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Mathematical Paradoxes

taught in Spring 2024 by

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Featuring an appendix by

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Mathematical Paradoxes

Wednesday, January 17

Baker 237B

"Textbook:" The Banach-Tarski Paradox,
Tomkowicz + Wagon

Assessment: 40% Participation
30% Homework (approx. two week cycle)
30% Final project on current research topic.

The fundamental question (antiquity)

When are two things the same size?

"Def:" Given two "shapes" $A, B \subseteq \mathbb{R}^2$, A and B are "scissors congruent" if you can "cut" A into finitely many "pieces" and "move" them to form B. ("ignoring" "boundaries").

"Fact:" If A and B are scissors congruent, they have the same "area."

Examples:

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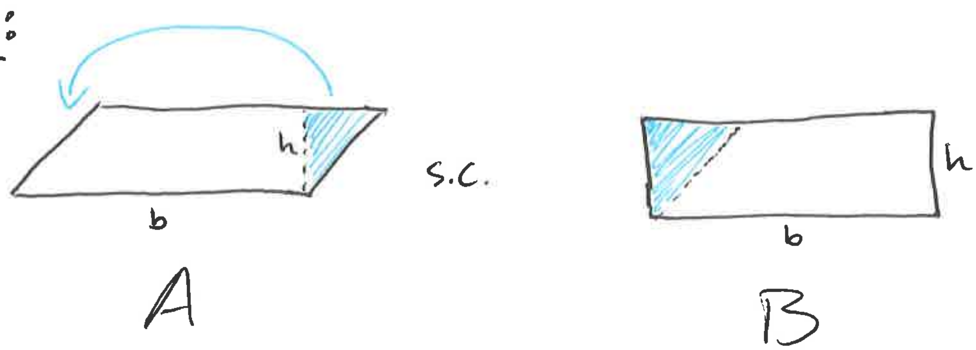
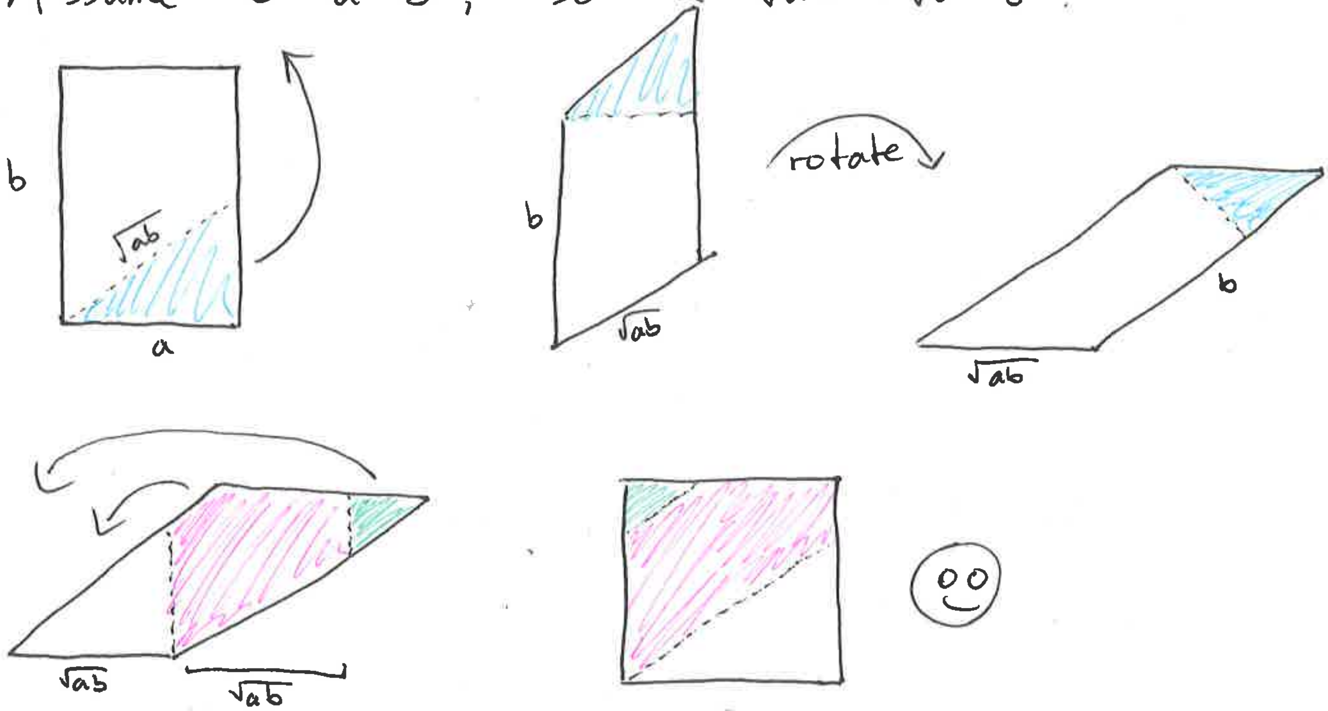


② $\boxed{1}$ Every rectangle is scissors congruent to a square.

Assume $0 < a < b$, so $a < \sqrt{ab} < \sqrt{a^2+b^2}$.

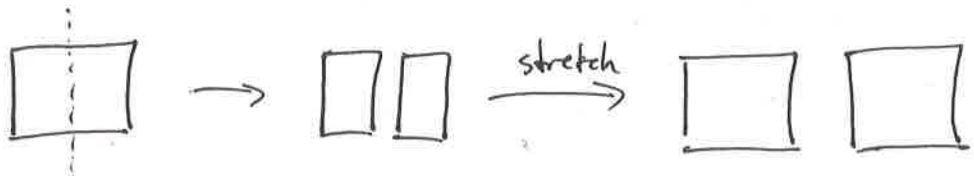


$\boxed{\text{HW}}$ Every polygon is scissors congruent to a square.
How about a disc?

<u>Problems:</u> Lots of words in the "<u>Def</u>" are inside scare quotes. What kinds of cuts and pieces are we considering? What moves are allowed? ...

③ Some pathologies to consider:

- Space filling curves exist. If you cut along one and "ignore boundaries," what's left?
- Consider this sequence of "moves:"



YAY! PARADOX

Clearly, "stretching" should be forbidden.

<u>Def</u>: Suppose that (X, d) is a metric space (e.g., $\mathbb{R}^n$ with usual Euclidean metric).
An <u>isometry</u> of (X, d) is a bijection $\varphi : X \to X$ s.t. $\forall x_0, x_1 \in X$ $d(\varphi(x_0), \varphi(x_1)) = d(x_0, x_1)$.

<u>Def</u>: An <u>isometric (self-)embedding</u> is any $\varphi : X \to X$ preserving d, i.e., $d(\varphi(x_0), \varphi(x_1)) = d(x_0, x_1)$.

So an isometry is a surjective isom emb.

$\boxed{\text{HW?}}$ Every isom (self-)emb of $\mathbb{R}^n$ is an isometry.

<u>Ex</u>: $\mathrm{Isom}(X, d) = \{\varphi : \varphi \text{ is an isometry of } (X, d)\}$ forms a group under composition.

<u>Prop</u>: Every isometry of $\mathbb{R}$ has the form $x \longmapsto ax + b$ for some $a \in \{-1, 1\}$, $b \in \mathbb{R}$.

④ pf (Prop):

Claim: If φ is an isometry with $\varphi : 0 \mapsto 0$, then $\exists a \in \{-1, 1\}$ with $\varphi : x \mapsto ax$.

pf (C): Put $a = \varphi(1)$. ■(C)

Now suppose that φ is arbitrary. Consider $\psi : x \mapsto \varphi(x) - \varphi(0)$. It is still an isometry, and $\psi : 0 \mapsto 0$. The claim says $\psi : x \mapsto ax$, and thus $\varphi : x \mapsto ax + \varphi(0)$ is as desired. ■(Prop)

Notation: $A = \bigsqcup_{i<k} B_i$ means □ $A = \bigcup_{i<k} B_i$, and

□ $i \neq j \Rightarrow B_i \cap B_j = \emptyset$.

Def: Given a metric space (X, d), we say that $A, B \subseteq X$ are equidecomposable (via isometries) if there is $k \in \mathbb{N}$ and sets $C_i \subseteq X$ and isometries $\gamma_i \in \mathrm{Isom}(X, d)$ s.t. □ $A = \bigsqcup_{i<k} C_i$

□ $B = \bigsqcup_{i<k} \gamma_i[C_i]$

Example: In $\mathbb{R}$, the half-open interval $[0, 1)$ and the open interval $(0, 1)$ are equidecomposable.

pf (sketch): Fix irrational $\alpha \in (0,1)$ and put

$D = \{ \overset{\text{frac part}}{(n\alpha)} : n \in \mathbb{N} \} \subseteq [0, 1)$. Now put

$C_0 = D \cap [0, 1-\alpha)$ $\gamma_0 : x \mapsto x + \alpha$

$C_1 = D \cap [1-\alpha, 1)$ $\gamma_1 : x \mapsto x + \alpha - 1$

$C_2 = [0, 1) \setminus (C_0 \cup C_1)$ $\gamma_2 : x \mapsto x$

■(Sketch)

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Paradoxes

Friday, Jan 19

Last time: $[0,1)$ and $(0,1)$ are equidecomposable (via isometries of $\mathbb{R}$).

Let's investigate some related examples.

Ex: Let $C = \{x \in \mathbb{R}^2 : d(x,0) = 1\}$ "the circle"
Then for all $x_0 \in C$, C is equidec. w/ $C \setminus \{x_0\}$.
Why? Pick some angle θ s.t. $\frac{\theta}{2\pi}$ is irrational.
Let r_θ be the isometry of "rotation by θ around 0."
Check: $\forall m,n \in \mathbb{N}$ $m \neq n \Rightarrow r_\theta^m(x_0) \neq r_\theta^n(x_0)$.

Put $D_0 = \{r_\theta^n(x_0) : n \in \mathbb{N}\}$ $\quad \gamma_0 = r_\theta$

$D_1 = C \setminus D_0$ $\quad \gamma_1 = id.$

You can also "produce points" with equidecomps:

Ex: Suppose that $y \in \mathbb{R}^2 \setminus C$. Then C is equidecomp. with $C \cup \{y\}$.

Similarly, whenever $F \subseteq \mathbb{R}^2 \setminus C$ is finite, C is equidecomp with $C \cup F$.

② <u>Prop:</u> Suppose that $A \subseteq C$ is countable. Then C is equidecomp. with $C \setminus A$.

<u>pf:</u> Say that an angle θ is GOOD if $\forall n \in \mathbb{N} \setminus \{0\}$ $A \cap r_\theta[A] = \emptyset$. Else, θ is BAD.

<u>Claim 1:</u> The set of BAD angles is countable.

<u>pf(C1):</u> θ is BAD iff

$$\exists n > 0 \ \exists a, b \in A \quad r_\theta^n(a) = b.$$

For each fixed $n > 0$ and $a, b \in A$, the set $\{\theta : r_\theta^n(a) = b\}$ is finite (in fact, of cardinality at most n). So the set of BAD angles is the union

$$\bigcup_{n>0} \bigcup_{a \in A} \bigcup_{b \in A} \{\theta : r_\theta^n(a) = b\},$$ thus is countable. $\blacksquare$(C1)

In particular, GOOD angles exist. Fix GOOD θ.

<u>Claim 2:</u> For all $m < n$, $r_\theta^m[A] \cap r_\theta^n[A] = \emptyset$.

<u>pf(C2):</u> $r_\theta^m[A] \cap r_\theta^n[A] = r_\theta^m\left[A \cap r_\theta^{n-m}[A]\right] = \emptyset$. $\blacksquare$(C2)

Now we can use r_θ to implement the previous example "in parallel" for all $a \in A$. That is,

$$D_0 = \bigcup_{n \in \mathbb{N}} r_\theta^n[A] \qquad \gamma_0 = r_\theta$$

$$D_1 = C \setminus D_0 \qquad \gamma_1 = id.$$

$\blacksquare$(Prop)

③ To spare us some repetition of labor, we discuss a more general setup.

<u>Def</u>: Suppose that Γ is a group and that X is a set. An <u>action</u> of Γ on X is a function

$$\Gamma \times X \longrightarrow X$$
$$(\gamma, x) \longmapsto \gamma \cdot x$$

satisfying
- $e_\Gamma \cdot x = x$
- $\gamma \cdot (\delta \cdot x) = (\gamma\delta) \cdot x$.

Equivalently, it's a group hom $\Gamma \to S_X$. group of permutations of X

Such an action $\Gamma \curvearrowright X$ induces an action $\Gamma \curvearrowright \mathcal{P}(X)$ via $\gamma \cdot A = \{\gamma \cdot a : a \in A\}$.

<u>Def</u>: Given an action $\Gamma \curvearrowright X$, we say that $A, B \subseteq X$ are <u>(Γ-) equidecomposable</u> if $\exists C_i \subseteq X$ $\exists \gamma_i \in \Gamma$ s.t.
- $A = \bigsqcup_{i<m} C_i$
- $B = \bigsqcup_{i<m} \gamma_i \cdot C_i$

Denote this by $A \approx B$. Or $A \underset{m}{\approx} B$ if we care about # of pieces.

<u>Def</u>: Given $\Gamma \curvearrowright X$ and $A, B \subseteq X$, we say that A is <u>(Γ-) embeddecomposable</u> into B if A is (Γ-) equidecomp with some subset of B.

Denote this by $A \precsim B$. Or $A \underset{m}{\precsim} B$.

④

<u>Remark:</u> If $A \underset{m}{\leq} B$, we obtain an injection from A to B, namely $\bigcup_{i<m} \gamma_i \restriction C_i$.

<u>Next goal:</u> A variation of Schröder-Bernstein:

$$(A \preceq B \text{ and } B \preceq A) \Rightarrow A \approx B.$$

First, let's carefully (re)prove the usual S-B theorem.

<u>Thm</u> (Careful Schröder-Bernstein):

Suppose that A and B are sets with injections $f: A \to B$, $g: B \to A$. Then there is a partition

$A = A_0 \sqcup A_1$ such that

$$f \restriction A_0 \cup g^{-1} \restriction A_1 : A \to B \text{ is a bijection.}$$

$$x \longmapsto \begin{cases} f(x) & x \in A_0 \\ g^{-1}(x) & x \in A_1 \end{cases}$$

<u>pf:</u> Put $B' = B \setminus f[A]$.

Put $A_1 = g\left[\bigcup_{n \in \mathbb{N}} (f \circ g)^n [B']\right]$

Put $A_0 = A \setminus A_1$. ■ (Careful S-B).

①

Paradoxes

Monday, Jan 22

Last time:

Thm (Careful Schröder-Bernstein):

Suppose that A, B are sets with injections $\boxed{\begin{array}{l} f: A \to B \\ g: B \to A \end{array}}$. Then there is a partition $A = A_0 \sqcup A_1$ with $A_1 \subseteq g[B]$ such that the map $f \restriction A_0 \cup g^{-1} \restriction A_1 : A \to B$ is a bijection.

Today we establish a variant for embeddecompositions.

Notation: Let's denote an embeddecomp witnessing $A \underset{m}{\precsim} B$ by $\mathcal{C} = \{(C_i, \gamma_i) : i < m\}$.

Recall: With such an embeddecomp $\mathcal{C}$, we obtain its associated injection $f_{\mathcal{C}} : A \to B$ by $f_{\mathcal{C}} = \bigcup_{i<m} \gamma_i \restriction A_i$.

So $f_{\mathcal{C}} : x \longmapsto \gamma_i \cdot x$ whenever $x \in C_i$.

Thm (Schröder-Bernstein for embeddecompositions):

Suppose we have a group action $\Gamma \curvearrowright X$ and subsets $A, B \subseteq X$ with $A \underset{m}{\precsim} B$ and $B \underset{n}{\precsim} A$.

Then $A \underset{m+n}{\approx} B$.

② pf (thm)

Fix embeddecomps $\mathcal{C} = \{(C_i, \gamma_i) : i<m\}$ for $A \preceq B$
$\mathcal{D} = \{(D_j, \delta_j) : j<n\}$ for $B \preceq A$.

Let $f : A \to B$, $g : B \to A$ be the associated injections.

Careful S-B yields a partition $A = A_0 \sqcup A_1$ such that $f \restriction A_0 \cup g^{-1} \restriction A_1 : A \to B$ is a bijection.

<u>Claim 1</u>: $\{A_0 \cap C_i : i<m\} \cup \{A_1 \cap g[D_j] : j<n\}$ forms a partition of A.

<u>pf (C1)</u>: We need to show two things:

ⓐ The family is pairwise disjoint
ⓑ The family covers A.

ⓐ We grind through three cases:

- $i \neq i' \Rightarrow (A_0 \cap C_i) \cap (A_1 \cap C_{i'}) = \emptyset$ as $C_i \cap C_{i'} = \emptyset$.
- $j \neq j' \Rightarrow (A_1 \cap g[D_j]) \cap (A_1 \cap g[D_{j'}]) = \emptyset$, as $D_j \cap D_{j'} = \emptyset$ and g is injective.
- $\forall i, j$ $(A_0 \cap C_i) \cap (A_1 \cap g[D_j]) = \emptyset$ as $A_0 \cap A_1 = \emptyset$.

ⓑ Suppose that $x \in A$ is arbitrary.

- If $x \in A_0$, fix $i<m$ with $x \in C_i$. Then $x \in A_0 \cap C_i$.
- If $x \in A_1$, since $x \in g[B]$ we can find $j<n$ with $x \in g[D_j]$. Then $x \in A_1 \cap g[D_j]$.

We did it! $\blacksquare$ (C1)

③ pf (thm, cont.)

Claim 2: $\{f[A_0 \cap C_i] : i<m\} \cup \{g^{-1}[A_1 \cap g[D_j]] : j<m\}$ forms a partition of B.

pf (C2): It is the image of the previous partition under the bijection $f \restriction A_0 \cup g^{-1} \restriction A_1$. ■(C2)

Finally, observe that the definitions of f and g yield

$$f \restriction (A_0 \cap C_i) = \gamma_i \restriction (A_0 \cap C_i)$$
$$g^{-1} \restriction (A_1 \cap g[D_j]) = \delta_j^{-1} \restriction (A_1 \cap g[D_j]).$$

Altogether, this means that

$$\{(A_0 \cap C_i, \gamma_i) : i<m\} \cup \{(A_1 \cap g[D_j], \delta_j^{-1}) : j<n\}$$

is an equidecomposition witnessing $A \underset{m+n}{\approx} B$. ■(Thm)

Def: Given an action $\Gamma \curvearrowright X$, we say that $A \subseteq X$ is paradoxical if there is a partition $A = A_0 \sqcup A_1$ with $A_0 \approx A$ and $A_1 \approx A$.

Intuitively, this means we can chop A into finitely many pieces, shuffle these around by the Γ-action, and end up with two copies of our original A.

(4) Our next major goal is

Thm (Hausdorff, Banach-Tarski)
The unit ball $\{x \in \mathbb{R}^3 : d(x,0) \leq 1\}$ is paradoxical (via isometries).

Prop: Given $\Gamma \curvearrowright X$ and $A \subseteq X$, TFAE:

$\boxed{\text{I}}$ A is paradoxical

$\boxed{\text{II}}$ There are disjoint $A_0, A_1 \subseteq A$ s.t. $A_0 \approx A$, $A_1 \approx A$.

pf: $\boxed{\text{I}} \Rightarrow \boxed{\text{II}}$ ☺

$\boxed{\text{II}} \Rightarrow \boxed{\text{I}}$ Fix disjoint A_i with $A_i \approx A$.

Put $A_1' = A \setminus A_0$, so $A = A_0 \sqcup A_1'$.

Claim: $A_1' \approx A$

pf(C): We know $A \approx A_1 \subseteq A_1'$, so $A \preccurlyeq A_1'$.

Also $A_1' \subseteq A$, so $A_1' \preccurlyeq A$.

Our S-B thm implies $A_1' \approx A$. ■(C)

This means $A = A_0 \sqcup A_1'$ witnesses paradoxicality.
■(Prop)

Remark: The proof shows that we can convert $\boxed{\text{II}}$ into $\boxed{\text{I}}$ using at most one more piece.

(1)

Paradoxes

Wednesday, Jan 24

FREE GROUPS

We fix a set S of "symbols."

Def: $S^{\pm} = S \sqcup \{s^{-1} : s \in S\}$ "formal inverses" (just new symbols)

Let's agree that $(s^{-1})^{-1}$ is a funny way to write s.

Def: An S-word is an element of $(S^{\pm})^{<\omega}$, i.e., a finite (possibly empty) sequence $s_0 \cdots s_{k-1}$ with each $s_i \in S^{\pm}$.

Def: An S-word $s_0 \cdots s_{k-1}$ is reduced if $\forall i\ s_i \neq s_{i+1}^{-1}$.

Let F_S denote the set of reduced S-words.

We endow F_S with a binary operation "Concatenate 'n reduce."

So $(s_0 \cdots s_{k-1})(t_0 \cdots t_{\ell-1}) = s_0 \cdots s_{k-1-m} t_m \cdots t_{\ell-1}$, where m is least s.t. $s_{k-1-m} \neq t_m^{-1}$.

Example: $(a^{-1}bba)(a^{-1}b^{-1}ab) = a^{-1}b\cancel{b}\cancel{a}\cancel{a^{-1}}\cancel{b^{-1}}ab$

$= a^{-1}bab \in F_{\{a,b\}}$.

We want this operation to make F_S into a group. So we need to check:

- associativity [annoying]
- identity ✓
- inverses ✓

② **Prop:** This operation on F_S is associative.

pf: We exploit associativity of $\boxed{\text{function composition}}$ to keep things organized.

With each $s \in S^{\pm}$, associate its left-multiplication map $\lambda_s : F_S \longrightarrow F_S$

$$w \longmapsto sw = \begin{cases} sw & \text{if } w \neq s^{-1}v \\ v & \text{if } w = s^{-1}v \end{cases}$$

Analogously, with each $u \in F_S$ we get a left-mult map $\lambda_u : w \mapsto uw$ [reducing as needed].

By a straightforward induction, if $u = s_0 \cdots s_k$ then $\lambda_u = \lambda_{s_0} \circ \lambda_{s_1} \circ \cdots \circ \lambda_{s_{k-1}}$.

Appealing to associativity of function composition,

$$\begin{aligned} \lambda_u \circ \lambda_v &= (\lambda_{s_0} \circ \cdots \circ \lambda_{s_{k-1}}) \circ (\lambda_{t_0} \circ \cdots \circ \lambda_{t_{\ell-1}}) \\ &= \lambda_{s_0} \circ \cdots \circ \lambda_{s_{k-1-m}} \circ \lambda_{t_m} \circ \cdots \circ \lambda_{t_{\ell-1}} \\ &= \lambda_{uv}. \end{aligned}$$

Next, we compute

$$\begin{aligned} \lambda_{(uv)w} &= (\lambda_u \circ \lambda_v) \circ \lambda_w \\ &= \lambda_u \circ (\lambda_v \circ \lambda_w) \\ &= \lambda_{u(vw)}. \end{aligned}$$

Finally,

$$\begin{aligned} (uv)w &= \lambda_{(uv)w}(\varnothing) \\ &= \lambda_{u(vw)}(\varnothing) \\ &= u(vw), \end{aligned}$$

establishing associativity. ▨ (Prop)

③ Here's an abstract account of what just happened:

- F_S has the following $\boxed{\text{universal property:}}$
  Given any group G and function $f: S \to G$, $\exists!$ function $\varphi: F_S \to G$ extending f s.t. $\forall v, w \in F_S$ $\varphi(vw) = \varphi(v)\varphi(w)$.

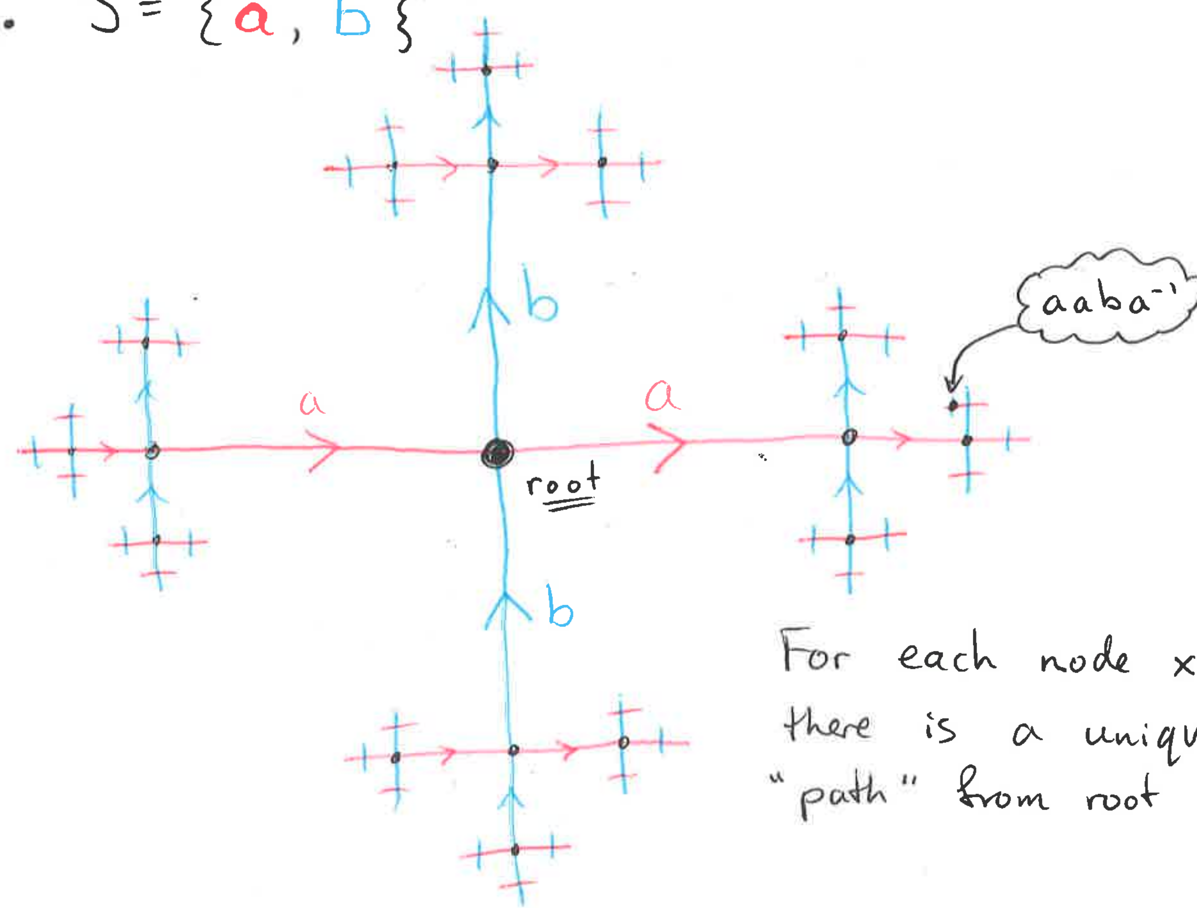
  Check: $\lambda_s^{-1} = \lambda_{s^{-1}}$

- Applying this property to $f: s \mapsto \lambda_s \in S_{F_S}$ yields an embedding of F_S into S_{F_S}, establishing associativity of our operation.

