Lecture 13

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March 4, 2014

1.
$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}$$

Let $b_n = \frac{n^2}{n^3+4}$. To see $b_{n+1} \geq b_n$, we look at the function $f(x) = \frac{x^2}{x^3+4}$. Then

$$f'(x) = \frac{(x^3+4)2x-(3x^2)x^2}{(x^3+4)^2} = \frac{2x^4+8x-3x^4}{(x^3+4)^2} = \frac{-x^4+8x}{(x^3+4)^2} = \frac{-x(x^3-8)}{(x^3+4)^2}$$

so $f'(x) < 0$ for $x > 2$. Therefore, for $n > 2$ we know that $b_{n+1} > b_n$, so the sequence is eventually decreasing.

We also check that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n^2}{n^3+4} = \lim_{n\to\infty}\frac{1}{n+\frac{4}{n^2}} = 0$$

Therefore, the alternating series converges.

2.
$$\sum_{n=0}^{\infty}\frac{\sin((n+\frac{1}{2})\pi)}{1+\sqrt{n}} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{1+\sqrt{n}}$$

we know that $\sin((n+\frac{1}{2})\pi) = (-1)^n$ for all integers n.

Clearly $\frac{1}{1+\sqrt{n+1}} < \frac{1}{1+\sqrt{n}}$ and $\lim_{n\to\infty}\frac{1}{1+\sqrt{n}} = 0$. Thus the alternating series converges.

3. Approximate the sum of the series correct to four decimal places.

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}$$

We know that if $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$ then $|R_n| = |s-s_n| \leq b_{n+1}$. We want $|R_n| < 0.0001$ so we need to find n such that $b_{n+1} < 0.0001$

$$\frac{1}{2!} = \frac{1}{2} \qquad \frac{1}{4!} = \frac{1}{24} \qquad \frac{1}{6!} = \frac{1}{720} \qquad \frac{1}{8!} = \frac{1}{40320}$$

so we need $n = 3$.

$$s_3 = -\frac{1}{2}+\frac{1}{24}-\frac{1}{720} = -0.4597$$

4.
$$1-\frac{1\cdot 3}{3!}+\frac{1\cdot 3\cdot 5}{5!}-\frac{1\cdot 3\cdot 5\cdot 7}{7!}+\cdots+(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2n-1)!}+\ldots$$

Trying the ratio test

$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{(2n+1)!}\frac{(2n-1)!}{1\cdot 3\cdot 5\cdots(2n-1)} = \lim_{n\to\infty}\frac{2n+1}{(2n)(2n+1)} = \lim_{n\to\infty}\frac{1}{2n} = 0 < 1$$

so by the ratio test the series absolutely converges.

5. The terms of a series are defined recursively by

$$a_1 = 2 \qquad a_{n+1} = \frac{5n+1}{4n+3}a_n$$

Determine whether $\sum a_n$ converges or diverges.

$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty}|\frac{5n+1}{4n+3}a_n\frac{1}{a_n}| = \lim_{n\to\infty}\frac{5n+1}{4n+3} = \frac{5}{4} > 1$$

so the series diverges.