∞.

2. Use the binomial series to expand the function as a power series. State the radius of convergence.

$$\begin{aligned} \frac{1}{(2+x)^3} &= \left(\frac{1}{2} \frac{1}{1+\frac{x}{2}}\right)^3 = \frac{1}{8}\left(1+\frac{x}{2}\right)^{-3} = \frac{1}{8} \sum_{n=0}^{\infty} \binom{-3}{n} \left(\frac{x}{2}\right)^n \\ &= \frac{1}{8}\left[1 + \frac{-3}{1!}\frac{x}{2} + \frac{(-3)(-4)}{2!}\left(\frac{x}{2}\right)^2 + \dots\right] \\ &= \frac{1}{8}\left[1 + \frac{-3}{1!}\frac{x}{2} + \frac{3 \cdot 4}{2!}\left(\frac{x}{2}\right)^2 + \dots + \frac{(-1)^n 3 \cdot 4 \cdots (n+2)}{n! 2^n} x^n + \dots\right] \\ &= \frac{1}{8}\left[\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2! 2^n} x^n\right] \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2^{n+4}} x^n \end{aligned}$$

has radius of convergenece $R = 2$ using the binomial series theorem.

3.

$$
\begin{aligned}
f(x) = \sin^2 x = \frac{1}{2}(1 - \cos(2x)) &= \frac{1}{2}(1 - \sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n}}{(2n)!}) \\
&= \frac{1}{2}(\sum_{n=1}^{\infty}(-1)^{n+1} \frac{(2x)^{2n}}{(2n)!}) \\
&= \sum_{n=1}^{\infty}(-1)^{n+1} \frac{2^{2n-1}x^{2n}}{(2n)!}
\end{aligned}
$$

with $R = \infty$.

We used the expansion

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

with $R = \infty$.

4.

$$
\begin{aligned}
\int \frac{\cos x - 1}{x} dx &= \int \frac{\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\right) - 1}{x} dx \\
&= \int \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n-1}}{(2n)!} dx \\
&= \sum_{n=1}^{\infty} \int (-1)^n \frac{x^{2n-1}}{(2n)!} dx \\
&= \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n}}{(2n)(2n)!} + C
\end{aligned}
$$

with $R = \infty$.