A geometrical perspective:

We consider a "rooted tree" such that each node has outgoing edges labeled with each $s \in S$, and incoming edges with the same labels.

E.g. $S = \{a, b\}$



For each node x, there is a unique "path" from root to x.

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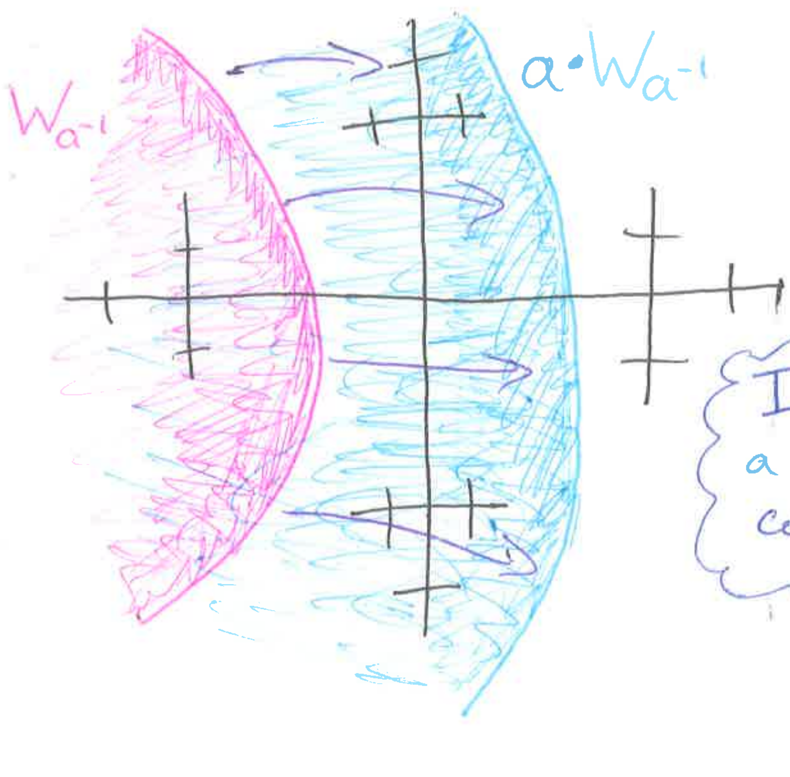
Identifying nodes of this tree by the paths from the root, we obtain another interpretation of multiplication in F_S:

> uv records the path from the root to the node reached by FIRST following the path labeled by u, and THEN following the path labeled by v.

In this fashion, we can interpret the left-multiplication map λ_u not only as a permutation of F_S, but also as a <u>graph automorphism</u> of this (labeled) tree. Under λ_u the root is "translated" to u.

<u>Example</u>: Working in F_S, for each $s \in S^{\pm}$ let $W_s \subseteq F_S$ denote the set of words beginning with s.

Then $a \cdot W_{a^{-1}} = \{e\} \cup \bigcup_{b \neq a} W_b$.



I smell a paradox coming...

①

Paradoxes

Friday, Jan 26

The paradoxicality of $\mathbb{F}_2$

Def: From now on, we write $\mathbb{F}_2$ for $F_{\{a,b\}}$, the free group on a 2-element set.

Prop: $\mathbb{F}_2$ is paradoxical under the left-mult action $\mathbb{F}_2 \curvearrowright \mathbb{F}_2$.

pf: Note that $W_a, W_{a^{-1}}, W_b, W_{b^{-1}}$ all pairwise disjoint.

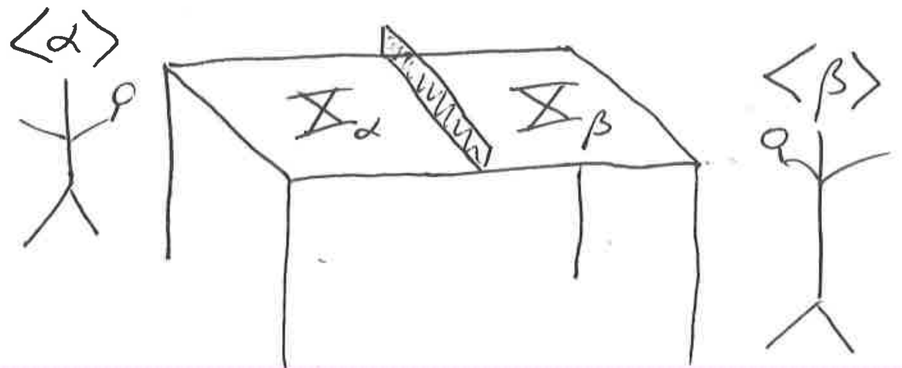
Behold! $\mathbb{F}_2 = W_a \sqcup a \cdot W_{a^{-1}}$ and $\mathbb{F}_2 = W_b \sqcup b \cdot W_{b^{-1}}$. ■(Prop)

Well, that was fun. How do we FIND $\mathbb{F}_2$ "in nature?" Let's play ping pong instead.

Def: Suppose that $\Gamma \curvearrowright X$ and that $\alpha, \beta \in \Gamma$.

A ping pong family (PPF) for α, β is a pair of disjoint non-$\emptyset$ sets $X_\alpha, X_\beta \subseteq X$ such that
- $\forall g \in \langle\alpha\rangle \setminus \{e\}$ $g \cdot X_\alpha \subseteq X_\beta$
- $\forall g \in \langle\beta\rangle \setminus \{e\}$ $g \cdot X_\beta \subseteq X_\alpha$.

Remark: $\langle\alpha\rangle = \{\alpha^z : z \in \mathbb{Z}\}$ is the cyclic subgroup generated by α.



② **Ping pong lemma:** Suppose that $\Gamma \curvearrowright X$ and that $\alpha, \beta \in \Gamma$ have infinite order and admit a ping pong family. Then $\langle \alpha, \beta \rangle \cong \mathbb{F}_2$.

Subgrp generated by $\{\alpha, \beta\}$

pf: There is a natural surjection of $\mathbb{F}_2$ onto $\langle \alpha, \beta \rangle$ given by the universal property from last time. It suffices to show that it has trivial kernel, hence is injective. Twds a contradiction, suppose that $w \in \mathbb{F}_2$ is a nontrivial element of the kernel.

By conjugating w by a large power of a, we may assume $w = a^{?} b^{?} a^{?} \cdots b^{?} a^{?}$ with each $? \in \mathbb{Z} \setminus \{0\}$. So $w \mapsto \alpha^{?} \beta^{?} \cdots \alpha^{?}$. This element moves X_α into X_β, thus cannot be trivial. This contradicts our choice of w. $\blacksquare$ (PPL).

The key technical result driving the Banach-Tarski paradox boils down to finding $\mathbb{F}_2$ within the isometries of the sphere.

Def: $M_3(\mathbb{Q})$ is the group of invertible (3×3)-matrices with entries in $\mathbb{Q}$. The operation is matrix multiplication.

③ <u>Thm</u> (Hausdorff): Here are two matrices:

$$\alpha = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}$$

Working in $M_3(\mathbb{Q})$, $\langle \alpha, \beta \rangle \cong \mathbb{F}_2$.

<u>pf:</u> Let's play ping pong! We'll find a PPF for the usual action $M_3(\mathbb{Q}) \curvearrowright \mathbb{Q}^3$.

Here's an obvious choice:

$$X_\alpha = \left\{ \begin{pmatrix} x/5^k \\ y/5^k \\ z/5^k \end{pmatrix} : k,x,y,z \in \mathbb{Z} \text{ and } \begin{array}{l} x \equiv 0 \\ y \not\equiv 0 \pmod 5 \\ z \equiv \pm 3y \end{array} \right\}$$

Disjoint!

$$X_\beta = \left\{ \begin{pmatrix} x/5^k \\ y/5^k \\ z/5^k \end{pmatrix} : k,x,y,z \in \mathbb{Z} \text{ and } \begin{array}{l} x \equiv \pm 3y \\ y \not\equiv 0 \pmod 5 \\ z \equiv 0 \end{array} \right\}$$

Let's check the ping pong condition inductively for positive powers of α, starting with α itself.

<u>$g=\alpha$:</u> Suppose we have $\vec{v} = \frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in X_\alpha$.

$$\text{Then } \alpha \circ \vec{v} = \frac{1}{5^{k+1}} \begin{pmatrix} 3x+4y \\ -4x+3y \\ 5z \end{pmatrix} \in X_\beta$$

Since:
- $3x+4y \equiv 4y \equiv 3(3y)$
- $-4x+3y \equiv 3y \not\equiv 0 \pmod 5$ ✓
- $5z \equiv 0$

④ <u>pf</u> (thm, cont.)

<u>$g = \alpha^{n+1}$</u>: Given $\vec{v} \in X_\alpha$, induction says $\alpha^n \cdot \vec{v} \in X_\beta^+$, i.e., that $\alpha^n \cdot \vec{v} = \frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with $\begin{matrix} x \equiv 3y \\ y \not\equiv 0 \\ z \equiv 0 \end{matrix}$ (mod 5).

Then $\alpha^{n+1} \cdot \vec{v} = \alpha \cdot \frac{1}{5^k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5^{k+1}}\begin{pmatrix} 3x+4y \\ -4x+3y \\ 5z \end{pmatrix} \in X_\beta^+$,

Since
- $3x + 4y \equiv 9y + 4y \equiv 3y$
- $-4x + 3y \equiv 3y + 3y \equiv y \not\equiv 0 \pmod 5$ ✓
- $5z \equiv 0$

<u>Note</u>: This also shows that α has infinite order, since $\forall n > 0 \;\; \alpha^n \cdot X_\alpha \subseteq X_\beta^+$.

A similar analysis handles negative powers of α and elements of $\langle\beta\rangle \setminus \{e\}$. ■(Thm).

<u>Cor</u> (Hausdorff): There is a subgroup of Isom(Sphere) isomorphic to $\mathbb{F}_2$.

<u>pf</u>: It suffices to check that α, β as above yield isometries $\vec{v} \mapsto \alpha\vec{v}$, etc., of $\mathbb{R}^3$ that fix 0. Compute $\alpha^T\alpha = \beta^T\beta = I$, use $\boxed{\text{HW}}$.

■(Cor)

1

Paradoxes

Monday, Jan 29

Paradoxicality for actions of free groups

Def: An action $\Gamma \curvearrowright X$ is <u>free</u> if every element of X has trivial stabilizer. I.e.,

$$\forall x \in X \ \forall \gamma \in \Gamma \ \gamma \cdot x = x \Rightarrow \gamma = e.$$

In other words, for each $x \in X$ the map

$$\begin{aligned} \Gamma &\longrightarrow X \\ \gamma &\longmapsto \gamma \cdot x \end{aligned} \quad \text{is injective.}$$

Def: Given an action $\Gamma \curvearrowright X$, the corresponding <u>orbit equivalence relation</u> E_Γ^X on X is given by $x \, E_\Gamma^X \, y$ iff $\exists \gamma \in \Gamma \ \gamma \cdot x = y$.

Lemma 1 [AC]: Suppose that $\mathbb{F}_2 \curvearrowright X$ is a free action. Then X is paradoxical.

pf($\mathcal{L}$1) Using AC, choose a transversal $T \subseteq X$ meeting each E_Γ^X-class in exactly one point. The translates $\{\gamma \cdot T : \gamma \in \Gamma\}$ partition X.

Now the sets $W_a \cdot T$, $W_{a^{-1}} \cdot T$, $W_b \cdot T$, $W_{b^{-1}} \cdot T$ witness paradoxicality as before, since

$$X = W_a \cdot T \sqcup a \cdot W_{a^{-1}} \cdot T, \text{ etc.} \quad \blacksquare(\mathcal{L}1)$$

Last time, we found matrices $\alpha, \beta \in \mathrm{Isom}(\text{Sphere})$ which generated a copy of $\mathbb{F}_2$. Sadly, the corresponding action $\mathbb{F}_2 \curvearrowright S$ is not free. But it's "almost" free!

Recall: $S = \{x \in \mathbb{R}^3 : d(x, 0) = 1\}$

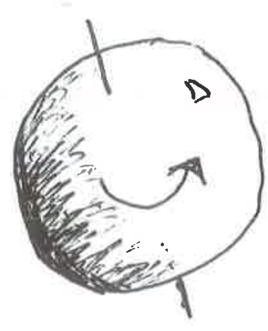
Prop: Suppose that $A \in M_3(\mathbb{R})$ satisfies
- $A^T A = I$
- $\det(A) = 1$.

If A stabilizes three points of S, then $A = I$.

pf: Ex: A preserves orthogonality, i.e., $u \perp v \Rightarrow Au \perp Av$.

Now suppose that x, y, z are distinct elements of S stabilized by A. WLOG, $y \neq -x$, so x & y are linearly independent. Consider the pair $v, -v$ of points orthogonal to both x and y. By above Ex, $Av = \pm v$. If $Av = -v$, then A has e.values $1, 1, -1$, contradicting $\det(A) = 1$. So $A \cdot v = v$, and thus $A = I$ as it fixes the basis x, y, v. ∎ (Prop)

Remark: Matrices as in the proposition are often called rotations of the sphere. The rotations form an index-two subgroup of the isometries, and it contains $\langle \alpha, \beta \rangle$ from last time. Typical rotation:



③

Lemma 2: There is a countable, $\mathbb{F}_2$-invariant set $C \subseteq S$ such that the action $\mathbb{F}_2 \curvearrowright S \setminus C$ is free.

pf($\mathcal{L}2$): Each nonidentity element of $\langle \alpha, \beta \rangle$ fixes at most two points of S. Thus, the set $C = \{x \in S : \exists \gamma \in \mathbb{F}_2 \setminus \{e\}\ \gamma \cdot x = x\}$

$$= \bigcup_{\gamma \in \mathbb{F}_2 \setminus \{e\}} \mathrm{Fix}(\gamma) \quad \text{is countable.}$$

Check: C is $\mathbb{F}_2$-invariant (or just use $\mathbb{F}_2 \cdot C$ instead).

Then the action $\mathbb{F}_2 \curvearrowright S \setminus C$ is free. ▣($\mathcal{L}2$)

Lemma 3: For any countable $C \subseteq S$, $S \approx S \setminus C$ via rotations (hence via isometries).

pf($\mathcal{L}3$): First, observe that $C \cup -C$ is still countable, so we may find $z \in S$ with both $z \notin C$ and $-z \notin C$. We consider rotations r_θ about the axis $\{z, -z\}$. As in our prior analysis of the circle, there is some GOOD θ s.t.

$$\forall n \in \mathbb{N} \setminus \{0\}\ C \cap r_\theta^n[C] = \emptyset.$$

Continuing the analogy, we get our equidecomposition witnessing $S \approx S \setminus C$ as follows:

$$D_0 = \bigcup_{n \in \mathbb{N}} r_\theta^n[C] \qquad \gamma_0 = r_\theta$$

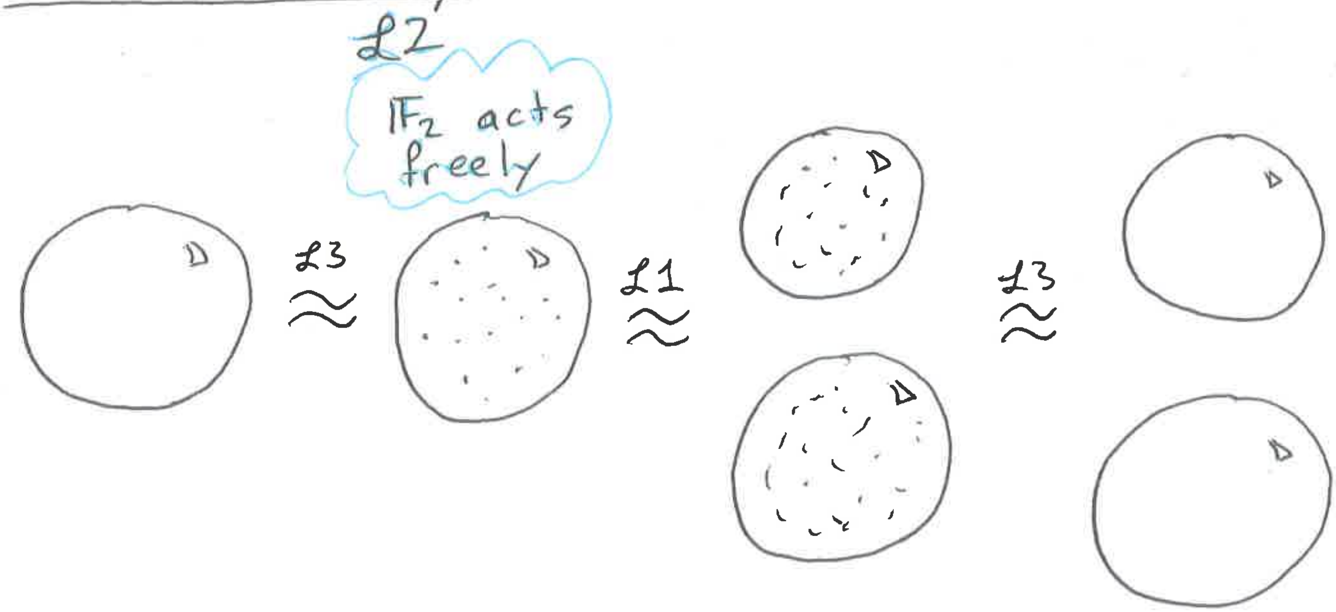
$$D_1 = S \setminus D_0 \qquad \gamma_1 = e. \qquad ▣(\mathcal{L}3)$$

④ **Thm** (Hausdorff, 1914): The sphere is paradoxical (via rotations, hence isometries).

pf: We work with the action $\mathbb{F}_2 \curvearrowright S$ induced by the matrices α, β from last time. Lemma 2 grants a countable $\mathbb{F}_2$-invariant set $C \subseteq S$ so that the action $\mathbb{F}_2 \curvearrowright S \setminus C$ is free. Lemma 1 then grants paradoxicality of $S \setminus C$. Finally, the equidecomposition in Lemma 3 allows us to transfer this paradoxicality back to S.

$\blacksquare$ (Thm)

Cartoon summary:



①

Paradoxes

Wednesday, Jan 31

Last time: (Hausdorff-ish 1914): The sphere $S = \{x \in \mathbb{R}^3 : d(x,0) = 1\}$ is paradoxical via rotations (hence isometries) of S.

Cor (Banach-Tarski I, 1924): The ball $B = \{x \in \mathbb{R}^3 : d(x,0) \le 1\}$ is paradoxical via isometries of $\mathbb{R}^3$.

pf: Fix a partition $S = C \sqcup D$ with $S \approx C$, $S \approx D$.
For $0 < r \le 1$, put $rS = \{x \in \mathbb{R}^3 : d(x,0) = r\}$, so $B \setminus \{0\} = \bigsqcup_r rS$. We can copy/paste the Hausdorff paradox, yielding $B \setminus \{0\} = (\bigsqcup_r rC) \sqcup (\bigsqcup_r rD)$ with $B \setminus \{0\} \approx \bigsqcup_r rC$ and $B \setminus \{0\} \approx \bigsqcup_r rD$.
This establishes paradoxicality of $B \setminus \{0\}$, hence of B since $B \approx B \setminus \{0\}$. ∎ (BTI)

For example, use a tiny circle containing 0 and use the previous calculation that a circle is equidecomp with circle $\setminus \{0\}$.

Cor (Banach-Tarski II): For all positive $r, s \in \mathbb{R}$, $B_r \approx B_s$, where $B_r = \{x \in \mathbb{R}^3 : d(x,0) \leq r\}$.

pf: WLOG $r \leq s$. Since $B_r \subseteq B_s$, certainly $B_r \preceq B_s$. By S-B, it suffices to
Fix a finite set $F \subseteq \mathrm{Isom}(\mathbb{R}^3)$ such that $B_s \subseteq \bigcup_{\gamma \in F} \gamma \cdot B_r$.
Iterate BT I to build a partition $B_r = \bigsqcup_{\gamma \in F} C_\gamma$ with each $C_\gamma \approx B_r$ with assoc. bijection g_γ. Then $B_s \subseteq \bigcup_{\gamma \in F} \gamma \cdot g_\gamma[C_\gamma]$, and thus $B_s \preceq B_r$ as desired. ∎ (BT II)

Remark: The analogous result holds for "open balls" $B_{<r} = \{x \in \mathbb{R}^3 : d(x,0) < r\}$ since $B_{r/2} \preceq B_{<r} \preceq B_r$.

Def: A set $A \subseteq \mathbb{R}^3$ is bounded if
$$\sup\{d(x,0) : x \in A\} < \infty.$$

Def: A set $A \subseteq \mathbb{R}^3$ has non-empty interior if $\exists x \in \mathbb{R}^3 \; \exists \varepsilon > 0$ s.t. $\{y \in \mathbb{R}^3 : d(y,x) < \varepsilon\} \subseteq A$.

Cor (Banach-Tarski III): Any two subsets of $\mathbb{R}^3$ that are bounded and have non-empty interior are equidecomposable via isometries.

pf: By BT II, it suffices to show that any such set A is equidecomposable with a ball.

- Boundedness yields $s \in \mathbb{R}$ with $A \preccurlyeq B_s$
- Non-empty interior yields $r \in \mathbb{R}$ with $B_r \preccurlyeq A$.

Since $B_r \approx B_s$, we are done! ■ (BT III)

So... is volume a *lie*?

Intermission: Why doesn't anybody "believe" in Banach-Tarski? Thinking back to the Greeks, if we chop a ball into finitely many pieces and move them, the "total volume" should not change.

The problem is that not every subset of $\mathbb{R}^3$ has well-defined volume. So you get

$$1 = \sum_{i<m} (???)_i = 2 .$$

Nevertheless, we can use these ideas to better probe the "threshold" of paradoxicality.

④ **Def:** A <u>finitely additive probability measure</u> (fapm) on a (non-$\emptyset$) set X is a function
$$m : \mathcal{P}(X) \longrightarrow [0,1]$$
satisfying
- $m(X) = 1$
- $A \cap B = \emptyset \Rightarrow m(A \cup B) = m(A) + m(B)$.

Def: An action $\Gamma \curvearrowright X$ is <u>amenable</u> if there is a Γ-invariant fapm m on X. I.e., for all $\gamma \in \Gamma$ and $A \subseteq X$, $m(\gamma \cdot A) = m(A)$.

<u>Observation:</u> If m is a Γ-invariant fapm on X, then $A \approx B \Rightarrow m(A) = m(B)$, since
$$m(A) = \sum_{i<n} m(C_i) = \sum_{i<n} m(\gamma_i \cdot C_i) = m(B).$$
Hence, if $m(A) > 0$ it $\boxed{\text{cannot}}$ be paradoxical.

<u>Upshot:</u> Amenability is an impediment to paradoxicality.
{Is it the only impediment??}

Def: A group Γ is <u>amenable</u> if the left-multiplication action $\Gamma \curvearrowright \Gamma$ is amenable.

$\boxed{\text{HW?}}$ If Γ is an amenable group and X is a non-$\emptyset$ set, then every action $\Gamma \curvearrowright X$ is amenable.

Paradoxes

Friday, Feb 2

Last time: We isolated amenability as the "canonical impediment" to paradoxicality. Before understanding/creating general fapms, we need to understand the simplest ones.

Def: An ultrafilter on a (non-$\emptyset$) set X is a fapm $\mathcal{P}(X) \longrightarrow \{0,1\}$.

Ex: For any $x \in X$, the Dirac measure

$$\delta_x : A \longmapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is an ultrafilter. It's often called the principal ultrafilter at x.

Remark: When X is finite, every ultrafilter is principal.

Three views of ultrafilters:

$\boxed{0}$ Functional analysis: ultrafilters are the exteme pts of the set of fapms on X, which the positive unit sphere of the dual of $\ell_\infty(X)$.

$\boxed{\mathrm{I}}$ Set theory

$\boxed{\mathrm{II}}$ Set theory wearing an algebra mask

(2) $\boxed{\text{I}}$ A set-theoretic perspective.

Def: A (proper) filter on a set X is a family $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying:

- $\emptyset \notin \mathcal{F}$ [properness]
- $X \in \mathcal{F}$
- $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
- $A \in \mathcal{F}$ and $A \subseteq B \Rightarrow B \in \mathcal{F}$.

Ex: (a) If X is an infinite set, we have a cofinite filter:

$$\{A \subseteq X : X \setminus A \text{ is finite}\}.$$

(b) If m is a fapm on X, then

$$m^{-1}(\{1\}) = \{A \subseteq X : m(A) = 1\}$$

is the corresponding measure 1 (or conull) filter.

Def: [Equiv] A (proper) filter is an ultrafilter if it cannot be extended to a larger (proper) filter.

Check: This means $\forall A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

Ultrafilter Lemma [AC]: Every (proper) filter on X extends to an ultrafilter on X.

pf: Zorn's Lemma (or transfinite induction, etc.). $\blacksquare$ (U.L.)

Cor: Every infinite set X admits a non-principal ultrafilter (i.e., $\mathcal{U}$ s.t. $\forall x \in X$ $\{x\} \notin \mathcal{U}$).

pf: Extend the cofinite filter. $\blacksquare$ (Cor.)

③ $\boxed{\text{II}}$ An algebraic perspective

Given a non-$\emptyset$ set X, we may endow $\mathcal{P}(X)$ with a ring structure via

- $A + B = A \triangle B$
- $A \times B = A \cap B$.

These operations make $\mathcal{P}(X)$ into a commutative ring with 1, and moreover $\forall A \in \mathcal{P}(X)$ $A \times A = A$.

Exercise: $\mathcal{I} \subseteq \mathcal{P}(X)$ is a proper (ring-theoretic) ideal iff $\check{\mathcal{I}} = \{A \subseteq X : X \setminus A \in \mathcal{I}\}$ is a proper filter.

Standard ring theory says that every proper ideal of $\mathcal{P}(X)$ extends to a maximal ideal of $\mathcal{P}(X)$.

Claim: If $\mathcal{I} \subseteq \mathcal{P}(X)$ is a maximal ideal, then for all $A \subseteq X$ exactly one of $A \in \mathcal{I}$, $X \setminus A \in \mathcal{I}$.

pf(C) (sketch): Given a max'l ideal $\mathcal{I}$, the quotient ring $\mathcal{P}(X)/\mathcal{I}$ is a field in which every element squares to itself. The only such field is $\mathbb{Z}/2\mathbb{Z}$. This means that

$$A \in \mathcal{I} \text{ iff } X \triangle A = X \setminus A \notin \mathcal{I}. \quad \blacksquare(C)(\text{sketch}).$$

So, using the above correspondence between filters and ideals, we recover the Ultrafilter Lemma.

④ Ultrafilters are central to various notions of $\boxed{\text{logical/topological compactness}}$, so it's time for a crash course in (set-theoretic) topology.

Def: A <u>topology</u> on a set X is a family $\tau \subseteq \mathcal{P}(X)$ s.t.

- $\emptyset \in \tau$ (redundant)
- $X \in \tau$
- $A, B \in \tau \Rightarrow A \cap B \in \tau$
- $\mathcal{A} \subseteq \tau \Rightarrow \bigcup \mathcal{A} \in \tau$

Remark: We think of elements of τ as "open sets." So the last two conditions assert that open sets are stable under <u>finite intersection</u> and <u>arbitrary union</u>.

Examples: ⓐ $X = \mathbb{R}$, $\tau = \{U \subseteq \mathbb{R} : U \text{ is a union of open intervals}\}$

ⓑ Metric space (X, d), $\tau = \{U \subseteq X : U \text{ is a union of open balls}\}$.

Warm-up: Sequence convergence.

Given a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X and some $y \in X$, we say $\lim_{n \to \infty} x_n = y$ if:

- $\forall$ open $U\, (y \in U \Rightarrow \exists M\ \forall n > M\ \ x_n \in U)$, equiv
- $\forall$ open $U\, (y \in U \Rightarrow \{n \in \mathbb{N} : x_n \in U\}$ is cofinite$)$.

Viewing a sequence properly as a function $f : \mathbb{N} \to X$, we can say $\lim f = y$ if

- $\forall$ open $U\, (y \in U \Rightarrow f^{-1}(U)$ is cofinite$)$.

We will use these ideas to formalize convergence in the language of $\boxed{\text{filters.}}$

Paradoxes

Monday, Feb 5

Last time: Two related definitions. Fix a set X:

A (proper) filter on X is $\mathcal{F} \subseteq \mathcal{P}(X)$ s.t.:
- $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$
- $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
- $A \in \mathcal{F}$ and $A \subseteq B \Rightarrow B \in \mathcal{F}$

A topology on X is $\tau \subseteq \mathcal{P}(X)$ s.t.
- $\emptyset \in \tau$, $X \in \tau$
- $A, B \in \tau \Rightarrow A \cap B \in \tau$
- $\mathcal{A} \subseteq \tau \Rightarrow \bigcup \mathcal{A} \in \tau$.

And a bonus definition: an ultrafilter is a maximal proper filter. Equiv., for all $A \subseteq X$ it contains one of A, $X \setminus A$.

So, what is the relationship?

Def: Given a topology τ on X and $x \in X$, define its neighborhood filter $\mathcal{N}_x$ on X by

$$A \in \mathcal{N}_x \text{ iff } \exists O \in \tau \ \underbrace{(x \in O \text{ and } O \subseteq A)}_{\text{abbrev. } x \in O \subseteq A}$$

Ex: In a metric space (X, d), $A \in \mathcal{N}_x$ iff $\exists \varepsilon > 0$ s.t. $\{y \in X : d(x,y) < \varepsilon\} \subseteq A$.

Prop: $\mathcal{N}_x$ is indeed a filter.

pf:
- $\emptyset \notin \mathcal{N}_x$, $X \in \mathcal{N}_x$ ✓
- Suppose $x \in O_A \subseteq A$, $x \in O_B \subseteq B$. Then $x \in O_A \cap O_B \subseteq A \cap B$ ✓
- Suppose $x \in O_A \subseteq A$ and $A \subseteq B$. Then $x \in O_A \subseteq B$. ✓

∎ (Prop)

Def: Given a function $\varphi : X \to Y$ and a filter $\mathcal{F}$ on X, we get a *push-forward* filter $\varphi_* \mathcal{F}$ defined by $B \in \varphi_* \mathcal{F}$ iff $\varphi^{-1}(B) \in \mathcal{F}$.

Ex: As discussed last time, given a function $\varphi : \mathbb{N} \to X$ and a topology τ on X,

$$\lim \varphi = y \text{ iff } \mathcal{N}_y \subseteq \varphi_*(\text{cofinite}).$$

Def: Given a topological space (X, τ) and a filter $\mathcal{F}$ on X, we say that $\mathcal{F}$ *converges* to $y \in X$ iff $\mathcal{N}_y \subseteq \mathcal{F}$.

Remark: Like sequences, filters generally fail to converge to anything. For example, if $X = \mathbb{R}$ (with usual τ) and $\mathcal{F} = \{\mathbb{R}\}$ there is no hope.

But some filters are ultra-special...

HOT TAKE These are controversial definitions:

Def: A topological space (X, τ) is *Hausdorff* if every ultrafilter on X converges to $\boxed{\text{at most one}}$ element of X.

Def: A topological space (X, τ) is *compact* if every ultrafilter on X converges to $\boxed{\text{at least one}}$ element of X.

③ Here are the standard definitions:

<u>Def</u>: (X, τ) is <u>Hausdorff</u> if for all $x \neq y \in X$ there are disjoint $O_x, O_y \in \tau$ with $x \in O_x$, $y \in O_y$.

<u>Def</u>: (X, τ) is <u>compact</u> if whenever $\mathcal{O} \subseteq \tau$ satisfies $\bigcup \mathcal{O} = X$, there is finite $\mathcal{A} \subseteq \mathcal{O}$ with $\bigcup \mathcal{A} = X$. "Every open cover admits a finite subcover."

Let's establish the equivalence of these two notions of compactness. Corresponding equivalence for Hausdorff on $\boxed{HW?}$

<u>Thm</u> [AC]: Suppose that (X, τ) is a topological space. TFAE:

$\boxed{I}$ Every ultrafilter on X converges to some element of X

$\boxed{II}$ Every open cover of X admits a finite subcover.

<u>pf</u>: It's a bit easier to work with the negations of $\boxed{I}$ and $\boxed{II}$, as this gives you objects to play with.

$\neg\boxed{I} \Rightarrow \neg\boxed{II}$: Fix an ultrafilter $\mathcal{U}$ on X that doesn't converge to any $x \in X$. Then for each $x \in X$ we may choose $A_x \in \mathcal{N}_x$ with $A_x \notin \mathcal{U}$. We may also choose $O_x \in \tau$ with $x \in O_x \subseteq A_x$. Note that $O_x \notin \mathcal{U}$. Put $\mathcal{O} = \{O_x : x \in X\}$, so $\bigcup \mathcal{O} = X$.

④ pf (thm, cont.):

Claim 1: No finite $\mathcal{A} \subseteq \mathcal{O}$ satisfies $\bigcup\mathcal{A} = X$.

pf (C1): Consider the (finite) dual
$$\check{\mathcal{A}} = \{X \setminus O : O \in \mathcal{A}\} \subseteq \mathcal{U}.$$
Then $\bigcap\check{\mathcal{A}} \in \mathcal{U}$ as well. Hence,
$$\bigcup\mathcal{A} = X \setminus \bigcap\check{\mathcal{A}} \notin \mathcal{U}$$
and in particular $\bigcup\mathcal{A} \neq X$. ■ (C1)

✓ $\neg\boxed{\mathrm{I}} \Rightarrow \neg\boxed{\mathrm{II}}$

$\neg\boxed{\mathrm{II}} \Rightarrow \neg\boxed{\mathrm{I}}$: Fix a cover $\mathcal{O}$ that admits no finite subcover. In other words, for all finite $\mathcal{A} \subseteq \mathcal{O}$, $\bigcap\check{\mathcal{A}} = X \setminus \bigcup\mathcal{A} \neq \emptyset$.

Define a proper filter $\mathcal{F}$ on X by declaring
$$B \in \mathcal{F} \text{ iff } \exists \text{ finite } \mathcal{A} \subseteq \mathcal{O} \text{ with } \bigcap\check{\mathcal{A}} \subseteq B.$$
Use the Ultrafilter Lemma [AC] to extend $\mathcal{F}$ to an ultrafilter $\mathcal{U}$ on X.

Claim 2: $\mathcal{U}$ does not converge to any $x \in X$.

pf (C2): Let $x \in X$ be arbitrary, and find some $O \in \mathcal{O}$ with $x \in O$. Certainly $O \in \mathcal{N}_x$. But
$$X \setminus O = \bigcap\check{\{O\}} \in \mathcal{F} \subseteq \mathcal{U},$$
so $O \notin \mathcal{U}$. We conclude that $\mathcal{N}_x \not\subseteq \mathcal{U}$. ■ (C2)

■ (Thm)

Paradoxes

Wednesday, Feb 7

Last time: ▫ A topological space is compact if every ultrafilter converges to at least one point.
▫ .... is Hausdorff .... at most one point.

Today we will see these ideas "in action." But first...

Handy observation: Suppose that $\mathcal{U}$ is an ultrafilter on X, that $A, B_0, \ldots, B_{k-1} \subseteq X$ s.t. ▫ $A \in \mathcal{U}$
▫ $A \subseteq \bigcup_i B_i$.

Then $\exists i < k$ with $B_i \in \mathcal{U}$.

pf (sketch): Else you could cover a measure 1 set with finitely many measure 0 sets. ∎ (H.O., sketch)

Remark: It is straightforward (and annoying) to "formalize" this in set-theoretic language like last time.

Thm: The unit interval $I = [0,1] \subseteq \mathbb{R}$ is compact.
[when given the topology induced by the usual metric]

pf: Fix an ultrafilter $\mathcal{U}$ on I. We want to find $x \in I$ with $\mathcal{N}_x \subseteq \mathcal{U}$. Given a finite sequence $s \in 10^{<\mathbb{N}}$ of "digits," let B_s denote the set of $x \in I$ admitting a decimal expansion beginning $0.s\ldots$

For example, $B_\emptyset = I = [0, 0.999\ldots]$

$B_{23} = [0.23, 0.24] = [0.23, 0.2399\ldots]$

② pf (thm, cont.):

Recursively apply the H.O. to find sequences $s_n \in 10^n$ satisfying: □ s_{n+1} extends s_n

□ $B_{s_n} \in \mathcal{U}$

use $B_s = \bigcup_{i<10} B_{si}$

Let $x \in I$ have decimal exp $\bigcup_n s_n$, equiv, $x \in \bigcap_n B_{s_n}$.

Claim: $\mathcal{N}_x \subseteq \mathcal{U}$.

pf (c): It suffices to show $\forall \varepsilon > 0$, $I \cap (x+\varepsilon, x-\varepsilon) \in \mathcal{U}$.

Find n with $10^{-n} < \varepsilon$, so $B_{s_n} \subseteq I \cap (x+\varepsilon, x-\varepsilon)$. ☒(c)

☒(Thm)

Remark: The same argument applies to any interval of the form $[a,b] \subseteq \mathbb{R}$. Of course these are all Hausdorff as well.

Def: A function $f : X \to \mathbb{R}$ is bounded if its image $f[X]$ is contained in a closed interval $[a,b] \subseteq \mathbb{R}$. Equivalently, if $\exists M \in \mathbb{N}$ s.t. $\forall x \in X$ $-M \le f(x) \le M$.

Easy exercises:

ⓐ If $f, g : X \to \mathbb{R}$ are bounded, so is
$$f+g : x \mapsto f(x) + g(x)$$

ⓑ If $r \in \mathbb{R}$ and $f : X \to \mathbb{R}$ is bounded, so is
$$rf : x \mapsto rf(x).$$

That is, the bounded functions $X \to \mathbb{R}$ form an $\mathbb{R}$-vector space, namely $\ell^\infty(X)$.

Suppose now that $\mathcal{U}$ is an ultrafilter on a set X.

Def: To any bounded $f : X \to \mathbb{R}$, we assign its *ultralimit* $\lim_{\mathcal{U}} f \in \mathbb{R}$, defined by

$$\lim_{\mathcal{U}} f = r \quad \text{iff} \quad f_* \mathcal{U} \text{ converges to } r.$$

Our analysis of compactness and Hausdorffness ensures that $\lim_{\mathcal{U}} f$ exists and is unique.

Prop: For all bounded $f, g : X \to \mathbb{R}$

$$\lim_{\mathcal{U}} (f+g) = \lim_{\mathcal{U}} f + \lim_{\mathcal{U}} g.$$

pf: Write $\lim_{\mathcal{U}} f = r$

We want to show that $\lim_{\mathcal{U}} (f+g) = r+s$.

I.e., that $\mathcal{N}_{r+s} \subseteq (f+g)_* \mathcal{U}$.

I.e., $\forall \varepsilon > 0 \quad (r+s-\varepsilon, r+s+\varepsilon) \in (f+g)_* \mathcal{U}$.

I.e., that $C = \{x \in X : r+s-\varepsilon < f(x) + g(x) < r+s+\varepsilon\} \in \mathcal{U}$.

Put $A = \{x \in X : r - \varepsilon/2 < f(x) < r + \varepsilon/2\}$

$B = \{x \in X : s - \varepsilon/2 < g(x) < s + \varepsilon/2\}$.

Now $A \in \mathcal{U}$ since $(r - \varepsilon/2, r + \varepsilon/2) \in \mathcal{N}_r \subseteq f_* \mathcal{U}$

and $B \in \mathcal{U}$ since $(s - \varepsilon/2, s + \varepsilon/2) \in \mathcal{N}_s \subseteq g_* \mathcal{U}$.

So $A \cap B \in \mathcal{U}$. But $A \cap B \subseteq C$,

and hence $C \in \mathcal{U}$ as desired. $\blacksquare$ (Prop)

④ A similar (easier) argument gives

<u>Prop:</u> For all $a \in \mathbb{R}$ and bounded $f : X \to \mathbb{R}$,

$$\lim_{\mathcal{U}} (af) = a \cdot \lim_{\mathcal{U}} f.$$

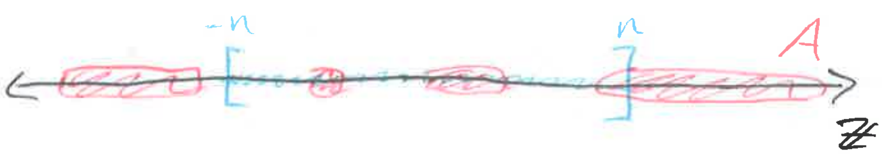
<u>pf:</u> $\boxtimes$ (Prop)

<u>Who cares?</u>

Recall our original motivation: witness amenability of a group (say $\mathbb{Z}$) by finding a left-invariant fapm on the group, precluding paradoxicality. Let's attempt to build such a fapm m on $\mathbb{Z}$:

<u>"Def:"</u> The <u>density</u> of $A \subseteq \mathbb{Z}$ is

$$\lim_{n\to\infty} \frac{|A \cap \{-n, \ldots, n\}|}{2n+1}.$$



<u>Ex:</u> densities of $2\mathbb{Z}$, $2\mathbb{Z}+1$ both $1/2$.

<u>Problem:</u> The density usually doesn't exist.

<u>Solution:</u> Take an $\boxed{\text{ultralimit}}$ instead!

Paradoxes

Friday, Feb 9

The amenability of the group of integers (and friends)

Recall: Def: A group Γ is amenable if there is a fapm $m : \mathcal{P}(\Gamma) \to [0,1]$ on Γ such that for all $\gamma \in \Gamma$ and $A \subseteq \Gamma$ $m(\gamma \cdot A) = m(A)$.

Before establishing the promised amenability of $\mathbb{Z}$, let's analyze the notion of "density" more carefully.

Def: For all $A \subseteq \mathbb{Z}$, define its density function

$$d_A : \mathbb{N} \longrightarrow [0,1]$$
$$n \longmapsto \frac{|A \cap F_n|}{|F_n|},$$

where $F_n = \{-n, \ldots, n\} \subseteq \mathbb{Z}$.

Obs. 1: For all disjoint $A, B \subseteq \mathbb{Z}$, $d_{A \cup B} = d_A + d_B$.

pf(O1): $d_{A \cup B} : n \longmapsto \frac{|(A \cup B) \cap F_n|}{|F_n|}$

$$= \frac{|A \cap F_n| + |B \cap F_n|}{|F_n|} = d_A(n) + d_B(n)$$

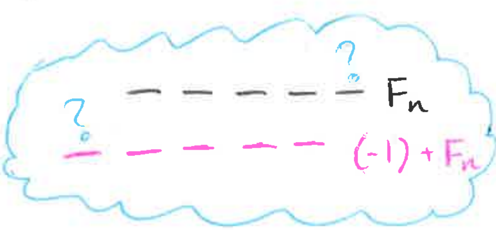
$\blacksquare$(O1)

Obs. 2: For all $A \subseteq \mathbb{Z}$ and $n \in \mathbb{N}$

$$|d_A(n) - d_{1+A}(n)| \leq \frac{2}{|F_n|}.$$

pf(O2): We see that $\big| |A \cap F_n| - |(1+A) \cap F_n| \big|$

$$= \big| |A \cap F_n| - |A \cap ((-1) + F_n)| \big|$$
$$\leq 2.$$



$\blacksquare$(O2)

② Thm: $(\mathbb{Z};+)$ is an amenable group.

pf: Fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

We define a function $m : \mathcal{P}(\mathbb{Z}) \to [0,1]$
$$A \mapsto \lim_{\mathcal{U}} d_A .$$

Claim 1: m is a fapm on $\mathbb{Z}$.

pf(C1): Two things to check:

- $m(\mathbb{Z}) = 1$: Note that $d_{\mathbb{Z}} : n \mapsto 1$, hence $m(\mathbb{Z}) = \lim_{\mathcal{U}} d_{\mathbb{Z}} = 1$. ✓
- $A \cap B = \emptyset \Rightarrow m(A \cup B) = m(A) + m(B)$:

$$\begin{aligned} m(A\cup B) &= \lim_{\mathcal{U}} d_{A\cup B} \\ &= \lim_{\mathcal{U}} (d_A + d_B) \quad [\text{Obs } 1] \\ &= \lim_{\mathcal{U}} d_A + \lim_{\mathcal{U}} d_B \\ &= m(A) + m(B). \ ✓ \qquad \blacksquare (C1) \end{aligned}$$

Claim 2: m is $\mathbb{Z}$-invariant.

pf(C2): It suffices to show for all $A \subseteq \mathbb{Z}$ that $m(A) = m(1+A)$. By Obs 2, we know for all $\varepsilon > 0$ that the set
$$\{n \in \mathbb{N} : -\varepsilon < d_A(n) - d_{1+A}(n) < \varepsilon\}$$
is cofinite, hence in $\mathcal{U}$ by nonprincipality. In other words, $\mathcal{N}_0 \subseteq (d_A - d_{1+A})_* \mathcal{U}$.

This means
$$\begin{aligned} m(A) - m(1+A) &= \lim_{\mathcal{U}} d_A - \lim_{\mathcal{U}} d_{1+A} \\ &= \lim_{\mathcal{U}} (d_A - d_{1+A}) \\ &= 0 \text{ as desired. } \blacksquare (C2) \end{aligned}$$

We did it! $\blacksquare$ (Thm)

③ Let's take a moment to reflect on this important argument. What was "special" about $\mathbb{Z}$?

<u>Reflection 1</u>: Claim 1 always works. More precisely, given a set X and an enumerated collection $\{F_n : n \in \mathbb{N}\}$ of non-$\emptyset$ finite subsets of X, we get for $A \subseteq X$ a corresponding density function
$$d_A : \mathbb{N} \longrightarrow [0,1]$$
$$n \longmapsto \frac{|A \cap F_n|}{F_n}.$$
Then $\lim_{\mathcal{U}} d_A$ is a fapm on X. (more on $\boxed{\text{HW3}}$)

<u>Reflection 2</u>: Claim 2 (via Obs 2) uses bonus geometrical information about the sets F_n: they have "small boundary."

Before formalizing this, let's quickly sketch:

<u>Thm</u>: $(\mathbb{Z}^2; +)$ is amenable.

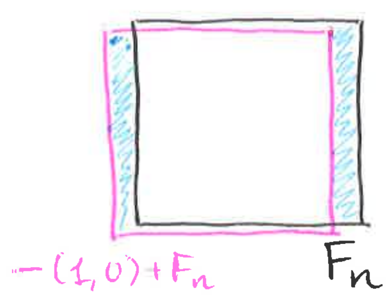
<u>pf (sketch)</u>: Put $F_n = \{0, \ldots, n\} \times \{0, \ldots, n\} \subseteq \mathbb{Z}^2$.

As before, for $A \subseteq \mathbb{Z}^2$ put $d_A : n \longmapsto \frac{|A \cap F_n|}{|F_n|}$

and (with nonprincipal $\mathcal{U}$ on $\mathbb{N}$) $m(A) = \lim_{\mathcal{U}} d_A$.

⋮

<u>"Obs 2:"</u> $|d_A(n) - d_{(1,0)+A}(n)| \leq \frac{2(n+1)}{(n+1)^2}$



At most $2(n+1)$ disagreements out of $(n+1)^2$ pts.

☒ (Sketch)

④ We are ready for the next big definition.

<u>Def:</u> Suppose that Γ is a group, that $S \subseteq \Gamma$ is finite, and that $\varepsilon > 0$. We say that a non-$\emptyset$ finite set $F \subseteq \Gamma$ is <u>(S, ε)-Følner</u> if

$$\forall \gamma \in S \quad \frac{|\gamma \cdot F \triangle F|}{|F|} \leq \varepsilon .$$

<u>Rmk:</u> There is an analogous notion for general actions $\Gamma \curvearrowright X$. $\boxed{\text{HW?}}$

<u>Examples:</u>

ⓐ In $\mathbb{Z}$, for $S = \{1\}$ the interval $F_n = \{-n, \dots, n\}$ is $(S, \frac{2}{2n+1})$-Følner

ⓑ In $\mathbb{Z}^2$, for $S = \{(1,0), (0,1)\}$ the square $F_n = \{0, \dots, n\} \times \{0, \dots, n\}$ is $(S, \frac{2}{n+1})$-Følner.

ⓒ In $\mathbb{F}_2$, for $S = \{a, b\}$ there $\boxed{\text{DOES NOT EXIST}}$ any $(S, \frac{1}{100})$-Følner set. $\boxed{\text{HW}}$

<u>Def:</u> We say that a group Γ satisfies the <u>Følner condition</u> if for all finite $S \subseteq \Gamma$ and $\varepsilon > 0$, there is an (S, ε)-Følner set.

(1)

Paradoxes

Monday, Feb 12

Last time:

Def: Suppose that Γ is a group.

(a) Given finite $S \subseteq \Gamma$ and $\varepsilon > 0$, we say that a non-$\emptyset$ finite $F \subseteq \Gamma$ is (S, ε)-Følner if
$$\forall \gamma \in S \quad \frac{|\gamma \cdot F \triangle F|}{|F|} \leq \varepsilon.$$

(b) Γ satisfies the Følner condition if for all finite $S \subseteq \Gamma$ and $\varepsilon > 0$, there is an (S, ε)-Følner set.

Fact $\boxed{\text{HW?}}$: If Γ admits a finite generating set T, it is enough to find (T, ε)-Følner sets.

Thm (Følner): Suppose that Γ is a countable group. If Γ satisfies the Følner condition, then it is amenable.

Remark: The assumption of countability is unnecessary.

pf: Fix an enumeration $\Gamma = \{\gamma_i : i \in \mathbb{N}\}$.
Put $S_n = \{\gamma_i : i < n\}$ and fix positive $\varepsilon_n \to 0$.
The Følner condition grants non-$\emptyset$ finite $F_n \subseteq \Gamma$ that are (S_n, ε_n)-Følner. As discussed last time, for $A \subseteq \Gamma$ we get a density function
$$d_A : \mathbb{N} \to [0,1]$$
$$n \mapsto \frac{|A \cap F_n|}{|F_n|}.$$

(2) <u>pf</u> (Thm, cont.)

Fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and consider the fapm m on Γ given by

$$m : A \mapsto \lim_{\mathcal{U}} d_A .$$

<u>Claim</u>: For all $\gamma \in \Gamma$ and $A \subseteq \Gamma$, $m(\gamma \cdot A) = m(A)$.

<u>pf(C)</u>: Let $\varepsilon > 0$ be arbitrary. By construction, the set $\{ n \in \mathbb{N} : \gamma^{-1} \in S_n \text{ and } \varepsilon_n < \varepsilon \}$ is cofinite.

Thus, so is $\left\{ n \in \mathbb{N} : \frac{|\gamma^{-1} \cdot F_n \triangle F_n|}{|F_n|} < \varepsilon \right\}$.

We compute
$$|d_{\gamma \cdot A}(n) - d_A(n)| = \frac{\big| |(\gamma \cdot A) \cap F_n| - |A \cap F_n| \big|}{|F_n|}$$
$$= \frac{\big| |A \cap (\gamma^{-1} \cdot F_n)| - |A \cap F_n| \big|}{|F_n|}$$
$$\leq \frac{|\gamma^{-1} \cdot F_n \triangle F_n|}{|F_n|} .$$

Thus, the set $\{ n \in \mathbb{N} : |d_{\gamma \cdot A}(n) - d_A(n)| < \varepsilon \}$ is cofinite, hence in $\mathcal{U}$.

In other words, $\mathcal{N}_0 \subseteq (d_{\gamma \cdot A} - d_A)_* \mathcal{U}$.

This means
$$m(\gamma \cdot A) - m(A)$$
$$= \lim_{\mathcal{U}} d_{\gamma \cdot A} - \lim_{\mathcal{U}} d_A$$
$$= \lim_{\mathcal{U}} (d_{\gamma \cdot A} - d_A)$$
$$= 0 \quad \text{as desired.} \qquad \blacksquare (C)$$

So m is our Γ-invariant fapm witnessing amenability. $\blacksquare$ (Thm)

③ To summarize, we have three properties of a (countable, for now) group Γ:

$\boxed{\mathrm{I}}$ Γ satisfies the Følner condition

$\boxed{\mathrm{II}}$ Γ is amenable

$\boxed{\mathrm{III}}$ Γ is not paradoxical.

Følner

Tarski

We have shown $\boxed{\mathrm{I}} \Rightarrow \boxed{\mathrm{II}}$ via "geometry"

and $\boxed{\mathrm{II}} \Rightarrow \boxed{\mathrm{III}}$ via "analysis."

Gromov found a secret backdoor for $\boxed{\mathrm{III}} \Rightarrow \boxed{\mathrm{I}}$:

show $\neg\boxed{\mathrm{I}} \Rightarrow \neg\boxed{\mathrm{III}}$ via "combinatorics."

The failure of $\boxed{\mathrm{I}}$ means that Γ is somehow expansive in the sense that finite sets have large boundary. We'll formalize this in the language of GRAPHs.

<u>Def</u>: Given a (possibly infinite) set V, a (simple, undirected) <u>graph</u> on V is a symmetric, irreflexive subset of V^2. So $G \subseteq V^2$ is a graph iff:

- $\forall x \quad (x,x) \notin G$
- $\forall x,y \quad (x,y) \in G \Rightarrow (y,x) \in G$.

<u>Rmk</u>: We may say the following for $(x,y) \in G$:

- x and y are <u>adjacent</u>
- (x,y) is a (G-) <u>edge</u>
- $x \sim y$
- $x \, G \, y$.

④ Some examples of graphs [relevant to us]

ⓐ If X, Y are disjoint sets and $f : X \to Y$ is a function, we obtain an associated graph G_f on $V = X \sqcup Y$ by $G_f = f \cup \check{f}$, i.e.,

$$G_f = \{(x,y) \in X \times Y : f(x) = y\} \cup \{(y,x) \in Y \times X : f(x) = y\}.$$

This is an example of a bipartite graph.

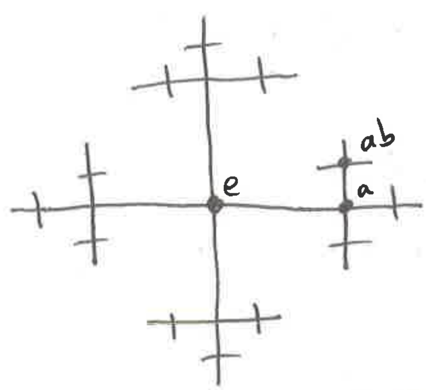
<u>**Def:**</u> G on V is <u>bipartite</u> if there is a partition $V = X \sqcup Y$ so that every G-edge has the form (x,y) or (y,x) for $x \in X$ and $y \in Y$.

Equivalently, $G \subseteq X \times Y \cup Y \times X$.

ⓑ If Γ is a group and $S \subseteq \Gamma \setminus \{e\}$, we get the (right) <u>Cayley graph</u>, $\mathrm{Cay}(\Gamma, S)$, on vertex set Γ by declaring edges

$$(\gamma, \gamma s) \text{ for each } \gamma \in \Gamma \text{ and } s \in S^{\pm}.$$

$\mathrm{Cay}(\mathbb{F}_2, \{a,b\})$:



Happens to be bipartite

$\mathrm{Cay}(\mathbb{Z}/3\mathbb{Z}, \{1 + 3\mathbb{Z}\})$:

Not bipartite

ⓒ $\Gamma \curvearrowright X$, $S \subseteq \Gamma$. Get a <u>Schreier graph</u> on X with edges $(x, s \cdot x)$ for $s \in S^{\pm}$ provided that $s \cdot x \neq x$.

Paradoxes

Wednesday, Feb 14

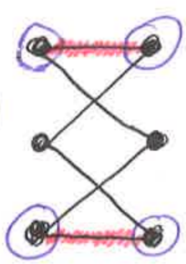
Matchings

The setup: G is a bipartite graph on $V = X \sqcup Y$.

Rmk: This makes sense in the non-bipartite setting, too.

Defs:
- Given $A \subseteq V$, its set of (G-)neighbors is
$$N_G(A) = \{w \in V : \exists a \in A \ (a,w) \in G\}.$$
- The (G-)degree of $v \in V$ is $|N_G(\{v\})|$.
  I.e., it counts the G-edges incident to v.
- G is locally finite if every vertex has finite degree.
- A matching is a subgraph $M \subseteq G$ in which every vertex has M-degree at most 1. Equivalently, M is a collection of p.w. non-incident G-edges. Equiv, it is a subgraph of G formed by a partial injection $i : X \to Y$.
- Given a matching $M \subseteq G$, its domain is
$$\operatorname{dom}(M) = \{v \in V : \exists w \in V \ (v,w) \in M\}.$$
- A matching M is perfect if $\operatorname{dom}(M) = V$. Equiv, if M is a subgraph of G formed by a bijection $i : X \to Y$.

dom(M). not perfect



perfect!

② We will prove this on Friday:

<u>Thm</u> (ess P. Hall, 1935) [AC]: Suppose that G is a locally finite bipartite graph on $V = X \sqcup Y$. Suppose further that G satisfies the <u>Hall condition</u>:

- $\forall$ finite $A \subseteq X$ $\quad |N_G(A)| \geq |A|$
- $\forall$ finite $B \subseteq Y$ $\quad |N_G(B)| \geq |B|$.

Then G admits a perfect matching.

<u>Rmk</u>: The Hall condition is clearly necessary as well.

Let's re-examine some prior work from this perspective.

<u>Thm</u> (Careful Schröder-Bernstein): Given injections $\boxed{\begin{array}{l} f: X \to Y \\ g: Y \to X \end{array}}$, there is $C \subseteq X$ s.t. $f \restriction C \cup g^{-1} \restriction (X \setminus C)$ is a bijection $X \to Y$.

<u>pf</u>: WLOG X and Y are disjoint.
[Work with $X \times \{0\}$ and $Y \times \{1\}$ if you like].
Define a graph G on $V = X \sqcup Y$ by

$$(x, y) \in G \text{ iff } f(x) = y \text{ or } g(y) = x.$$

degree at most 2

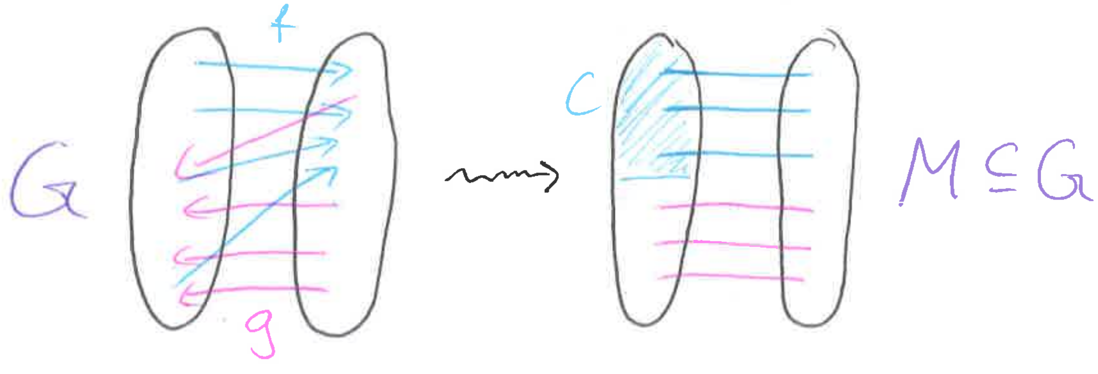
Check Hall condition: for $A \subseteq X$ finite we see

$$f[A] \subseteq N_G(A), \text{ hence } |A| = |f[A]| \leq |N_G(A)| \checkmark$$

A symmetric argument handles $B \subseteq Y$. ✓

So G admits a perfect matching M.

Put $C = \{x \in X : (x, f(x)) \in M\}$. ∎ (Careful S-B)



③ Similarly, we can use matchings to detect equidecomposability via a prescribed set of group elements.

Prop: Suppose that $\Gamma \curvearrowright Z$ and that $S \subseteq \Gamma$ is finite. Given $X, Y \subseteq Z$, there is an equidecomposition

$$\boxed{\begin{aligned} X &= \bigsqcup_{\gamma \in S} C_\gamma \\ Y &= \bigsqcup_{\gamma \in S} \gamma \cdot C_\gamma \end{aligned}}$$

iff the graph G on $X \sqcup Y$:

$$G = \{(x, \gamma \cdot x) : x \in X \text{ and } \gamma \cdot x \in Y \text{ and } \gamma \in S\}$$

admits a perfect matching.

G technically a multigraph unless action is free

pf: For $\gamma \in S$, put $C_\gamma = \{x \in X : (x, \gamma \cdot x) \in M\}$. ■(Prop)

It's worth reconsidering past equidecompositions...

The following is central in our proof of Hall's theorem:

Lemma: Suppose that G is a locally finite bipartite graph on $V = X \sqcup Y$ satisfying the Hall condition. Then for all finite $F \subseteq V$ there is a finite matching M with $F \subseteq \mathrm{dom}(M)$.

pf($\mathcal{L}$): By induction on $|F|$, it suffices to prove:

☆ For each finite matching $M \subseteq G$ and $x \in V$, there is a finite matching $M' \subseteq G$ with $\{x\} \cup \mathrm{dom}(M) \subseteq \mathrm{dom}(M')$.

By symmetry we may assume $x \in X$, and $x \notin \mathrm{dom}(M)$.

Def: An *M-alternating path* from x is a sequence of G-edges like this:

length = # of edges

Such a path is *augmenting* if $y \notin \mathrm{dom}(M)$.

Note: "Flipping" the matched edges along an augmenting path results in a valid matching.

④ <u>pf</u> (Lemma, cont.)

<u>Claim</u>: There is an M-augmenting path from x of length at most $2m+1$, where $m = |\mathrm{dom}(M) \cap Y|$

<u>pf(c)</u>: Suppose otherwise. We examine how many $y \in \mathrm{dom}(M)$ we can "reach" by short alternating paths. Recursively define finite sets $B_i \subseteq Y$ by

$$B_0 = N_G(\{x\}) \qquad \text{Note: } |B_0| > 0$$

$$B_{i+1} = N_G(N_M(B_i) \cup \{x\})$$

So for each $y \in B_i$ there is an alternating path from x to y of length at most $2i+1$. By assumption, we then know for $i \le m$ that $B_i \subseteq \mathrm{dom}(M)$. The Hall condition ensures for $i < m$ that

$$|B_{i+1}| \geq |N_M(B_i) \cup \{x\}| = |B_i| + 1,$$

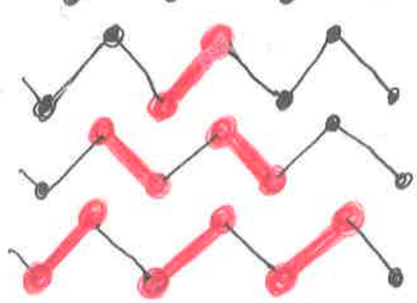
so $|B_i| > i$. But then $B_m \subseteq \mathrm{dom}(M)$ is too big! ∎(c)

So the claim grants an M-augmenting path from x to some y. Flip edges to obtain a matching M' with $\mathrm{dom}(M') = \mathrm{dom}(M) \cup \{x, y\}$. This establishes ✪, and proves the Lemma. ∎($\mathcal{L}$)

Sadly, these finite matchings typically fail to "converge."

For example, on this graph: ...

a typical sequence of augmentations is:



The domains are growing, but the matched edges themselves keep flipping back and forth.

Paradoxes

Friday, Feb 16

From finite combinatorics to locally finite combinatorics...

As promised, we prove Hall's theorem for locally finite bipartite graphs. We do so by a method that readily generalizes to many other situations.

Def: Given a set V, let $\mathrm{FIN}(V) = \{F \subseteq V : F \text{ is finite}\} \subseteq \mathcal{P}(V)$ denote its collection of finite subsets.

Def: Given $F \in \mathrm{FIN}(V)$, define the cone above F to be the set $C_F = \{A \in \mathrm{FIN}(V) : F \subseteq A\}$.

Def: The cone filter $\mathcal{F}$ on $\mathrm{FIN}(V)$ is defined by

$$P \in \mathcal{F} \text{ iff } \exists F \in \mathrm{FIN}(V)\ C_F \subseteq P.$$

Prop: $\mathcal{F}$ is indeed a proper filter on $\mathrm{FIN}(V)$.

pf: We check the definition:

- $\emptyset \notin \mathcal{F}$: Cones are nonempty, as $F \in C_F$.
- $\mathrm{FIN}(V) \in \mathcal{F}$: Observe that $\mathrm{FIN}(V) = C_\emptyset$.
- $P, Q \in \mathcal{F} \Rightarrow P \cap Q \in \mathcal{F}$: Suppose that $C_E \subseteq P$, $C_F \subseteq Q$. Then $C_{E \cup F} = C_E \cap C_F \subseteq P \cap Q$, so $P \cap Q \in \mathcal{F}$.
- $P \in \mathcal{F}$ and $P \subseteq Q \Rightarrow Q \in \mathcal{F}$: If $C_F \subseteq P$, then $C_F \subseteq Q$.

■ (Prop)

<u>Thm</u> (Hall) [AC]: Suppose that G is a locally finite bipartite graph on $V = X \sqcup Y$ satisfying the Hall condition: ▫ $\forall A \in \mathrm{FIN}(X)$ $|N_G(A)| \geq |A|$
▫ $\forall B \in \mathrm{FIN}(Y)$ $|N_G(B)| \geq |B|$.

Then G admits a perfect matching.

Last time we proved:

<u>Lemma</u>: Given a graph G as above, for all $F \in \mathrm{FIN}(V)$ there is a (finite) matching $M \subseteq G$ with $F \subseteq \mathrm{dom}(M)$.

<u>pf (Thm)</u>: For each $F \in \mathrm{FIN}(V)$, use the Lemma to choose a matching $M_F \subseteq G$ with $F \subseteq \mathrm{dom}(M_F)$. Using the Ultrafilter Lemma, fix an ultrafilter $\mathcal{U}$ on $\mathrm{FIN}(V)$ extending the cone filter.
Finally, define M by

$$(v,w) \in M \text{ iff } \{F \in \mathrm{FIN}(V) : (v,w) \in M_F\} \in \mathcal{U}.$$

We will show that M is the desired perfect matching. To do so, we check the following four claims in turn:

<u>Claim 0</u>: $M \subseteq G$.

<u>Claim 1</u>: M is symmetric.

<u>Claim 2</u>: M is a matching.

<u>Claim 3</u>: $\mathrm{dom}(M) = V$.

③ pf (Thm, cont.):

pf(C0): Suppose that $(v,w)\in M$, i.e., that
$$\{F\in FIN(V) : (v,w)\in M_F\}\in\mathcal{U}.$$
Since $\emptyset\notin\mathcal{U}$, the above set is non-$\emptyset$.
So there is some $F\in FIN(V)$ with $(v,w)\in M_F$.
As $M_F\subseteq G$, we conclude that $(v,w)\in G$. ∎(C0)

pf(C1): Suppose that $(v,w)\in M$. As each M_F is symmetric, we see that
$$\{F\in FIN(V) : (w,v)\in M_F\}$$
$$=\{F\in FIN(V) : (v,w)\in M_F\}\in\mathcal{U}.$$
This shows that $(w,v)\in M$ as well. ∎(C1)

pf(C2): Towards a contradiction, suppose that M is $\boxed{\text{not}}$ a matching. I.e., that there are $u,v,w\in V$ with $v\neq w$ and
- $(u,v)\in M$
- $(u,w)\in M$.

That is,
$$P=\{F\in FIN(V) : (u,v)\in M_F\}\in\mathcal{U}$$
$$Q=\{F\in FIN(V) : (u,w)\in M_F\}\in\mathcal{U}$$
The intersection of these two sets is also in $\mathcal{U}$, hence non-$\emptyset$. But for any $F\in P\cap Q$ we have
- $(u,v)\in M_F$
- $(u,w)\in M_F$

contradicting the fact that M_F is a matching.
∎(C2)

④ <u>pf</u> (thm, cont.)

<u>pf(C3)</u>: Consider arbitrary $v \in V$.

We know the cone $C_{\{v\}} \in \mathcal{U}$.

For all $F \in C_{\{v\}}$, we know that M_F must match v with a neighbor. In symbols,

$$C_{\{v\}} = \bigsqcup_{w \in N_G(\{v\})} \{F \in C_{\{v\}} : (v,w) \in M_F\}.$$

We have covered $C_{\{v\}}$ with finitely many sets, thus one is also in $\mathcal{U}$. Fix $w \in N_G(\{v\})$ with

$$\{F \in C_{\{v\}} : (v,w) \in M_F\} \in \mathcal{U}.$$

This means

$$\{F \in FIN(V) : (v,w) \in M_F\} \in \mathcal{U}$$

and thus $(v,w) \in M$ as desired. ∎(C3)

So M is a perfect matching after all! ∎(Thm)

<u>Remarks</u>:

ⓐ This can all be cast in topological language, by placing a natural compact Hausdorff topology on the set of matchings for G.

ⓑ This can also be cast in the language of propositional logic...

ⓒ Similar arguments work for MANY combinatorial problems [colorings, flows, etc.].

Paradoxes

Monday, Feb 19

Recall some Defs: Suppose that Γ is a group.

(a) Γ satisfies the Følner condition if for all finite $S \subseteq \Gamma$ and $\varepsilon > 0$ there is a non-$\emptyset$ finite (S, ε)-Følner set $F \subseteq \Gamma$, i.e., $\forall \gamma \in S \ \dfrac{|\gamma \cdot F \triangle F|}{|F|} \leq \varepsilon$.

(b) Γ is paradoxical if there is a partition $\Gamma = A_0 \sqcup A_1$ with $\Gamma \approx A_0$ and $\Gamma \approx A_1$ (via left-mult action).

Today's goal:

Thm (Følner + Tarski): Suppose that Γ is a group. $\neg \boxed{\text{I}} \Rightarrow \neg \boxed{\text{III}}$:

$\neg \boxed{\text{I}}$ Γ does $\boxed{\text{not}}$ satisfy the Følner condition.

$\neg \boxed{\text{III}}$ Γ is paradoxical.

To prove this, we pass through the following lemma, referred to as $\boxed{\text{Gromov's doubling condition}}$.

Lemma (Gromov): Suppose that Γ is a group that does not satisfy the Følner condition. Then there is a finite set $T \subseteq \Gamma$ such that for all finite $F \subseteq \Gamma$, $|T \cdot F| \geq 2|F|$.

Recall: $T \cdot F = \{\tau\delta : \tau \in T \text{ and } \delta \in F\}$.

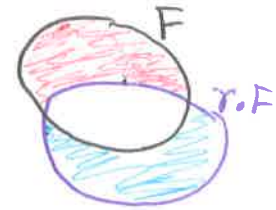
$= \bigcup \{\tau \cdot F : \tau \in T\}$.

② **pf($\mathcal{L}$):** Fix finite $S \subseteq \Gamma$ and $\varepsilon > 0$ such that no (S, ε)-Følner set exists. This means for all finite $F \subseteq \Gamma$, $\exists \gamma \in S$ $|\gamma \cdot F \triangle F| \geq \varepsilon \cdot |F|$.

Put $S_1 = S^{\pm} \cup \{e\}$, so S_1 is a finite subset of Γ.

Claim 1: For all finite $F \subseteq \Gamma$, $|S_1 \cdot F| \geq (1 + \varepsilon/2)|F|$.

pf(C1): Fix F, and fix $\gamma \in S$ s.t.



$$|(\gamma \cdot F \setminus F) \cup (F \setminus \gamma \cdot F)| = |\gamma \cdot F \triangle F| \geq \varepsilon \cdot F$$

Case $\boxed{1}$: $|\gamma \cdot F \setminus F| \geq \frac{\varepsilon}{2}|F|$.

Then $|\gamma \cdot F \cup F| \geq (1 + \varepsilon/2)|F|$, and we're done since $\gamma \cdot F \cup F \subseteq S_1 \cdot F$. ✓

Case $\boxed{2}$: $|F \setminus \gamma \cdot F| \geq \frac{\varepsilon}{2}|F|$.

But $|\gamma^{-1} \cdot F \setminus F| = |F \setminus \gamma \cdot F| \geq \frac{\varepsilon}{2}|F|$, so we may apply the above logic to $\gamma^{-1} \in S_1$ ✓ ∎(C1)

Define finite subsets $S_n \subseteq \Gamma$ recursively:

- $S_1 = S^{\pm} \cup \{e\}$ [as above]
- $S_{n+1} = S_1 \cdot S_n$

Claim 2: For all finite $F \subseteq \Gamma$, $|S_n \cdot F| \geq (1 + \varepsilon/2)^n |F|$.

pf(C2): By induction on n. Base case is C1.

$$|S_{n+1} \cdot F| = |S_1 \cdot (S_n \cdot F)| \geq (1 + \varepsilon/2)|S_n \cdot F| \geq (1 + \varepsilon/2)^{n+1}|F|. \text{ ∎(C2)}$$

Picking n large enough so that $(1 + \varepsilon/2)^n \geq 2$, we see that $T = S_n$ satisfies Gromov's doubling condition. ∎($\mathcal{L}$)

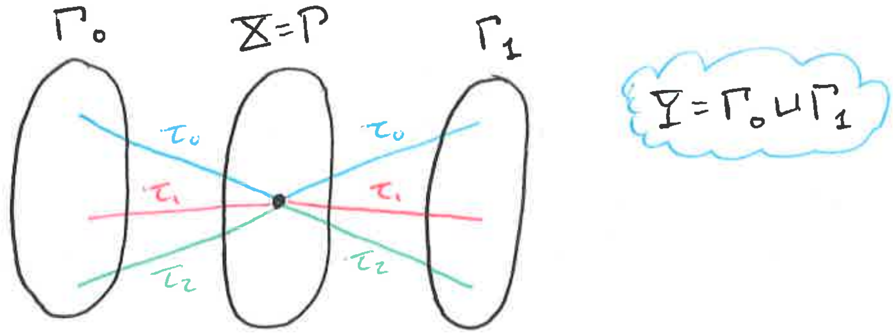
③ <u>pf</u> ($\neg\boxed{\mathrm{I}} \Rightarrow \neg\boxed{\mathrm{III}}$, Gromov):

Since Γ does not satisfy the Følner condition, we use the Lemma to find finite $T \subseteq \Gamma$ satisfying the doubling condition: for all finite $F \subseteq \Gamma$ $|T \cdot F| \geq 2|F|$.

We declare
- $X = \Gamma$
- $Y = \Gamma \times 2 = \{(\gamma, i) : \gamma \in \Gamma, i < 2\} = \Gamma_0 \sqcup \Gamma_1$.

We define a locally finite bipartite graph G on $X \sqcup Y$:

$$(\gamma, (\delta, i)) \in G \text{ iff } \exists \tau \in T^{\pm} \ \tau \cdot \gamma = \delta$$



<u>Claim:</u> G satisfies the Hall condition.

<u>pf (C):</u> Suppose first that $A \subseteq X$ is finite.

Then $T \cdot A \times \{0\} \subseteq N_G(A)$, so $|N_G(A)| \geq |A|$. ✓

Suppose now that $B \subseteq Y$ is finite. Put $B_i = B \cap \Gamma_i$ and by symmetry assume $|B_0| \geq |B_1|$.

Put $C = \{\gamma \in \Gamma : (\gamma, 0) \in B_0\}$, and observe that $T \cdot C \subseteq N_G(B_0)$. We compute

$$|N_G(B)| \geq |T \cdot C| \geq 2|C| = 2|B_0| \geq |B|. \text{ ✓}$$

$\blacksquare$(C)

<u>pf</u> ($\neg\boxed{\text{I}} \Rightarrow \neg\boxed{\text{III}}$, cont.)

By Hall's theorem, G admits a perfect matching $M \subseteq G$. We define our partition $\Gamma = A_0 \sqcup A_1$ by

$$A_0 = \{\gamma \in X : M \text{ matches } \gamma \text{ to } \Gamma_0\}$$
$$A_1 = \{\gamma \in X : M \text{ matches } \gamma \text{ to } \Gamma_1\},$$

and observe that M encodes equidecompositions $A_0 \approx \Gamma$, $A_1 \approx \Gamma$ as discussed last week.

$\blacksquare$ ($\neg\boxed{\text{I}} \Rightarrow \neg\boxed{\text{III}}$)

<u>Remark</u>: We proved something a bit stronger. Namely, if $T \subseteq \Gamma$ is a finite set satisfying the doubling condition, then there is a paradoxical decomposition of Γ in which the equidecomps can be realized by elements of T. In other words:

LOCAL doubling $\Rightarrow$ GLOBAL doubling [Paradoxicality]

In summary, we have finally proved:

<u>Thm</u> (Følner + Tarski) [AC]: Suppose that Γ is a (countable) group. TFAE:

$\boxed{\text{I}}$ Γ satisfies the Følner condition.

$\boxed{\text{II}}$ Γ is amenable.

$\boxed{\text{III}}$ Γ is not paradoxical.

Paradoxes

Wednesday, Feb 21

What we did over the past few weeks:

Thm (Følner + Tarski) [AC]: Suppose that Γ is a (countable) group.

TFAE:

$\boxed{\text{I}}$ Γ satisfies the Følner condition.

$\boxed{\text{II}}$ Γ is amenable.

$\boxed{\text{III}}$ Γ is not paradoxical.

Remark: We only assumed countability of Γ in the proof of $\boxed{\text{I}} \Rightarrow \boxed{\text{II}}$. This is unnecessary as we shall discuss soon.

Today we analyze some $\boxed{\text{algebraic}}$ aspects of amenability.

Def: A group Γ is finitely generated if there is a finite subset $S \subseteq \Gamma$ with $\Gamma = \langle S \rangle = \{\gamma \in \Gamma : \gamma \text{ expressible as an } S\text{-word}\}$.

Remark: Finitely generated groups are countable.

Prop: Suppose that Γ is an amenable group. Then every finitely generated subgroup of Γ is amenable.

pf: Fix a f.g. subgroup $\Delta \leq \Gamma$, and towards a contradiction assume Δ is not amenable. Then Δ is paradoxical, thus $\boxed{\text{HW}}$ so is Γ. This contradicts amenability of Γ. ■ (Prop).

$\boxed{\text{HW}}$ Converse: If every finitely generated subgroup of a group Γ is amenable, then Γ itself is amenable.

② This allows us to finally drop that countability hypothesis.

pf ($\boxed{I} \Rightarrow \boxed{II}$): Suppose that Γ satisfies the Følner condition. Then Γ cannot be paradoxical: $\boxed{\text{HW?}}$. Thus, every finitely generated subgroup of Γ is non-paradoxical, hence amenable. This implies amenability of Γ. ∎ ($\boxed{I} \Rightarrow \boxed{II}$)

In turn, we can upgrade our earlier work.

Thm: If Γ is an amenable group, then every subgroup of Γ is amenable.

pf: No such subgroup can be paradoxical. ∎

Dually, amenability passes to quotients.

Def: Given groups Γ and Δ, we say that Δ is a quotient of Γ if there is a surjective hom $\Gamma \to \Delta$.

Thm: If Γ is an amenable group, then every quotient of Γ is amenable.

pf: Suppose that $\varphi : \Gamma \to \Delta$ is a surjective hom, and fix a Γ-invariant fapm m on Γ. We consider the pushforward $n = \varphi_* m$ on Δ,

so $\quad n : A \mapsto m(\varphi^{-1}(A))$

Certainly n is a fapm on Δ.

③ pf (thm, cont.)

Claim: n is Δ-invariant.

pf (C): Suppose that $A \subseteq \Delta$ and $\delta \in \Delta$ are arbitrary. Fix $\gamma \in \Gamma$ with $\varphi(\gamma) = \delta$. Note then for $\eta \in \Gamma$ we have $\varphi(\gamma\eta) \in$ iff $\varphi(\eta) \in A$.

In other words, $\varphi^{-1}(\delta \cdot A) = \gamma \cdot \varphi^{-1}(A)$.

Finally, we compute
$$\begin{aligned} n(\delta \cdot A) &= m(\varphi^{-1}(\delta \cdot A)) \\ &= m(\gamma \cdot \varphi^{-1}(A)) \\ &= m(\varphi^{-1}(A)) \\ &= n(A). \quad \blacksquare (C) \end{aligned}$$

$\blacksquare$ (Thm)

Thm: If Γ, Δ are amenable groups, so is $\Gamma \times \Delta$.

pf: (sketch). $\boxed{\text{HW}}$ shows how to obtain the Følner condition for $\Gamma \times \Delta$ from appropriate Følner sets in Γ and Δ. $\blacksquare$ (Thm, sketch)

Corollary: Abelian groups are amenable.

pf: It suffices to check that finitely generated abelian groups are amenable. This follows from the preceding theorem, as every finitely generated abelian group is a direct product of (finitely many) cyclic groups. $\blacksquare$ (Cor)

④

Def: A group is <u>virtually [blah]</u> if it has a finite index subgroup that is [blah].

Thm: Suppose that Γ is virtually amenable. Then Γ is amenable.

pf: Fix a finite index $\Delta_0 \leq \Gamma$ that is amenable. Thin down to finite index $\Delta \leq \Delta_0$ with $\Delta \trianglelefteq \Gamma$ (i.e., $\forall \gamma \in \Gamma$ $\gamma \Delta \gamma^{-1} = \Delta$). Note that Δ is amenable.

Fix a Δ-invariant fapm n on Δ.

Fix coset reps $\{\gamma_i : i < k\}$ for Δ.

Define Δ-inv. fapms m_i on Γ by

$$m_i : A \longmapsto n(\gamma_i^{-1} \cdot A \cap \Delta) \quad = \gamma_i * n$$

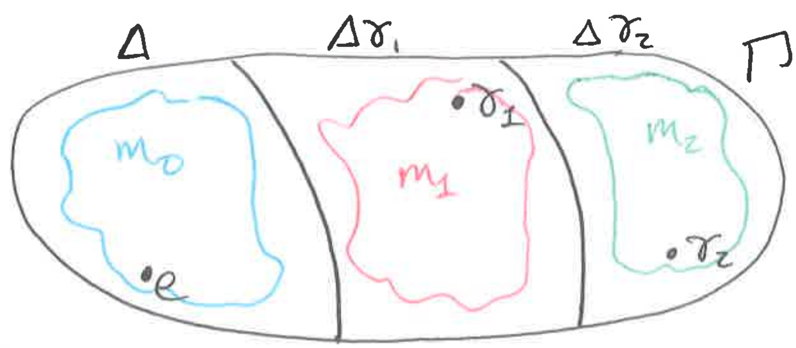
and average these to a fapm m on Γ by

$$m : A \longmapsto \frac{1}{k} \sum_{i<k} m_i(A).$$

Claim: m is Γ-invariant.

<u>pf</u> (C, by picture ... real argument coming later)

Each m_i is a Δ-inv fapm with $m_i(\Delta\gamma_i) = 1$.



Then m is the average of these.

Left multiplication by $\gamma \in \Gamma$ does two things:

- shuffles the cosets around
- translates each coset by an element of Δ.

Neither affects the average value, m. ∎ (C)

∎ (Thm)

Paradoxes

Friday, Feb 23

Growth vs amenability

Guest star: the $\boxed{\text{LAMPLIGHTER}}$

Suppose that Γ is a group with finite generating set S. We define by recursion finite sets $B_n \subseteq \Gamma$ for $n > 0$:

- $B_1 = S^{\pm} \cup \{e\}$
- $B_{n+1} = B_1 B_n$

So $\gamma \in B_n$ iff it has a representation as an S-word with at most n characters.

<u>Def</u>: The <u>growth function</u> for (Γ, S) is the map

$$g : n \longmapsto |B_n|.$$

<u>Examples</u>: $\boxed{0}$ If Γ is finite, g is eventually constant

$\boxed{1}$ If $\Gamma = \mathbb{Z}$, $S = \{1\}$, $g : n \longmapsto n + (n+1)$

$\boxed{2}$ If $\Gamma = \mathbb{Z}^2$, $S = \{(1,0), (0,1)\}$, $g : n \longmapsto n^2 + (n+1)^2$

$\boxed{3}$ If $\Gamma = \mathbb{F}_2$, $S = \{a, b\}$, $g : n \longmapsto 2 \cdot 3^n - 1$

$\boxed{\text{HW3}}$ paradoxical $\Rightarrow$ exponential growth

Converse? Our new friend shows us the light...

② We first put $\Delta = \bigoplus_{z\in\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. Formally,

$$\Delta = \{ f : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \text{ s.t. } \underbrace{f^{-1}(\{0+2\mathbb{Z}\}) \text{ is cofinite}}_{\text{finite support}} \}$$

The operation is pointwise: $(f+g) : i \mapsto f(i)+g(i)$.
We can think of elements of Δ as bi-infinite tapes of 0s and 1s with finitely many 1s, and the operation of bitwise XOR. But we won't.

<u>Remark</u>: Δ is abelian (hence amenable) but NOT fin gen.

There is a natural action $\mathbb{Z} \curvearrowright \Delta$ by automorphisms

$$z \cdot f : i \mapsto f(z+i).$$

So $1 \cdot f$ slides the tape f $\boxed{\text{left}}$ one click.

We consider the corresponding <u>semidirect product</u>

$$\Gamma = \Delta \rtimes \mathbb{Z} = \{(\delta, z) : \delta \in \Delta \text{ and } z \in \mathbb{Z}\}.$$

<u>Recall</u>: the operation on $\Delta \rtimes \mathbb{Z}$ is given by

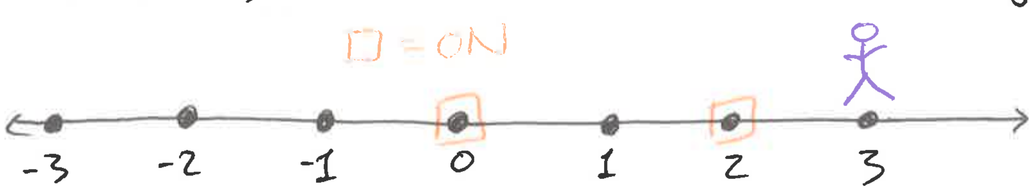
$$(\delta_0, z_0)(\delta_1, z_1) = (\delta_0 + (z_0 \cdot \delta_1), \; z_0 + z_1)$$

Γ is called the <u>lamplighter group</u>.

How to think about Γ: Given $(\delta, z) \in \Gamma$

δ codes the lamp configuration from the perspective of the lamplighter

z codes the location of the lamplighter



$$(\chi_{\{-3,-1\}}, 3)$$

③ The group Γ has a generating set $S = \{\ell, w\}$

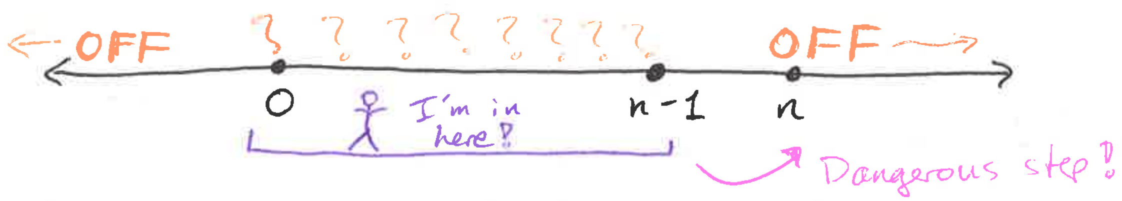
ℓ: Change the status of the lamp at the lamplighter's location. $(\chi_{\{0\}}, 0)$

w: Lamplighter walks one unit right. $(0, 1)$

<u>Prop</u>: Γ satisfies the Følner condition.

<u>pf</u>: By $\boxed{\text{HW3}}$ it suffices to find (S, ε)-Følner sets for the generating set S above.

For $n > 0$, consider the non-$\emptyset$ finite set $F_n \subseteq \Gamma$ consisting of configurations like so:



<u>Claim</u>: F_n is $(S, 2/n)$-Følner.

<u>pf(C)</u>: Observe first that $|F_n| = n2^n$.

- $\ell \cdot F_n = F_n$, so $\dfrac{|\ell \cdot F_n \,\Delta\, F_n|}{|F_n|} = 0 \leq 2/n$ ✓

- $|w \cdot F_n \setminus F_n| = 2^n$, and a symmetric calculation shows $|F_n \setminus w \cdot F_n| = 2^n$.

So $\dfrac{|w \cdot F_n \,\Delta\, F_n|}{|F_n|} \leq \dfrac{2^n + 2^n}{n2^n} = 2/n$ ✓

∎(C)

Since $2/n$ can be made arbitrarily small, we can find (S, ε)-Følner sets. ∎(Prop)

So Γ is amenable!

<u>Prop</u>: Γ has exponential growth.

<u>pf</u>: We shall show $g(2n) \geq 2^n$, i.e., that

$$|B_{2n}| \geq 2^n.$$

Consider words of the form

$$\underbrace{w\ell^? w\ell^? \cdots w\ell^?}_{n\text{-many } w\text{s}} \in B_{2n}$$

where $\ell^? \in \{e, \ell\}$ [independently for each $\ell^?$].

The resulting configurations look like:

More formally, the function $2^n \longrightarrow B_{2n}$

$$s \longmapsto w\ell^{s(n-1)} w\ell^{s(n-2)} \cdots w\ell^{s(0)}$$

is an injection. ▨ (Prop).

Paradoxes

Monday, Feb 26

Integration against fapms

Motivation: Given a fapm m on X, we want an operation $f \longmapsto \int f \, dm$ which assigns to every $f : X \to [0,1]$ an "average value" satisfying:

(a) [Positivity]. $\int f \, dm \in [0,1]$

(b) [Respects m] For $A \subseteq X$, $\int \chi_A \, dm = m(A)$.

As usual, $\chi_A : x \longmapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$

(c) [Linearity] $\int (af + bg) \, dm = a \int f \, dm + b \int g \, dm$

For all $a, b \in \mathbb{R}$ so that this makes sense. I.e., so that $af + bg : X \to [0,1]$.

Example: If $m = \mathcal{U}$, an ultrafilter on X, then $f \mapsto \lim_{\mathcal{U}} f$ satisfies these axioms.

Remark: If we can integrate all $f : X \to [0,1]$, we may extend the integral to all bounded $g : X \to \mathbb{R}$ by writing $g = af_+ - bf_-$ for appropriate $f_\pm : X \to [0,1]$.

Remark: It turns out that (a), (b), (c) completely determine the map $f \mapsto \int f \, dm$.

② Note that ⓑ + ⓒ already determine $\int f\,dm$ for some f.

Def: A function $f: X \to [0,1]$ is simple if it is a (finite) linear combination of characteristic functions. I.e., if there are $a_i \in [0,1]$ and $A_i \subseteq X$ with

$$f = \sum_{i<k} a_i \chi_{A_i}$$

Fact: f is simple iff its image $f[X]$ is finite.

Def: Given a simple function $f = \sum_i a_i \chi_{A_i}$, put $I_m(f) = \sum_i a_i\, m(A_i)$.

Fact: $0 \leq I(f) \leq 1$ and all representations of f yield the same value for $I(f)$. This implies linearity of I (among simple functions).

$\boxed{\text{Goal:}}$ Extend I to integrate all $f: X \to [0,1]$.

The key observation: positivity yields order-theoretic info

Def: Given functions $f, g: X \to [0,1]$, we say $f \leq g$ iff $\forall x \in X\ f(x) \leq g(x)$.
Equivalently, iff $g - f : X \to [0,1]$.

Remark: So we want $f \leq g \Rightarrow \int f\,dm + \int (g-f)\,dm = \int g\,dm$
$\Rightarrow \int f\,dm \leq \int g\,dm$.

Def: ⓐ For $g: X \to [0,1]$, put $S_g = \{f : f \text{ simple and } f \leq g\}$.

ⓑ Given a fapm m on X and $g: X \to [0,1]$, put

$$\int g\,dm = \sup I_m[S_g] = \sup\{I_m(f) : f \in S_g\}.$$

③ We need to show that $g \mapsto \int g\,dm$ satisfies ⓐ ⓑ ⓒ.

Prop ⓐ: $\int g\,dm \in [0,1]$

pf: Immediate. ■ ⓐ

Prop ⓑ: For all $A \subseteq X$, $\int \chi_A\,dm = m(A)$.

pf: ▫ $m(A) \leq \int \chi_A\,dm$: Observe that χ_A is simple, and thus $\chi_A \in S_{\chi_A}$. So $m(A) = I(\chi_A) \leq \sup I[S_{\chi_A}]$
$= \int \chi_A\,dm$. ✓

▫ $\int \chi_A\,dm \leq m(A)$: It suffices to show that $m(A)$ is an upper bound of $I[S_{\chi_A}]$. Towards that end, suppose $f = \sum_i b_i \chi_{B_i} \in S_{\chi_A}$, with $b_i \in [0,1]$. WLOG, the B_is are pairwise disjoint. Note that $b_i > 0 \Rightarrow B_i \subseteq A$, since $f \leq \chi_A$. We compute

$$I(f) = \sum_{b_i=0} b_i\,m(B_i) + \sum_{b_i>0} b_i\,m(B_i)$$
$$\leq 0 + \sum_{B_i \subseteq A} m(B_i) \leq m(A)$$ as desired.

■ ⓑ

Half of linearity is easy:

Prop ⓒ1: If $g, ag : X \to [0,1]$ for some $a > 0$, then
$$\int (ag)\,dm = a \int g\,dm.$$

pf: Observe that $f \leq g$ iff $af \leq ag$.
Thus, $S_{ag} = a S_g$. Then $I[S_{ag}] = a I[S_g]$, and finally $\int (ag)\,dm = a \int g\,dm$. ■ ⓒ1

④ So all that remains is the troublesome:

<u>Prop C2</u>: For all $g_0, g_1 : X \to [0,1]$

$$\int (g_0 + g_1)\, dm = \int g_0\, dm + \int g_1\, dm.$$

<u>pf</u>: ∘ $\int (g_0+g_1)\, dm \geq \int g_0\, dm + \int g_1\, dm$:

Note that $S_{g_0+g_1} \supseteq S_{g_0} + S_{g_1} = \{ f_0 + f_1 : f_i \in S_{g_i} \}$.

So $\sup I[S_{g_0+g_1}] \geq \sup I[S_{g_0}] + \sup I[S_{g_1}]$

i.e., $\int (g_0+g_1)\, dm \geq \int g_0\, dm + \int g_1\, dm$. ✓

▫ $\int (g_0+g_1)\, dm \leq \int g_0\, dm + \int g_1\, dm$:

As before, it suffices to show that $\int g_0\, dm + \int g_1\, dm$ is an upper bound for $I[S_{g_0+g_1}]$.

<u>Claim</u>: If $f \in S_{g_0+g_1}$ then $I(f) \leq \int g_0\, dm + \int g_1\, dm$.

<u>pf(C)</u>: For each $N > 0$, and $h : X \to [0,1]$, put

$$\lfloor h \rfloor_N : x \longmapsto \sup \{ \tfrac{k}{N} : \tfrac{k}{N} \leq h(x) \}.$$

"round down to $\frac{k}{N}$." So $h - \frac{1}{N} \leq \lfloor h \rfloor_N \leq h$.

Then $\lfloor g_0 \rfloor_N + \lfloor g_1 \rfloor_N \geq g_0 + g_1 - \frac{2}{N} \geq f - \frac{2}{N}$.

So for all $N > 0$ we see

$$\begin{aligned} \int g_0\, dm + \int g_1\, dm &\geq I(\lfloor g_0 \rfloor_N) + I(\lfloor g_1 \rfloor_N) \\ &\geq I(f) - \tfrac{2}{N} \end{aligned}$$

Hence $I(f) \leq \int g_0\, dm + \int g_1\, dm$. ∎(C)

So $\int g_0\, dm + \int g_1\, dm$ is indeed an upper bound for $I[S_{g_0+g_1}]$. ✓ ∎ C2

Paradoxes

Wednesday, Feb 28

Last time: Given a fapm m on X, there is an operation $f \mapsto \int f \, dm$ defined for $f: X \to [0,1]$ s.t.:

ⓐ $\int f \, dm \in [0,1]$

ⓑ $\int \chi_A \, dm = m(A)$

ⓒ $\int (af + bg) \, dm = a\int f \, dm + b \int g \, dm.$

$\boxed{\text{HW}}$ (1) This operation is unique, i.e., if some φ satisfies ⓐⓑⓒ for all $f: X \to [0,1]$, then $\varphi: f \mapsto \int f \, dm$.

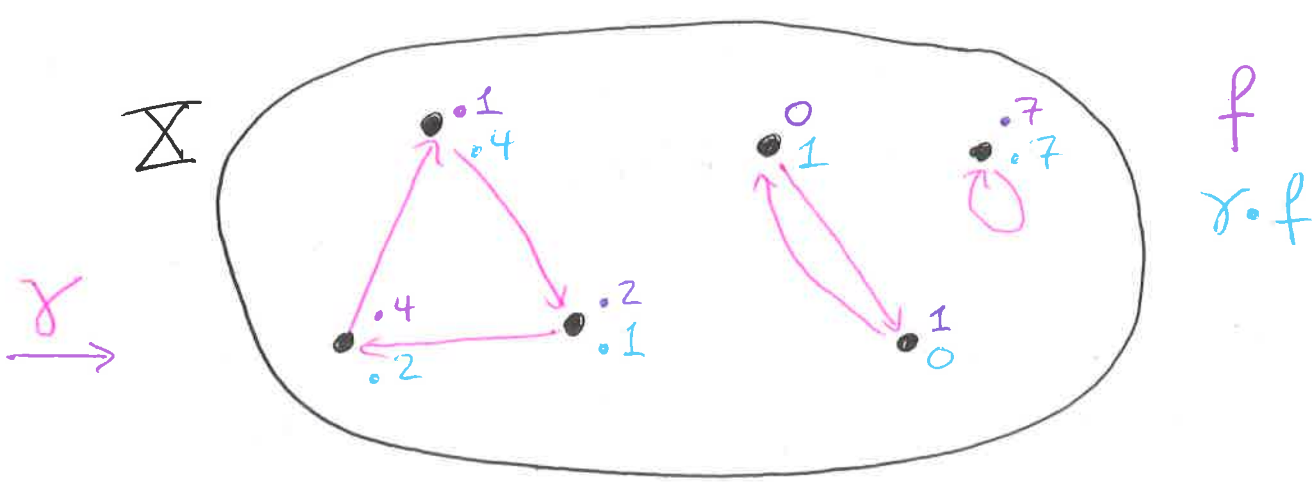
(2) If φ satisfies ⓐ+ⓒ, there is some fapm m on X with $\varphi: f \mapsto \int f \, dm$.

Def: $[0,1]^X$ denotes the set of all functions $X \to [0,1]$

Suppose now that $\Gamma \curvearrowright X$. This induces a Bernoulli shift action $\Gamma \curvearrowright [0,1]^X$ via $\gamma \cdot f : x \mapsto f(\gamma^{-1} \cdot x)$

Equivalently: $f(x) = r$ iff $(\gamma \cdot f)(\gamma \cdot x) = r$.

Note: $\gamma \cdot \chi_A = \chi_{\gamma \cdot A}$



② **Prop:** Suppose that m is a Γ-invariant fapm on X. Then for all $f \in [0,1]^X$, $\int f\, dm = \int \gamma \cdot f\, dm$.

pf: Consider the operation $f \mapsto \int \gamma \cdot f\, dm$.

ⓐ $\int \gamma \cdot f\, dm \in [0,1]$

ⓑ $\int \gamma \cdot \chi_A\, dm = \int \chi_{\gamma \cdot A}\, dm = m(\gamma \cdot A) = m(A)$

ⓒ $\int \gamma \cdot (af + bg)\, dm = a \int \gamma \cdot f\, dm + b \int \gamma \cdot g\, dm$.

By uniqueness $\boxed{\text{HW}}$, we conclude $\int \gamma \cdot f\, dm = \int f\, dm$. ■ (Prop)

Thm: Suppose that Γ is a group, and that $\Delta \leq \Gamma$ is an amenable subgroup. Suppose further that the action $\Gamma \curvearrowright \Gamma/\Delta$ is amenable. Then Γ is amenable.

pf: Fix fapms m_Δ on Δ, n on Γ/Δ witnessing the assumed amenability. The main steps are:

ⓐ Push forward m_Δ to each coset in Γ/Δ

ⓑ Use n to average these.

Claim 1: Given any $C \in \Gamma/\Delta$ and $\beta, \gamma \in C$ $\beta_* m_\Delta = \gamma_* m_\Delta$.

pf (C1): Since $\beta\Delta = C = \gamma\Delta$, we know $\gamma^{-1}\beta \in \Delta$.

For all $A \subseteq C$ we compute:

$$\begin{aligned} \beta_* m_\Delta(A) &= m_\Delta(\beta^{-1} \cdot A) \\ &= m_\Delta((\gamma^{-1}\beta)\beta^{-1} \cdot A) \quad [\Delta\text{-invariance}] \\ &= m_\Delta(\gamma^{-1} \cdot A) \\ &= \gamma_* m_\Delta(A) \quad \text{as desired. ■ (C1).} \end{aligned}$$

We may thus ease notation by writing m_C for the pushforward via any $\gamma \in C$, for $C \in \Gamma/\Delta$.

③ pf (Thm, cont.)

We next check that these measures m_C play nicely with the action $\Gamma \curvearrowright \Gamma/\Delta$.

<u>Claim 2</u>: For all $\gamma \in \Gamma$, $C \in \Gamma/\Delta$, and $A \subseteq C$,
$$m_C(A) = m_{\gamma \cdot C}(\gamma \cdot A).$$

<u>pf (C2)</u>: Suppose $C = \beta\Delta$, so $\gamma \cdot C = \gamma\beta\Delta$. Then

- $m_C(A) = \beta_* m_\Delta(A) = m_\Delta(\beta^{-1} \cdot A)$.
- $m_{\gamma \cdot C}(\gamma \cdot A) = (\gamma\beta)_* m_\Delta(\gamma \cdot A) = m_\Delta(\beta^{-1}\gamma^{-1}\gamma \cdot A) = m_C(A)$. ■(C2)

Now, suppose $A \subseteq \Gamma$. We define a "mass function" $f_A \in [0,1]^{\Gamma/\Delta}$ in analogy with density functions earlier.
$$f_A : C \longmapsto m_C(A \cap C).$$
These functions also play nicely with the action $\Gamma \curvearrowright \Gamma/\Delta$:

<u>Claim 3</u>: For all $\gamma \in \Gamma$ and $A \subseteq \Gamma$, $\gamma \cdot f_A = f_{\gamma \cdot A}$.

<u>pf (C3)</u>: By the definition of the Bernoulli shift
$$\gamma \cdot f_A : \gamma \cdot C \longmapsto m_C(A \cap C)$$
$$= m_{\gamma \cdot C}((\gamma \cdot A) \cap (\gamma \cdot C)). \quad \boxed{C2}$$
Renaming the input variable yields
$$\gamma \cdot f_A : C \longmapsto m_C((\gamma \cdot A) \cap C)$$
$$= f_{\gamma \cdot A}(C) \text{ as desired. } ■(C3)$$

Finally, we define $m : \mathcal{P}(\Gamma) \longrightarrow [0,1]$ by
$$A \longmapsto \int f_A \, d\eta.$$

pf (Thm, cont.)

Claim 4: m is a fapm on Γ.

pf (C4): ▫ $m(\Gamma)=1$: Note that $f_\Gamma : C \mapsto 1$

so $m(\Gamma) = \int \chi_{\Gamma/\Delta}\, dn = 1.$ ✓

▫ $A \cap B = \emptyset \Rightarrow m(A \cup B) = m(A) + m(B)$:

We compute $(A \cup B) = \int f_{A\cup B}\, dn$

$= \int (f_A + f_B)\, dn$

$= m(A) + m(B).$ ✓ ∎(C4)

One last thing to check:

Claim 5: m is Γ-invariant.

pf (C5): Given our preparation, this is mostly symbolic manipulation. For $\gamma \in \Gamma$ and $A \subseteq \Gamma$ we compute

$$m(\gamma \cdot A) = \int f_{\gamma \cdot A}\, dn$$

$$= \int \gamma \cdot f_A\, dn \quad \boxed{C3}$$

$$= \int f_A\, dn \quad \boxed{\text{Prop}}$$

$$= m(A). \quad ∎(C5)$$

So the measure m witnesses amenability of Γ after all, proving the theorem! ∎(Thm)

Paradoxes

Friday, Mar 1

Amenability of isometry groups

Last time, we showed:

Thm: Suppose that Γ is a group with subgroup $\Delta \leq \Gamma$ s.t.

- Δ is amenable: there is a Δ-inv fapm on Δ
- Δ is co-amenable: there is a Γ-inv fapm on Γ/Δ.

Then Γ is amenable.

Cor: If $\Delta \trianglelefteq \Gamma$ is a normal subgroup with both Δ and Γ/Δ amenable groups, then Γ is amenable.

pf: Any Γ/Δ-inv fapm on Γ/Δ witnesses the co-amenability of Δ, since $\gamma \cdot (\beta\Delta) = (\gamma\Delta)(\beta\Delta)$. ■ (Cor)

Let's use this to investigate some isometry groups.

Prop: $\mathrm{Isom}(\mathbb{R})$ is an amenable group.

pf: We know every isometry of $\mathbb{R}$ has the form

$$r \longmapsto ar+b, \quad \begin{matrix} a \in \{-1,1\} \\ b \in \mathbb{R}. \end{matrix}$$

Consider the hom

$$\varphi : \mathrm{Isom}(\mathbb{R}) \longrightarrow (\{\pm 1\}; \times)$$
$$(r \mapsto ar+b) \longmapsto a.$$

$\Delta = \ker(\varphi) \cong \mathbb{R}$ is abelian, hence amenable.
$\mathrm{Isom}(\mathbb{R})/\Delta \cong (\{\pm 1\}; \times)$ is also abelian, hence amenable.

So $\mathrm{Isom}(\mathbb{R})$ is amenable by the above corollary. ■ (Prop)

Recall: The circle is $C = \{x \in \mathbb{R}^2 : d(x,0) = 1\}$.

There is a natural surjection

$$\mathbb{R} \longrightarrow C$$
$$r \longmapsto \begin{pmatrix} \cos 2\pi r \\ \sin 2\pi r \end{pmatrix}$$

This surjection does NOT preserve distance but coincidentally it induces a quotient of isometry groups.

Recall: $\mathrm{Isom}(C) = \{x \mapsto Ax : A \in M_{2\times 2}(\mathbb{R}),\ A^T A = I\}$

Given an isometry $r \mapsto ar + b$ of $\mathbb{R}$, the induced isometry of C is

$$x \longmapsto \begin{pmatrix} \cos 2\pi b & -a\sin 2\pi b \\ \sin 2\pi b & a\cos 2\pi b \end{pmatrix} x .$$

A tedious calculation reveals that this is indeed a surjective group hom $\mathrm{Isom}(\mathbb{R}) \to \mathrm{Isom}(C)$.

Prop: $\mathrm{Isom}(C)$ is an amenable group.

pf: Quotients of amenable groups are amenable. ■(Prop)

Remark: You can also prove this "directly" by building a hom

$$\mathrm{Isom}(C) \longrightarrow (\{\pm 1\}; \times)$$
$$(x \mapsto Ax) \longmapsto \det(A),$$

and proceeding as in the previous Proposition. These two arguments are identical.

(3) Let's continue bootstrapping up in dimension.

<u>Prop:</u> $\mathrm{Isom}(\mathbb{R}^2)$ is an amenable group.

<u>pf:</u> Every isometry of $\mathbb{R}^2$ has the form

$$x \mapsto Ax + b \qquad \begin{array}{l} A \in M_{2\times 2}(\mathbb{R}) \text{ with } A^T A = I \\ b \in \mathbb{R}^2. \end{array}$$

We build a hom $(x \mapsto Ax+b) \mapsto A$ and declare $\Delta \trianglelefteq \mathrm{Isom}(\mathbb{R}^2)$ to be its kernel.

Then $\Delta \cong \mathbb{R}^2$ and $\mathrm{Isom}(\mathbb{R}^2)/\Delta \cong \mathrm{Isom}(C)$.

This implies amenability of $\mathrm{Isom}(\mathbb{R}^2)$. ■(Prop)

We immediately glean various non-paradoxicality results:

<u>Thm:</u> The following metric spaces are NOT paradoxical via isometries: $\mathbb{R}$, C, $\mathbb{R}^2$.

<u>pf:</u> This is the whole point of amenability. $1 \neq 1+1$. ■(Thm)

<u>Remark:</u> There is no hope of bootstrapping this further, as $S = \{x \in \mathbb{R}^3 : d(x,0) = 1\}$ and $\mathbb{R}^d$ $(d \geq 3)$ are paradoxical via isometries.

It is more subtle to check whether <u>subsets</u> of such spaces are paradoxical via isometries. The issue is, given an amenable action $\Gamma \curvearrowright X$ with invariant fapm m on X, a given subset $A \subseteq X$ may have $m(A) = 0$. The measure m fails to preclude paradoxicality of A, due to the unfortunate arithmetic: $0 = 0 + 0$.

Sometimes things work out.

Thm: The half-open interval $[0,1) \subseteq \mathbb{R}$ is NOT paradoxical under the action $\mathrm{Isom}(\mathbb{R}) \curvearrowright \mathbb{R}$.

pf: Any such paradoxical decomposition could be converted into a paradoxical decomposition of the circle by pushing forward through the surjection from before. Such a decomposition doesn't exist. ∎(Thm)

The following example shows that this is not an "abstract consequence" of the amenability of $\mathrm{Isom}(\mathbb{R})$.

Def: The affine group of $\mathbb{R}$, $\mathrm{Aff}(\mathbb{R})$, consists of maps of the form $r \mapsto ar+b$ $\quad a \in \mathbb{R}\setminus\{0\}$, $b \in \mathbb{R}$.

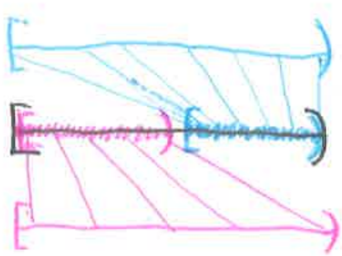
The operation is composition (of course).

Prop: $\mathrm{Aff}(\mathbb{R})$ is an amenable group.

pf: Examine $(r \mapsto ar+b) \longmapsto a$ as before. ∎(Prop)

Prop: $[0,1)$ IS paradoxical under $\mathrm{Aff}(\mathbb{R}) \curvearrowright \mathbb{R}$.

pf:



$r \mapsto 2r-1$

$r \mapsto 2r$

∎(Prop)

Sneak peek: After break we will:

- Dispel all paradoxes in $\mathbb{R}$
- Analyze $\mathbb{R}^2$ more thoroughly
- Consider more exotic geometries (?)

Paradoxes

Monday, March 11

Dispelling paradoxes in $\mathbb{R}$

Recall: (a) The isometry group $\mathrm{Isom}(\mathbb{R})$ is amenable, hence $\mathbb{R}$ itself is not paradoxical via isometries.

(b) The interval $[0,1] \subseteq \mathbb{R}$ is not paradoxical via the action of $\mathrm{Isom}(\mathbb{R})$. This was an ad hoc argument using the circle.

Q: Is ANY non-$\emptyset$ subset of $\mathbb{R}$ paradoxical?

Remark: Amenability of $\mathrm{Isom}(\mathbb{R})$ does not immediately resolve this. We know for amenable groups Γ that all actions $\Gamma \curvearrowright X$ are amenable (hence non-paradoxical). The issue is that actions typically don't "restrict" down to subsets. We abstractify a bit.

Def: A monoid is a structure $(M; \circ)$ such that

- The binary operation $\circ$ on M is associative
- There is a two-sided identity element:

$$\exists e \in M \ \forall m \in M \quad \begin{array}{l} e \circ m = m \\ m \circ e = m. \end{array}$$

Prop: In any monoid, the identity element is unique.

pf: Given identities e, f, we compute

$$e = e \circ f = f. \qquad \blacksquare \text{(Prop)}$$

② Examples:

ⓐ If X is any set, then $X^X = \{f : X \to X\}$ is a monoid under composition. The identity element is $id : x \mapsto x$.

ⓑ The subset of injective functions is also a monoid.

ⓒ The partial functions $\{f : X \rightharpoonup X\}$ form a monoid with the operation of composition on largest possible domain. The partial injections form a submonoid.

ⓓ If $\Gamma \curvearrowright X$, the (partial) embeddicomps form a monoid.

ⓔ Given a set S, the free monoid on S has base set $M_S = S^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N}} S^n$ of finite sequences from S.
The operation is concatenation.

Def: Given a monoid M and $B \subseteq M$, the monoid generated by B is the image of the monoid hom $M_B \longrightarrow M$

$$\begin{aligned} b &\longmapsto b \\ \emptyset &\longmapsto e. \end{aligned}$$

Monoid Ping Pong Lemma: Suppose that X is a non-empty set, and that $f, g \in X^X$ are injections such that $f[X] \cap g[X] = \emptyset$. Then the monoid generated by $\{f, g\}$ in X^X is isomorphic to M_2.

③ pf (MPPL): We need to show for any two words $v \neq w$ in $\{f, g\}$, the corresponding compositions are distinct functions. I.e., $\exists x \in X$ $v(x) \neq w(x)$. WLOG v is non-empty with first letter f. We shall proceed by induction on the length of v.

Case 0: $w = \emptyset$. Then for any $x \in g[X]$ we see:

$$v(x) = f \circ v'(x) \in f[X]$$
$$w(x) = x \in g[X]$$

So $v(x) \neq w(x)$. ✓

Case 1: w starts with g. Then for any $x \in X$

$$v(x) \in f[X]$$
$$w(x) \in g[X]$$

So $v(x) \neq w(x)$. ✓

Case 2: w starts with f. Write $v = fv'$
$w = fw'$

By induction, $\exists x \in X$ with $v'(x) \neq w'(x)$

Since f is an injection, $v(x) \neq w(x)$. ✓

■ (MPPL).

Cor: Suppose that $\Gamma \curvearrowright X$ is a group action and that $Y \subseteq X$ is a non-$\emptyset$ paradoxical set. Then Γ contains a fin gen subgroup of exponential growth.

pf (sketch): Paradoxicality yields embeddicomps $f, g \in Y^Y$ with disjoint images. MPPL yields an isomorphism with M_2, then the subgroup generated by relevant group elements has exponential growth as in $\boxed{\text{HW}}$. ■ (Cor, sketch)

<u>Thm</u>: No non-$\emptyset$ subset of $\mathbb{R}$ is paradoxical via isoms.

<u>pf</u>: It suffices to show that every finitely generated subgroup of $\mathrm{Isom}(\mathbb{R})$ has sub-exponential growth.

Consider finite $S = \{x \mapsto \pm x + b_i\}$ and put $S_+ = \{x \mapsto x + b_i\}$.

Then $\langle S_+ \rangle$ is abelian, thus $\boxed{\text{HW}}$ has subexponential growth. Each ball in $\langle S \rangle$ is at most twice as big as the corresponding ball in $\langle S_+ \rangle$, so $\langle S \rangle$ has subexponential growth as well. ∎ (Thm)

<u>Def</u>: A group Γ is <u>supramenable</u> if no action $\Gamma \curvearrowright X$ admits a non-empty paradoxical set.

<u>Equiv</u>

The left-mult action $\Gamma \curvearrowright \Gamma$ admits no non-$\emptyset$ paradoxical set

<u>Equiv</u>

For any non-$\emptyset$ $A \subseteq \Gamma$, there is a left-inv fin additive measure on Γ with $m(A) = 1$.

<u>Open questions</u>:

$\boxed{A}$ Is supramenability equivalent to every fin gen subgroup having subexponential growth?

$\boxed{B}$ If Γ, Δ are supramenable, is $\Gamma \times \Delta$?

Paradoxes

Wednesday, Mar 13

Paradoxicality in $\mathbb{R}^2$

Recall: $\mathrm{Isom}(\mathbb{R}^2) = \{x \mapsto Ax + v\}$, where
- A is a (2×2)-matrix with $A^T A = I$
- $v \in \mathbb{R}^2$.

There is a "natural" identification of $\mathbb{R}^2$ with $\mathbb{C}$ via $\binom{a}{b} \mapsto a+bi$.

Now if $z = c+di \in \mathbb{C}$, we observe that

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ac-bd \\ ad+bc \end{pmatrix} \mapsto z(a+bi)$$

So "multiplication by $z = c+di$" in $\mathbb{C}$ corresponds to "multiplication by $A_z = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$" in $\mathbb{R}^2$.

Finally, $A_z^T A_z = \begin{pmatrix} c^2+d^2 & 0 \\ 0 & c^2+d^2 \end{pmatrix}$, which equals I whenever $|z|=1$. In summary, we have:

Prop: Whenever $z, w \in \mathbb{C}$ with $|z|=1$, the map $x \mapsto zx+w$ is an isometry of $\mathbb{C}$, or equivalently of $\mathbb{R}^2$.

pf: See above discussion. $\blacksquare$(Prop)

<u>Thm</u> (Mazurkiewicz, Sierpiński):

There is a subset $\Upsilon \subseteq \mathbb{R}^2$ that is paradoxical via isometries.

<u>pf</u>: Fix any $z \in \mathbb{C}$ satisfying:

- $|z| = 1$
- z is not a root of any nonzero polynomial in $\mathbb{Z}[t]$, polynomials with integer coefficients.

<u>Claim 1</u>: The map $\begin{aligned} \mathbb{N}[t] &\longrightarrow \mathbb{C} \\ f &\longmapsto f(z) \end{aligned}$ is injective.

<u>pf (C1)</u>: For distinct $f, g \in \mathbb{N}[t]$, $f(z) - g(z) \neq 0$. $\blacksquare$(C1)

We shall put $\Upsilon = \{f(z) : f \in \mathbb{N}[t]\} \subseteq \mathbb{C}$.

Define $A = \{f(z) : f \in \mathbb{N}[t]$ has zero const term$\}$

$B = \{f(z) : f \in \mathbb{N}[t]$ has nonzero const term$\}$.

So $\Upsilon = A \sqcup B$.

<u>Claim 2</u>: $\Upsilon \approx A$.

<u>pf (C2)</u>: $x \longmapsto zx$ $\blacksquare$(C2)

<u>Claim 3</u>: $\Upsilon \approx B$.

<u>pf (C3)</u>: $x \longmapsto x + 1$ $\blacksquare$(C3)

So Υ is indeed paradoxical. $\blacksquare$(Thm).

<u>Remark</u>: This example is kind of silly, as Υ is countable, has empty interior, is unbdd (in fact is dense in $\mathbb{R}^2$), etc. etc.

(3) **Next goal:** No bounded subset of $\mathbb{R}^2$ with non-empty interior is paradoxical via isometries.

We will need some analytic tools...

Def: Given a set X, put $\ell^\infty(X) = \{f : X \to \mathbb{R} \text{ bounded}\}$.
$\ell^\infty(X)$ carries two important structures:

- It is a real vector space.
- There is a partial order $f \le g$ iff $\forall x \in X \ f(x) \le g(x)$.

Def: $\mathbb{1} \in \ell^\infty(X)$ is the function $\mathbb{1} : x \mapsto 1$.

Recall: $\boxed{\text{HW}}$ if $\Phi : \ell^\infty(X) \to \mathbb{R}$ is a linear function satisfying

- $\Phi(\mathbb{1}) = 1$
- $f \geq 0 \Rightarrow \Phi(f) \geq 0$

then $\exists$ fapm m on X with $\Phi(f) = \int f \, dm$.

This lets us use *linear algebra* to build measures!

Def: A partial positive linear functional (pplf) is a function $\varphi : V \to \mathbb{R}$ where

- $V \le \ell^\infty(X)$ is a subspace containing $\mathbb{1}$
- $\varphi(\mathbb{1}) = 1$
- $f \geq 0 \Rightarrow \varphi(f) \geq 0$.

Such a pplf is total if $V = \ell^\infty(X)$.
As discussed above, a total pplf is essentially a fapm on X.

④ <u>Thm</u> (Riesz) [AC]: Any pplf may be extended to a total pplf.

<u>pf</u>: By Zorn, we may extend to a maximal pplf, so it suffices to argue that any max'l pplf is total. Suppose $\varphi: V \to \mathbb{R}$ is max'l, and towards a cont. that $V \neq \ell^\infty(X)$. Fix $h \in [0,1]^X$ with $h \notin V$. We will argue that φ can be extended to $W = \operatorname{span}(V \cup \{h\})$.

Towards that end, put $\Psi(h) = \sup\{\varphi(f) : f \leq h\}$ and extend to W by $\Psi(g + ah) = \varphi(g) + a\Psi(h)$. Clearly, Ψ is linear, so we check:

<u>Claim</u>: For $f \in W$, if $f \geq 0$ then $\Psi(f) \geq 0$.

<u>pf(C)</u>: Write $f = g + ah$ with $g \in V$, $a \in \mathbb{R}$.

<u>Case 0</u>: $a = 0$. Then $\Psi(f) = \varphi(f) \geq 0$. ✓

<u>Case 1</u>: $a > 0$. Since $f \geq 0$ we know $ah \geq -g$. Thus, by def of $\Psi(h)$, $\Psi(ah) \geq \varphi(-g)$. Hence, $\Psi(f) = \Psi(ah) - \varphi(-g) \geq 0$. ✓

<u>Case 2</u>: $a < 0$. Rewrite as $f = g - bh$ with $b > 0$. For each $n > 0$, choose $g_n \in V$ with

$$\varphi(g_n) \geq \Psi(bh) - \frac{1}{n}, \text{ and } g_n \leq bh \leq g.$$

We compute for all $n > 0$

$$\begin{aligned} 0 \leq \varphi(g - g_n) &= \varphi(g) - \varphi(g_n) \\ &\leq \varphi(g) - \Psi(bh) + \frac{1}{n} \\ &= \Psi(f) + \frac{1}{n} \end{aligned}$$

And thus $\Psi(f) \geq 0$. ✓ ■(C) ■(Thm)

Paradoxes

Friday, Mar 15

Today's goal:

Thm (Banach? Riesz?) [AC]: There is a finitely additive measure $m : \mathcal{P}_b(\mathbb{R}^2) \longrightarrow [0, \infty)$ such that:

- For every square $\square \subseteq \mathbb{R}^2$, $m(\square) = \text{area}(\square)$
- For every $A \in \mathcal{P}_b(\mathbb{R}^2)$ and isometry $\gamma \in \text{Isom}(\mathbb{R}^2)$, $m(\gamma \cdot A) = m(A)$.

Def: $\mathcal{P}_b(\mathbb{R}^2)$ is the set of bounded subsets of $\mathbb{R}^2$.

Def: A fam is $m : \mathcal{P}_b(\mathbb{R}^2) \longrightarrow [0, \infty)$ s.t. for $A, B \in \mathcal{P}_b(\mathbb{R}^2)$ disjoint, $m(A \cup B) = m(A) + m(B)$.

Remark: The theorem precludes paradoxicality of the unit square (or any other square) via isometries.

pf (outline): We will proceed in four steps:

Step A: Define a useful partial fapm on unit square
Step B: Extend to a fapm on the unit square
Step C: Extend to a fam on $\mathbb{R}^2$
Step D: Ensure invariance under isometries. $\blacksquare$ (Thm)

Def: (a) Given a set X, we say that nonempty $\mathcal{A} \subseteq \mathcal{P}(X)$ is a subalgebra if it is stable under finite unions and complements (hence also intersections).

(b) A partial fapm on X is a function $m : \mathcal{A} \longrightarrow [0,1]$ satisfying the usual fapm axioms, for some subalg $\mathcal{A}$.

(2) **Step A:** For $\mathbb{X} = [0,1) \times [0,1) \subseteq \mathbb{R}^2$, there is a partial fapm m_A on $\mathbb{X}$ such that:

- m_A : point or line segment $\longmapsto 0$
- m_A : polygon without boundary $\longmapsto$ its area.

pf: Sketch 1: Let $\mathcal{A}$ consist of sets that are finite disjoint unions of pts/lines/polygons, and let m_A be as defined. This involves tedious verification that $\mathcal{A}$ is an algebra and that m_A is additive...

Sketch 2: Let $\mathcal{A}$ be the Borel sets, and let m_A be Lebesgue measure. ■ (A)

Step B: There is a fapm m_B on $\mathbb{X}$ extending m_A.

pf: Consider $\ell^\infty(\mathbb{X})$, and define a subspace

$$V = \operatorname{span}\{\chi_A : A \in \mathcal{A}\} \subseteq \ell^\infty(\mathbb{X}).$$

So elements of V are "simple $\mathcal{A}$-measurable functions."

We define a pplf $\varphi : V \longrightarrow \mathbb{R}$ via

$$\textstyle\sum_i a_i \chi_{A_i} \longmapsto \sum_i a_i\, m(A_i)$$

Note that $\varphi(\mathbb{1}) = 1$. By Riesz' theorem from last time, this extends to a total pplf

$$\Phi : \ell^\infty(\mathbb{X}) \longrightarrow \mathbb{R}.$$

By $\boxed{\text{HW}}$, there is a fapm m_B on $\mathbb{X}$ with

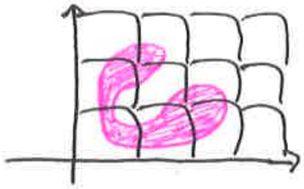
$$\Phi : f \longmapsto \int f \, dm_B.$$

Since Φ extends φ, we know m_B extends m_A.

■ (B)

<u>Step C</u>: There is a fam $m_C : \mathcal{P}_b(\mathbb{R}^2) \longrightarrow [0,\infty)$ so that for every square $\square \subseteq \mathbb{R}^2$, $m_C(\square) = \text{area}(\square)$.

<u>pf</u>: We tile $\mathbb{R}^2$ by translates of $\mathbb{X} = [0,1) \times [0,1)$.



Formally, $\mathbb{R}^2 = \mathbb{Z}^2 \cdot \mathbb{X}$.

Any bounded $A \subseteq \mathbb{R}^2$ intersects only finitely many tiles. So we may "put a copy of m_B on each tile and add them up."

Formally, $m_C : A \longmapsto \sum_{(i,j) \in \mathbb{Z}^2} m_B\left((-i,-j)\cdot A \cap \mathbb{X}\right)$. $\blacksquare$(C)

We are almost done... we just need to ensure that our measure is invariant under isometries.

<u>Recall</u>: $\Gamma = \mathrm{Isom}(\mathbb{R}^2)$ is an amenable group.

So we may fix a Γ-invariant fapm η on Γ.

<u>Step D</u>: There is a fam $m : \mathcal{P}_b(\mathbb{R}^2) \longrightarrow [0,\infty)$ s.t.

ⓐ $m(\square) = \text{area}(\square)$.

ⓑ For $\gamma \in \Gamma$ and $A \in \mathcal{P}_b(\mathbb{R}^2)$, $m(\gamma \cdot A) = m(A)$.

<u>pf</u>: We already have a fam m_C satisfying ⓐ.

To get ⓑ, we average!

pf (Step D, cont.)

Given a bounded set $A \subseteq \mathbb{R}^2$, we may find a big square $\square$ with $A \subseteq \square$. Say $\text{area}(\square) = K$. Then for all $\gamma \in \Gamma$ we have

$$m_c(\gamma \cdot A) \leq m_c(\gamma \cdot \square) = K.$$

So for such A we obtain a function

$$f_A : \Gamma \longrightarrow [0, K]$$
$$\gamma \longmapsto m_c(\gamma \cdot A).$$

We declare $m : A \longmapsto \int f_A \, dn$.

Claim 1: m is a Γ-invariant fam on $\mathbb{R}^2$.

pf (C1): Checking additivity is routine. For Γ-invariance, we compute

$$m(\gamma \cdot A) = \int f_{\gamma \cdot A} \, dn = \int \gamma \cdot f_A \, dn = \int f_A \, dn = m(A). \quad \blacksquare(\text{C1})$$

Claim 2: For all squares $\square \subseteq \mathbb{R}^2$, $m(\square) = \text{area}(\square)$.

pf (C2): Put $\text{area}(\square) = a$. Then

$$f_\square : \gamma \longmapsto m_c(\gamma \cdot \square) = a, \text{ a constant function.}$$

Thus, $m(\square) = \int f_\square \, dn = a$. $\blacksquare$(C2) $\blacksquare$(D)

So the outline works! Replace $\square$ with $\blacksquare$

Cor: No bounded subset of $\mathbb{R}^2$ with nonempty interior is paradoxical via isometries.

pf: Any such set contains a tiny square, thus has positive m-measure. $\blacksquare$(Cor)

Paradoxes

Monday, Mar 18

The von Neumann Paradox, pt I.

Recap: We have been studying paradoxicality in the Euclidean plane $\mathbb{R}^2$ via isometries.

ⓐ There is a non-empty paradoxical set. For example, $\{f(z) : f \in \mathbb{N}[t]\}$ with $z \in \mathbb{C}$ a transcendental complex number of modulus 1.

ⓑ The unit square $[0,1)\times[0,1)$ is NOT paradoxical via isometries. We constructed an isometry-invariant fam on $\mathbb{R}^2$ which gave the unit square measure 1.

Question: How does this analysis change if we act by other reasonable "size-preserving" groups?

Def: The affine group of the plane is the set

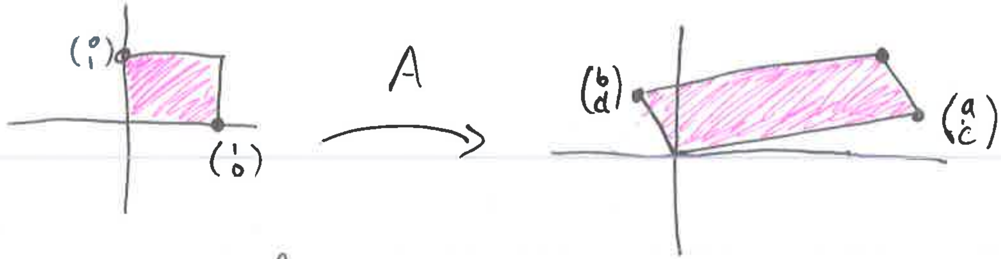
$$\mathrm{Aff}(\mathbb{R}^2) := \{x \longmapsto Ax + b : A \text{ an invertible } (2\times 2)\text{-mat}, \; b \in \mathbb{R}^2\}$$

equipped with the operation of composition.

Remark: Like in one dimension, many affine maps do NOT preserve size.

2

Example: Consider the map $x \longmapsto Ax$ with $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. So $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} c \\ d \end{pmatrix}$.



The image of the unit square is a parallelogram of area $|\det(A)|$. So for A to "preserve size," it should have $\det(A) = \pm 1$.

Def: The group of area-preserving affine maps of the plane is the subgroup $Apam(\mathbb{R}^2) \leq Aff(\mathbb{R}^2)$:

$$Apam(\mathbb{R}^2) = \{x \mapsto Ax + b : \det(A) = \pm 1\}.$$

Note: $Isom(\mathbb{R}^2) \leq Apam(\mathbb{R}^2)$. But $Apam$ contains non-isometries as well, for example

$$x \longmapsto \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$$

Thm (von Neumann): The unit square in $\mathbb{R}^2$ is paradoxical via area preserving affine maps.

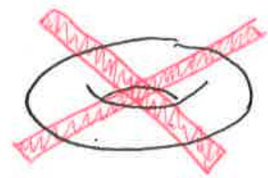
Remark: In one dimension, $Isom(\mathbb{R}) = Apam(\mathbb{R})$, so von Neumann's paradox is "new" at dimension two.

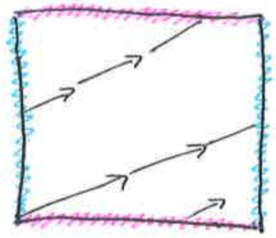
3

Recall: When analyzing equidecomposability for the interval $[0,1) \subseteq \mathbb{R}$, it was convenient to shift context to the circle instead.

Remark: The corresponding "homogeneous object" for the unit square $[0,1)\times[0,1) \subseteq \mathbb{R}^2$ is the torus:

Def: The **torus** is $T = \mathbb{R}^2/\mathbb{Z}^2$.



$T =$ 

Problem: $\mathrm{Apam}(\mathbb{R}^2)$ doesn't act on T in a reasonable fashion.

Solution: Restrict attention to matrices with integer entries.

Prop: If A is a (2×2)-matrix with integer entries and $\det(A) = \pm 1$, then $A\cdot\mathbb{Z}^2 = \mathbb{Z}^2$.

pf: Certainly, $A\cdot\mathbb{Z}^2 \subseteq \mathbb{Z}^2$. On the other hand, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is another matrix of the same type, so $A^{-1}\cdot\mathbb{Z}^2 \subseteq \mathbb{Z}^2$. I.e., $\mathbb{Z}^2 \subseteq A\cdot\mathbb{Z}^2$. ■ (Prop)

④ **Def:** If $\Gamma \leq \mathrm{Apam}(\mathbb{R}^2)$ is the subgroup

$$\Gamma = \{ x \mapsto Ax : A \text{ has integer entries} \}$$

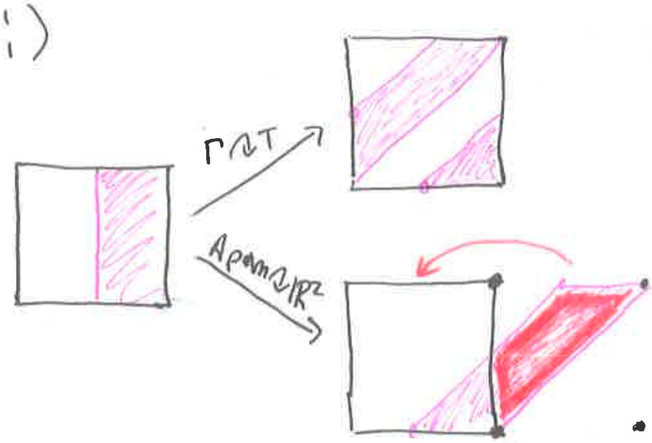
then $\Gamma \curvearrowright T$ via $\gamma \cdot (x + \mathbb{Z}^2) = \gamma(x) + \mathbb{Z}^2$.

Prop: Suppose that $B, C \subseteq T$ are equidecomp via the action $\Gamma \curvearrowright T$. Viewing them as subsets of the unit square in the obvious way, we have $B \approx C$ via $\mathrm{Apam}(\mathbb{R}^2)$.

pf: Working piece by piece, it suffices to show for $\gamma \in \Gamma$ that $\gamma \cdot B = C$ in T implies $B \approx C$ via $\mathrm{Apam}(\mathbb{R}^2)$. Suppose $\gamma = x \mapsto Ax$. Then $A \cdot B$ meets finitely many "tiles" of the unit square, and each may be moved back to the square by an element of $\mathrm{Apam}(\mathbb{R}^2)$ of the form $x \mapsto Ax + b$ with $b \in \mathbb{Z}^2$.

■ (Prop)

Ex: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$



Cor: To establish the von Neumann paradox, it suffices to establish paradoxicality of $\Gamma \curvearrowright T$. We won't do this (because it is false), but we will do so "off a small set."

①

Paradoxes

Wednesday, Mar 20

The von Neumann Paradox, pt II

Last time: We considered the group

$$\Gamma = \{x \mapsto Ax : A \text{ a } (2\times2)\text{-matrix over } \mathbb{Z} \text{ w/ } \det(A) = \pm 1\} \leq \mathrm{Apam}(\mathbb{R}^2).$$

We have an action $\Gamma \curvearrowright T = \mathbb{R}^2/\mathbb{Z}^2$ via

$$\gamma \cdot (x + \mathbb{Z}^2) = \gamma(x) + \mathbb{Z}^2.$$

And we "reduced" the question of paradoxicality of $\square \subseteq \mathbb{R}^2$ via $\mathrm{Apam}(\mathbb{R}^2)$ to the question of paradoxicality of T via Γ.

Sadly, T isn't paradoxical. But it almost is...

Recall: $\boxed{\text{HW}}$ If $\alpha : x \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} x$, $\beta : x \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} x$, then $\langle \alpha, \beta \rangle \cong \mathbb{F}_2$.

The issue, like with Hausdorff's paradox, is that the resulting action $\mathbb{F}_2 \curvearrowright T$ is not free.

Ex:

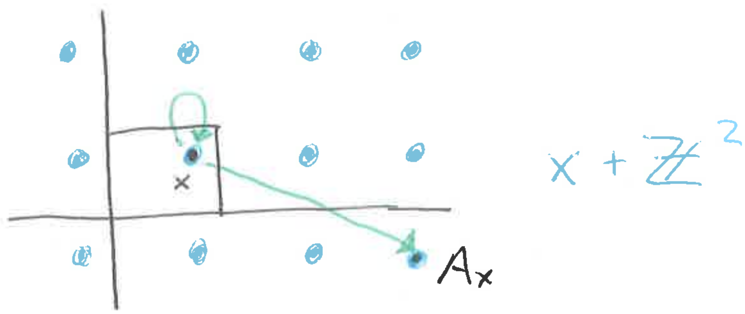
α : 

So $\mathrm{Fix}(\alpha) =$ 

②

Prop: Working with this action $\Gamma \curvearrowright T$, for all $\gamma \in \Gamma \setminus \{e\}$, the set $\mathrm{Fix}(\gamma) = \{t \in T : \gamma \cdot t = t\}$ can be covered by countably many lines.

pf: Suppose $\gamma : x \longmapsto Ax$ for some $A \neq I$. Consider $t = x + \mathbb{Z}^2 \in \mathrm{Fix}(\gamma)$. Then $\gamma \cdot t = t$ implies that $Ax \in x + \mathbb{Z}^2$.



Claim: For each $z \in \mathbb{Z}^2$, the set
$$F_z = \{x \in \mathbb{R}^2 : Ax = x + z\}$$
can be covered by a line.

pf(C): We see that $x \in F_z$ iff $(A - I)x = z$. $A - I$ is a nonzero (2×2)-matrix, hence its kernel is at most one-dimensional. Since F_z is either empty or a translation of $\ker(A-I)$, we're done! ■(C)

This analysis shows that $\mathrm{Fix}(\gamma) = \bigcup \{F_z : z \in \mathbb{Z}^2\}$ can be covered by countably many lines. ■(Prop)

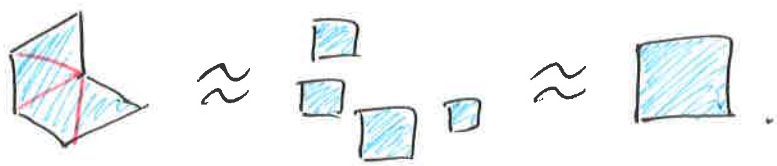
④ We will show something better next time:

<u>Thm</u>: Suppose that $C \subseteq \square$ can be covered by countably many lines. Then $\square \approx \square \setminus C$ via <u>translations</u>.

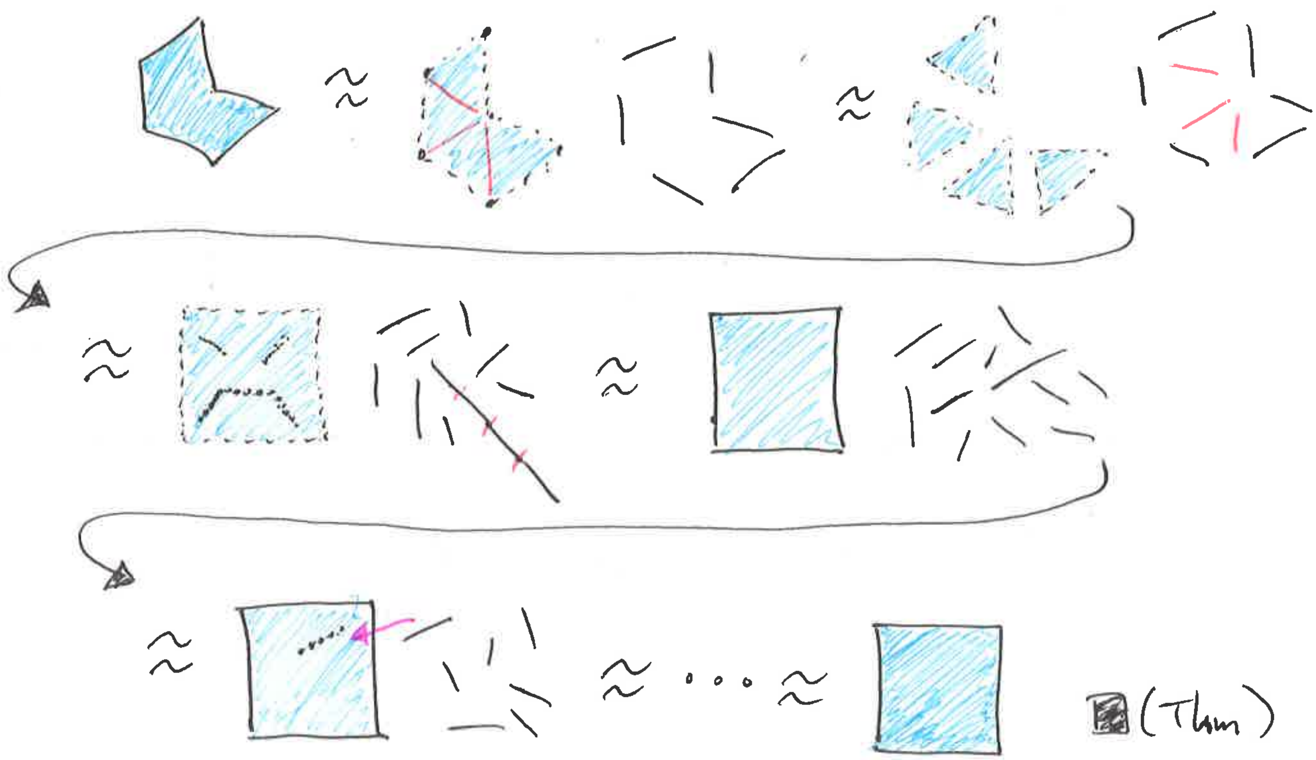
<u>Remark</u>: In retrospect, this justifies our cavalier attitude about boundaries of polygons:
★ nonchalance ★

<u>Thm</u>: Any polygon is equidecomposable to a square of the same area via $\mathrm{Isom}(\mathbb{R}^2)$.

<u>pf</u>: We showed this "modulo boundaries" on $\boxed{\text{HW}}$:



The above thm helps us deal with boundaries:



∎ (Thm)

③

Notation: Let T^* denote the free part of the action $\Gamma \curvearrowright T$. So $T^* = \{t \in T : \forall \gamma \in \Gamma \setminus \{e\} \ \gamma \cdot t \neq t\}$.

Prop: $T \setminus T^*$ can be covered by countably many lines.

pf: We know for all $\gamma \in \Gamma \setminus \{e\}$ that $\mathrm{Fix}(\gamma)$ can be covered by countably many lines. Now write $T \setminus T^* = \bigcup \{\mathrm{Fix}(\gamma) : \gamma \in \Gamma \setminus \{e\}\}$. ■ (Prop)

Remark: This means that T^* is "large" in various senses. For example, there is a natural version of Lebesgue measure on T, which gives every line measure 0 and thus T^* full measure.

Prop: T^* is paradoxical via the Γ-action.

pf: We know that the subgroup $\langle \alpha, \beta \rangle \cong \mathbb{F}_2$ acts freely on T^*, and free actions of $\mathbb{F}_2$ are paradoxical. ■ (Prop).

So we almost have the von Neumann paradox. The missing ingredient is something like $T \approx T^*$. Then we can mimic the trick from the Hausdorff paradox and get

$$T \approx T^* \approx \begin{matrix} T^* \approx T \\ T^* \approx T \end{matrix}.$$

①

Paradoxes

Friday, Mar 22

Ignoring sets of "small dimension"

Recall from week 1:

Thm: If $C \subseteq [0,1)$ is countable, then $[0,1) \approx [0,1) \setminus C$ via isometries (in fact, translations) of $\mathbb{R}$.

pf (sketch): By a counting argument, we find a "rotation" $\sigma_r : [0,1) \longrightarrow [0,1)$

$$x \longmapsto \begin{cases} x+r & \text{if } x < 1-r \\ x+r-1 & \text{if } x \geq 1-r \end{cases}$$

s.t. for all $m \neq n \in \mathbb{Z}$ $\sigma^m[C] \cap \sigma^n[C] = \emptyset$.
I.e., each orbit of σ contains at most one element of C. Each such orbit looks like:

(Sketch)

Today's goal is to bump this up a dimension:

Thm: Suppose that $C \subseteq \square$ can be covered by countably many lines, i.e., $\exists \mathcal{L} = \{ l_i : i \in \mathbb{N} \}$ where each $l_i \subseteq \mathbb{R}^2$ is a line and $C \subseteq \bigcup \mathcal{L}$. Then $\square \approx \square \setminus C$ via translations.

But first we prove a warm-up lemma that is really a "parallel" version of the week 1 result:

② **Lemma:** If $C \subseteq \square$ is countable, then $\square \approx \square \setminus C$.

pf($\mathcal{L}$, sketch): Recall $\square = [0,1) \times [0,1)$ which we may regard as a stack of intervals.

We may find a single rotation

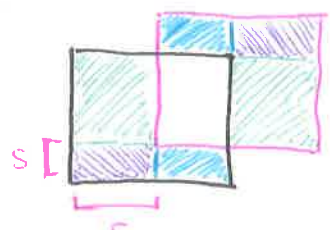
$$\sigma : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x+r \\ y \end{pmatrix} \pmod 1$$

so that each orbit of σ contains at most one element of C, using the same counting argument. $\blacksquare$($\mathcal{L}$, sketch)

pf(Thm): Recall $\mathcal{L} = \{\ell_i : i \in \mathbb{N}\}$, $C \subseteq \bigcup \mathcal{L}$.

We want to show $\square \approx \square \setminus C$. We will analyze "rotations" of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x+r \\ y+s \end{pmatrix} \pmod 1$$



Such rotations can be implemented by equidecompositions built from translations.

Claim 1: There is a rotation $\sigma : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+r \\ y+s \end{pmatrix} \pmod 1$ such that for all $m \neq n \in \mathbb{Z}$ the set $\sigma^m[C] \cap \sigma^n[C]$ is countable.

pf(C1): We first declare a "slope," finding $\begin{pmatrix} r_0 \\ s_0 \end{pmatrix}$ so that the segment

is not parallel to any $\ell_i \in \mathcal{L}$. This is possible, as there are only countably many slopes to avoid.

3 <u>pf</u> (Thm, cont.)

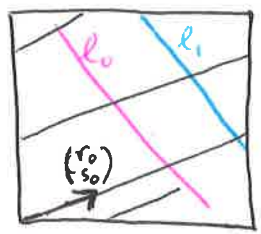
<u>pf</u> (C1, cont.)

We will search among rotations $\sigma_\alpha : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \alpha r_0 \\ y + \alpha s_0 \end{pmatrix}$ (mod 1)

Let's declare such a rotation BAD if $\exists m \neq n \in \mathbb{Z}$ with $\sigma^m[C] \cap \sigma^n[C]$ uncountable.

Note that when ℓ_0 and ℓ_1 are lines of different slope, $|\sigma^m[\ell_0] \cap \sigma^n[\ell_1]| \leq 1$.

Thus, if σ is a BAD rotation, there are parallel $\ell_0, \ell_1 \in \mathcal{L}$ with $\sigma^m[\ell_0] \cap \sigma^n[\ell_1] \neq \emptyset$.



For each such pair, there are countably many BAD rotations. Thus, there are countably many BAD rotations altogether, and at least one GOOD one. ∎(C1)

So fix a rotation σ as in the claim. Put

$$D = \bigcup_{m \neq n} \sigma^m[C] \cap \sigma^n[C] \subseteq \square,$$ so D is ctbl.

By the Lemma, we know $\square \approx \square \setminus D$ via translations. To conclude the proof of the theorem, it suffices to show that

$\square \setminus D \precsim \square \setminus C$ via translations.

Note that $\square \setminus D$ consists exactly of those points t with $|\{n \in \mathbb{Z} : \sigma^n(t) \in C\}| \leq 1$.

In particular, $\square \setminus D$ is σ-invariant.

④ pf (Thm, cont.)

Claim 2: $\square\setminus D \precsim \square\setminus C$.

pf (C2): Since each orbit of $\sigma\restriction(\square\setminus D)$ contains at most one element of C, we can recycle our old trick:

■ (C2)

So we have shown $\square \precsim \square\setminus C$. Since of course $\square\setminus C \precsim \square$, the Schröder-Bernstein property completes the proof. ■ (Thm)

Remark: Naturally, this inductive approach generalizes to higher dimensions. If $\square \subseteq \mathbb{R}^d$ is the unit cube of d-dimensional Euclidean space, and $C \subseteq \square$ is covered by countably many translations of proper subspaces, then $\square \approx \square\setminus C$.

Paradoxes

Monday, Mar 25

Actually, we are working with the Riemann Sphere!

Möbius transformations

Def: A Möbius transformation is a function $f: \mathbb{C} \to \mathbb{C}$ of the form $f: z \longmapsto \dfrac{az+b}{cz+d}$ with $a,b,c,d \in \mathbb{C}$ satisfying $ad-bc \neq 0$.

Given a (2×2)-matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with complex entries and $\det(A) \neq 0$, we obtain a Möbius transf f_A by

$$f_A: z \longmapsto \frac{az+b}{cz+d}.$$

Examples:

- $f_{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}: z \longmapsto z+1$
- $f_{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}: z \longmapsto \dfrac{z}{z+1}$
- $f_{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}: z \longmapsto \dfrac{2z+1}{z+1} = \dfrac{z}{z+1} + 1 = f_{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} \circ f_{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}(z)$

Prop: Given (2×2)-matrices A, B over $\mathbb{C}$ with nonzero determinants, $f_A \circ f_B = f_{AB}$.

pf: Tedious calculation. ∎

Remark: This implies that the collection of Möbius transformations forms a group under composition, and that $A \mapsto f_A$ is a group hom.

Prop: The kernel of $A \mapsto f_A$ is $\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{C} \setminus \{0\} \right\}$.

pf: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the kernel iff $\dfrac{az+b}{cz+d} = z$. for all $z \in \mathbb{C}$. Shuffle this around to

$$\forall z \in \mathbb{C} \quad cz^2 + (d-a)z - b = 0,$$

implying $a = d$, $b = c = 0$ as desired. ∎(Prop)

In other words, $f_A = f_B$ iff B is a (nonzero) scalar multiple of A.

Prop: The group of Möbius transformations is generated by the following three types of maps:

- Translations: $z \longmapsto z + b$ $\quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$
- Dilations: $z \longmapsto az$ $\quad \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ $\quad (a \neq 0)$
- Inversion: $z \longmapsto 1/z$ $\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

pf: This is basically the Gauss-Jordan algorithm. Alternatively, we may explicitly handle $z \mapsto \dfrac{az+b}{cz+d}$ for nonzero c like so (if $c = 0$ it's easy):

$$z \overset{T}{\longmapsto} z + d/c \overset{I}{\longmapsto} \frac{1}{z + d/c} \overset{D}{\longmapsto} \frac{\left(\frac{bc-ad}{c^2}\right)}{z + d/c} \overset{T}{\longmapsto} \frac{\left(\frac{bc-ad}{c^2}\right)}{z + d/c} + \frac{a}{c}$$

$$= \frac{az+b}{cz+d}$$ ∎(Prop)

Def: A generalized circle is a subset of $\mathbb{C}$ of the form $\{z \in \mathbb{C} : A z\bar{z} + Bz + C\bar{z} + D = 0\}$, where $A, D \in \mathbb{R}$, $B = \overline{C} \in \mathbb{C}$, and $AD < BC$.

Examples: Let's identify objects with (A, B, C, D):

- Unit circle $\rightsquigarrow (1, 0, 0, -1)$.
- Real axis $\rightsquigarrow (0, i, -i, 0)$.
- Imaginary axis $\rightsquigarrow (0, 1, 1, 0)$.

Exercise: Generalized circles are exactly circles and lines ("circles through ∞").

Prop: Möbius transformations map gcircles to gcircles.

pf: It suffices to handle the generating types.

Translation: circle $\to$ circle, line $\to$ line ✓

Dilation: ✓

Inversion: $z \mapsto 1/z$. Typical gcircle gets sent to:

$$\left\{z \in \mathbb{C} : \frac{A}{z\bar{z}} + \frac{B}{z} + \frac{C}{\bar{z}} + D = 0\right\}$$

$$= \{z \in \mathbb{C} : A + B\bar{z} + Cz + Dz\bar{z} = 0\}$$

$$= \{z \in \mathbb{C} : A' z\bar{z} + B' z + C' \bar{z} + D' = 0\}$$

with $(A', B', C', D') = (D, C, B, A)$.

This is a valid gcircle. ✓ $\blacksquare$ (Prop)

4

<u>Def</u>: Given four complex numbers, $w, x, y, z \in \mathbb{C}$, the corresponding <u>cross-ratio</u> is

$$(w, x ; y, z) = \frac{(w-y)(x-z)}{(w-z)(x-y)} \in \mathbb{C}.$$

<u>Prop</u>: If $f : \mathbb{C} \to \mathbb{C}$ is a Möbius transf, then for all $w, x, y, z \in \mathbb{C}$

$$(f(w), f(x) ; f(y), f(z)) = (w, x ; y, z).$$

<u>pf</u>: We handle the generators:

Translation: $z \mapsto z + b$. All of the $+b$s will cancel in the differences. ✓

Dilation: $z \mapsto az$. All of the as will cancel in the ratio ✓

Inversion: $z \mapsto 1/z$. Note: $\frac{1}{w} - \frac{1}{y} = \frac{1}{wy}(y-w)$.

So we compute

$$\left(\frac{1}{w}, \frac{1}{x} ; \frac{1}{y}, \frac{1}{z}\right)$$

$$= \frac{\frac{1}{wy}(y-w)\frac{1}{xz}(z-x)}{\frac{1}{wz}(z-w)\frac{1}{xy}(y-x)}$$

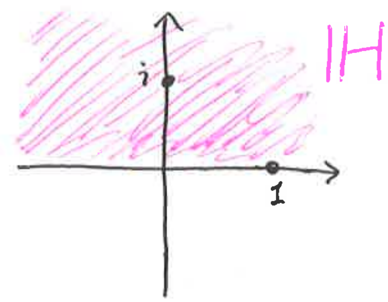
$$= (w, x ; y, z).$$ ✓ ■ (Prop)

1

Paradoxes

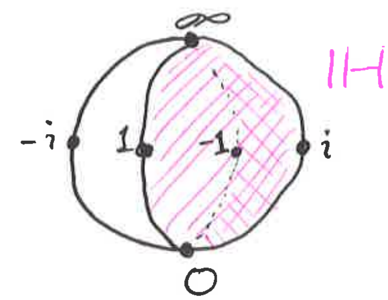
Wednesday, Mar 27

The hyperbolic plane, pt I



Def: $\mathbb{H} \subseteq \mathbb{C}$ is the upper half-plane

$$\mathbb{H} = \{ z \in \mathbb{C} : \mathrm{im}(z) > 0 \}$$



Remark: You can also think of $\mathbb{H}$ as a hemisphere of the Riemann sphere with boundary circle $\mathbb{R} \cup \{\infty\}$.

Def: A Möbius transformation $f_A : z \mapsto \frac{az+b}{cz+d}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is special if
- $a, b, c, d \in \mathbb{R}$
- $\det(A) > 0$.

Remarks: (a) By rescaling, we may assume $\det(A) = 1$ without changing the transformation f_A.

(b) The special Möbius transformations form a group (under composition). Let's call it SMT.

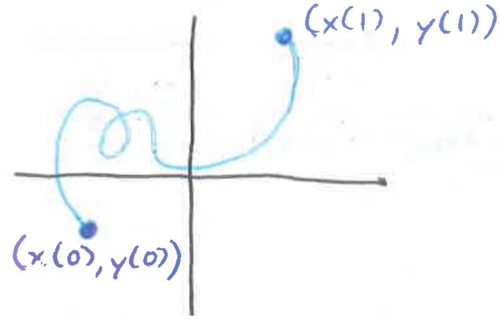
Prop: For all $f \in \mathrm{SMT}$, $f[\mathbb{H}] = \mathbb{H}$.

pf: Exercise. ☺ (Prop)

2 Crash course in Riemannian manifolds!

Example: Suppose we have a "smooth" curve in the Euclidean plane.

$$[0,1] \longrightarrow \mathbb{R}^2$$
$$t \longmapsto (x(t), y(t))$$



Calculus says: The length of this curve is given by $\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

Idea: The infinitesimal length $ds = \sqrt{(dx)^2 + (dy)^2}$, and we integrate this.

Now the metric on the Euclidean plane is given by infimizing the length of a curve with prescribed endpoints.

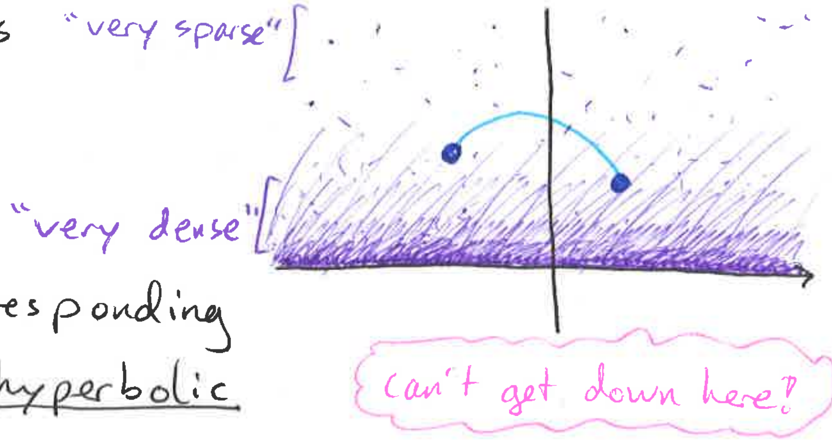
Of course, in this case the infimum is realized by a straight line segment.

In general, this search for a length minimizer involves fancy existence/uniqueness results about differential equations, calculus of variations, etc. etc.

Somehow we will make do with algebra...

(3) Def: The hyperbolic infinitesimal length on the upper half-plane is

$$ds = \frac{\sqrt{(dx)^2 + (dy)^2}}{y}$$



As before, the corresponding integral yields the hyperbolic length of smooth curves.

Infimizing (in fact minimizing) this length yields the hyperbolic metric on $\mathbb{H}$.

Today's goal: Special Möbius transformations are hyperbolic isometries.

Prop: The group SMT is generated by the following three types:

- Translations: $z \longmapsto z+b$ $\quad b \in \mathbb{R}$ $\quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$
- Dilations: $z \longmapsto az$ $\quad a>0$ $\quad \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$
- Neg inversion: $z \longmapsto -1/z$ $\quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

pf: Same as last time. $\blacksquare$ (Prop)

(4)

Thm: The hyperbolic metric is invariant under the action SMT $\curvearrowright \mathbb{H}$.

pf: It suffices to check that the infinitesimal length $ds = \frac{\sqrt{(dx)^2+(dy)^2}}{y}$ is invariant under SMTs $z = x+iy \longmapsto \hat{z} = \hat{x}+i\hat{y}$ using change of variable formulas from calculus.

- Translation: $\hat{x} = x+b \qquad \hat{y} = y$

  $d\hat{x} = dx \qquad d\hat{y} = dy$

  So $d\hat{s} = \frac{\sqrt{d\hat{x}^2+d\hat{y}^2}}{\hat{y}} = ds$ ✓

- Dilation: $\hat{x} = ax \qquad \hat{y} = ay$

  $d\hat{x} = a\,dx \qquad d\hat{y} = a\,dy$

  So $d\hat{s} = \frac{\sqrt{d\hat{x}^2+d\hat{y}^2}}{\hat{y}}$

  $= \frac{\sqrt{(a\,dx)^2+(a\,dy)^2}}{ay} = ds$ ✓

- Neg inv.: $\hat{z} = \frac{-1}{z} = \frac{-1}{x+iy} = \frac{-x+iy}{x^2+y^2}$

  $\hat{x} = \frac{-x}{x^2+y^2} \qquad \hat{y} = \frac{y}{x^2+y^2}$

  $d\hat{x} = \frac{(x^2-y^2)\,dx+2xy\,dy}{(x^2+y^2)^2} \qquad d\hat{y} = \frac{(x^2-y^2)\,dy-2xy\,dx}{(x^2+y^2)^2}$

  So $d\hat{s} = \frac{\sqrt{d\hat{x}^2+d\hat{y}^2}}{\hat{y}} = \frac{1}{(x^2+y^2)}\,\frac{\sqrt{((x^2-y^2)^2+4x^2y^2)(dx^2+dy^2)}}{y}$

  $= ds$ ✓

$\blacksquare$ (Thm)

Paradoxes

Friday, Mar 29

The hyperbolic plane pt II

Last time: We equipped the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\} = \{(x,y) \in \mathbb{R}^2 : y > 0\}$$

with an infinitesimal length $ds = \dfrac{\sqrt{(dx)^2 + (dy)^2}}{y}$.

We showed that this is invariant under the action $SMT \curvearrowright \mathbb{H}$, where SMT is the group of special Möbius transformations:

$$\left\{ z \longmapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R} \text{ and } ad-bc=1 \right\}.$$

Today's goal: Understand the resulting metric on $\mathbb{H}$.

Warm-up: What is the distance between two points $(0, y_0)$ and $(0, y_1)$ on the imaginary axis? Let's say $y_0 < y_1$.

Claim 1: This distance is at most $\log(y_1) - \log(y_0)$.

pf(C1): Consider the linear interpolation

$$[0,1] \longrightarrow \mathbb{H}$$
$$t \longmapsto (0,\ y_0 + t(y_1 - y_0))$$

Its length is $\displaystyle\int_0^1 \frac{y_1 - y_0}{y_0 + t(y_1 - y_0)}\,dt = \log(y_1) - \log(y_0)$

$\blacksquare$(C1)

②

Claim 2: This distance is at least $\log(y_1) - \log(y_0)$.

pf (C2): Consider an arbitrary smooth curve

$$[0,1] \longrightarrow \mathbb{H} \qquad 0 \longmapsto (0, y_0)$$
$$t \longmapsto (x(t), y(t)) \qquad 1 \longmapsto (0, y_1).$$

We compute $$\int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)}\, dt$$

This "informal" argument can be made formal with a little analysis

$$\geq \int_0^1 \frac{\sqrt{\left(\frac{dy}{dt}\right)^2}}{y(t)}\, dt$$

$$\geq \int_{y_0}^{y_1} \frac{dy}{y} = \log(y_1) - \log(y_0) \quad \blacksquare (C2)$$

So the distance is exactly $\log(y_1) - \log(y_0)$! $\blacksquare$ (warm-up)

That was exhausting. How can we compute other distances without doing more work? We know that SMT acts by isometries, so let's move the axis around.

Prop: The image of the imaginary axis w/ ∞

$$i\mathbb{R} = \{z \in \mathbb{C} : \mathrm{re}(z) = 0\} \cup \{\infty\}$$

under any $f \in SMT$ is a gcircle symmetric about the real axis $\mathbb{R}$.

pf: We already know that $f[i\mathbb{R}]$ is a gcircle. It is invariant under complex conjugation, as $f : \bar{z} \longmapsto \overline{f(z)}$. So it is symmetric about $\mathbb{R}$. $\blacksquare$

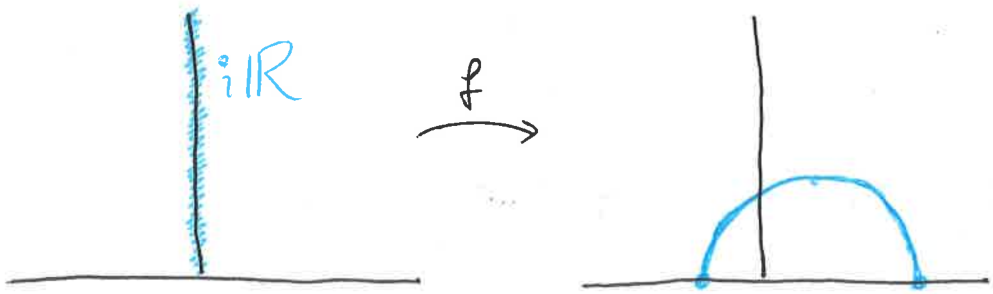
③ **Prop:** Given any gcircle C symmetric about $\mathbb{R}$, there is $f \in SMT$ with $f[i\mathbb{R}] = C$.

pf: If C is a vertical line, an appropriate translation works. So let's assume that C is a circle with $C \cap \mathbb{R} = \{w_0, w_1\}$.

The SMT $g: z \longmapsto \frac{z - w_0}{z - w_1}$ sends $\begin{matrix} w_0 \mapsto 0 \\ w_1 \mapsto \infty, \end{matrix}$

thus its inverse f must map $\begin{matrix} 0 \mapsto w_0 \\ \infty \mapsto w_1 \end{matrix}$.

So $f[i\mathbb{R}]$ is a real-symmetric gcircle containing $\{w_0, w_1\}$, and hence $f[i\mathbb{R}] = C$. ▨(Prop).



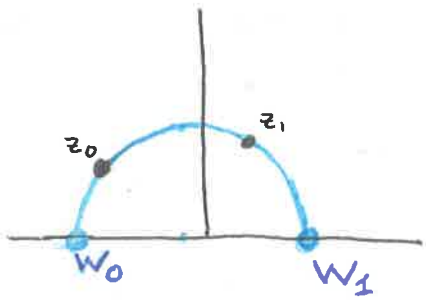
Since any two points of $\mathbb{H}$ are caught in such a circle, we can transport our analysis of $i\mathbb{R}$ to compute arbitrary distances.

Sneaky trick: Given two points $iy_0, iy_1 \in i\mathbb{R}$ we can rewrite what we know:

$$d_{\mathbb{H}}(iy_0, iy_1) = |\log(y_1) - \log(y_0)|$$
$$= \left|\log\left(\frac{y_1}{y_0}\right)\right|$$
$$= \left|\log\left(\frac{iy_1 \cdot (iy_0 - \infty)}{iy_0 \cdot (iy_1 - \infty)}\right)\right|$$
$$= |\log(iy_0, iy_1 ; 0, \infty)|.$$

④ **Thm:** Suppose that $z_0, z_1 \in \mathbb{H}$ are distinct. Let $w_0, w_1 \in \mathbb{R} \cup \{\infty\}$ be the two elements of $\mathbb{R} \cup \{\infty\}$ on the unique real-symmetric gcircle through z_0, z_1. Then

$$d_{\mathbb{H}}(z_0, z_1) = |\log(z_0, z_1 ; w_0, w_1)|.$$



pf: Consider $f \in SMT$ mapping this gcircle to $i\mathbb{R}$. Say $f: \begin{array}{l} w_0 \mapsto 0 \\ w_1 \mapsto \infty \end{array}$. We compute

$$\begin{aligned} d_{\mathbb{H}}(z_0, z_1) &= d_{\mathbb{H}}(f(z_0), f(z_1)) \\ &= |\log(f(z_0), f(z_1) ; 0, \infty)| \\ &= |\log(f(z_0), f(z_1) ; f(w_0), f(w_1))| \\ &= |\log(z_0, z ; w_0, w_1)|. \end{aligned}$$ ■(Thm)

Cor: The arc of the above gcircle connecting z_0 to z_1 is a length-minimizing curve.

pf: The corresponding segment of the imaginary axis is, by our warm-up calculation. ■(Cor)

Paradoxes
Monday, Apr 1

Paradoxicality in the hyperbolic plane

Recall: SMT is the group $\left\{ z \mapsto \frac{az+b}{cz+d} : \begin{array}{l} a,b,c,d \in \mathbb{R} \\ ad-bc=1 \end{array} \right\}$.

We know that SMT $\curvearrowright \mathbb{H}$ by hyperbolic isometries.

Def: $SL_2(\mathbb{R})$ is the group of (2×2)-matrices with real entries and determinant 1.

Observation: We have a natural hom $SL_2(\mathbb{R}) \longrightarrow SMT$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(z \mapsto \frac{az+b}{cz+d} \right).$$

Its kernel is $\{I, -I\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$.

Proposition: The SMTs

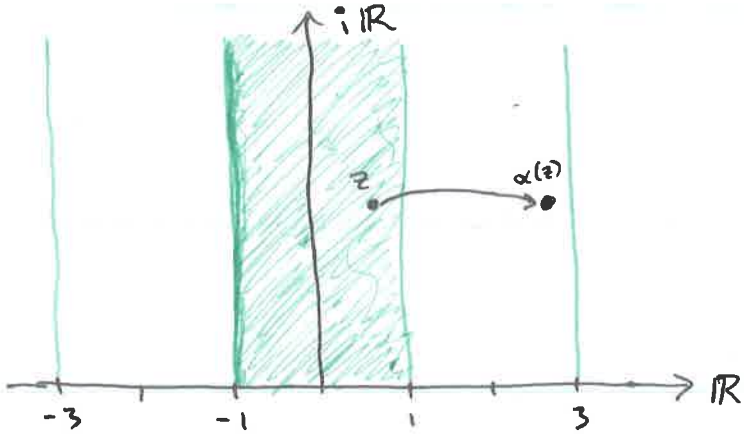
$$\alpha : z \longmapsto z+2 \qquad \beta : z \longmapsto \frac{z}{2z+1}$$

generate a subgroup of SMT isomorphic to $\mathbb{F}_2$.

pf: We know from $\boxed{\text{HW}}$ that $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \leq SL_2(\mathbb{R})$ is isomorphic to $\mathbb{F}_2$. As $\langle \alpha, \beta \rangle$ is the image of this subgroup under the above hom, it suffices to argue that $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \longrightarrow \langle \alpha, \beta \rangle$ is an embedding. For this, it suffices to argue that $-I \notin \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$, which follows from the fact that no element of $\mathbb{F}_2$ has order 2. ∎ (Prop)

② We carefully analyze the action of $\langle \alpha, \beta \rangle \leq SMT$ on $\mathbb{H}$ (and on $\mathbb{C} \cup \{\infty\}$).

$\alpha : z \mapsto z+2$ is quite easy to visualize:



The strip $\{z : -1 \leq re(z) < 1\}$ meets each α-orbit exactly once.

$\beta : z \mapsto \frac{z}{2z+1}$ is a bit trickier to visualize:

Let's run some calculations on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ to get our bearings.

$$\beta : 0 \mapsto \frac{0}{2 \cdot 0 + 1} = 0$$

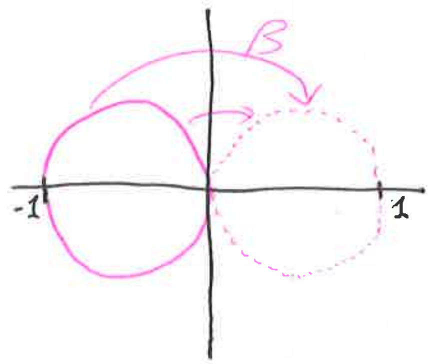
$$-1 \mapsto \frac{-1}{2(-1)+1} = 1$$

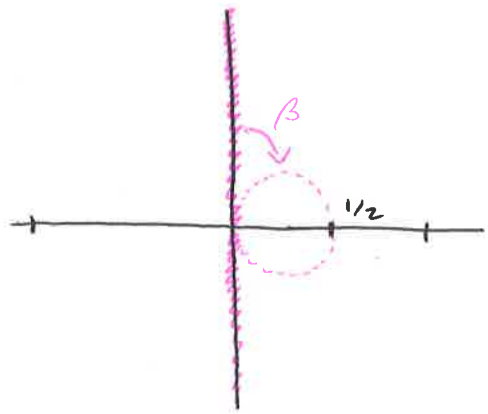
$$1 \mapsto \frac{1}{2(1)+1} = 1/3$$

$$\infty \mapsto \frac{\infty}{2(\infty)+1} = 1/2$$

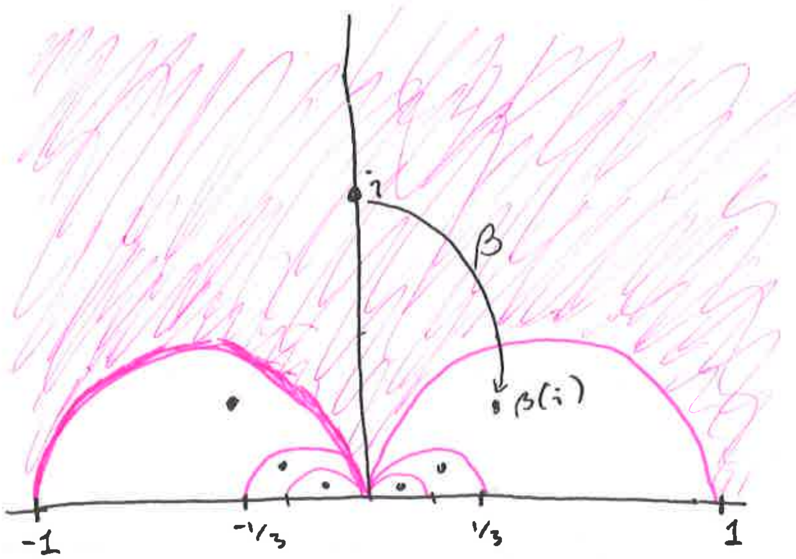
$$i \mapsto \frac{i}{2i+1} = \frac{i(-2i+1)}{5} = \frac{2}{5} + \frac{1}{5} i$$

③ It follows that β maps the $\mathbb{R}$-symmetric gcircle through $\{-1, 0\}$ to that through $\{0, 1\}$, and also the imaginary axis $i\mathbb{R}$ to the $\mathbb{R}$-symm gcircle through $\{0, 1/2\}$.

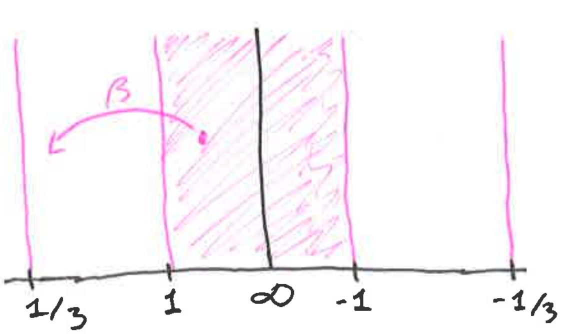


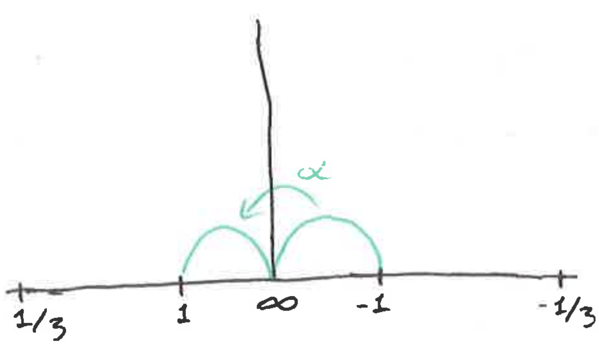


So we may visualize the action of β on $\mathbb{H}$:



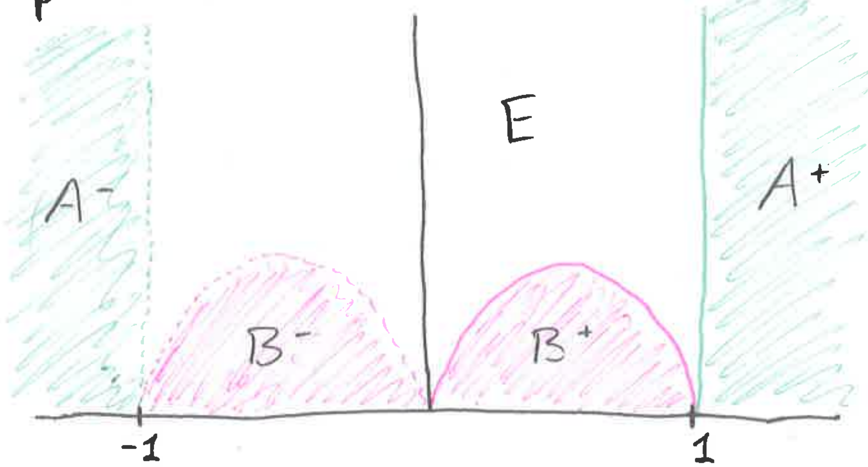
If we instead "puncture" the sphere at 0, we get





If $\gamma : z \mapsto {}^{-1}/_{z}$, then $\beta^{-1} = \gamma \alpha \gamma^{-1}$.

④ To analyze $\langle \alpha, \beta \rangle \curvearrowright \mathbb{H}$, we superpose these pictures:



We've built a ping-pong family! This yields another proof of $\langle \alpha, \beta \rangle \cong \mathbb{F}_2$.

But more importantly, we see

<u>Thm</u> [ZF]: $\mathbb{H}$ is paradoxical via isometries.

<u>pf</u>: $\mathbb{H} = \alpha \cdot A^- \sqcup A^+$.

$\mathbb{H} = \beta \cdot B^- \sqcup B^+$.

Schröder-Bernstein handles E to make it a true paradox. ∎ (Thm).

Next time, we will see that each orbit of $\langle \alpha, \beta \rangle$ hits E in exactly one point, and (hence) that the action is free. This will allow us to "port over" more delicate analyses of $\mathbb{F}_2$...

①

Paradoxes

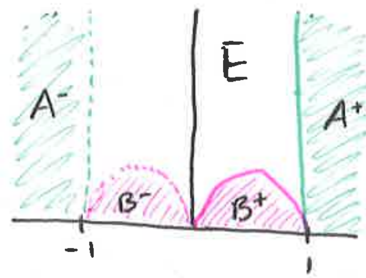
Wednesday, Apr 3

Fundamental domains in the hyperbolic plane

Setup: $\alpha : z \longmapsto z+2$ $\quad \beta : z \longmapsto \dfrac{z}{2z+1}$

We know $\Gamma = \langle \alpha, \beta \rangle \cong \mathbb{F}_2$.

We have a ping-pong family, so each $\Gamma \curvearrowright \mathbb{H}$ orbit meets E in at most one point.



Today's goal: E meets every orbit (in one point).

Strategy: Given $z \in \mathbb{H}$, first find $\gamma \in \Gamma$ maximizing $\mathrm{im}(\gamma \cdot z)$, then translate into E using α.

Prop: Suppose that $f : z \longmapsto \dfrac{az+b}{cz+d}$ is an SMT with $ad-bc=1$. Then for all $z \in \mathbb{H}$,

$$\mathrm{im}(f(z)) = \frac{1}{|cz+d|^2}\,\mathrm{im}(z).$$

pf: Compute $\mathrm{im}(f(z)) = \mathrm{im}\left(\dfrac{az+b}{cz+d}\right)$

$$= \boxed{\frac{1}{|cz+d|^2}}\ \mathrm{im}\big((az+b)(c\bar{z}+d)\big)$$

$$= \boxed{}\ \mathrm{im}\left(acz\bar{z} + adz + bc\bar{z} + bd\right)$$

$$= \boxed{}\ \mathrm{im}\big((ad-bc)z\big)$$

$$= \boxed{\frac{1}{|cz+d|^2}}\ \mathrm{im}(z). \qquad \blacksquare\,(\text{Prop})$$

Now we know how SMTs affect imaginary parts. The subtle thing is ensuring that a maximum imaginary part exists.

<u>Lemma:</u> Suppose that $\vec{w}, \vec{v} \in \mathbb{R}^2$ are $\mathbb{R}$-linearly independent. Then there is $r > 0$ s.t. for all $(a,b) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, $d(a\vec{w} + b\vec{v}, \vec{0}) > r$.

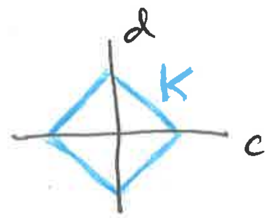
<u>Remark</u>: This phenomenon is sometimes called <u>lacunarity</u>.

<u>pf($\mathcal{L}$)</u>: Suppose otherwise, and build a sequence $(a_n, b_n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ with $a_n\vec{w} + b_n\vec{v} \longrightarrow \vec{0}$.

Normalize, putting $c_n = \dfrac{a_n}{|a_n| + |b_n|}$, $d_n = \dfrac{b_n}{|a_n| + |b_n|}$.

We see
- $|c_n| \le 1$, $|d_n| \le 1$
- $|c_n| + |d_n| = 1$
- $c_n\vec{w} + d_n\vec{v} = \dfrac{1}{|a_n| + |b_n|}(a_n\vec{w} + b_n\vec{v}) \longrightarrow \vec{0}$.

So (c_n, d_n) live here:



By compactness, for a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ there is $(c,d) \in K$ with $(c,d) = \lim_{\mathcal{U}} (n \mapsto (c_n, d_n))$. . But then $c\vec{w} + d\vec{v} = \vec{0}$, contradicting $\mathbb{R}$-linear independence of $\vec{w}$ and $\vec{v}$. ■($\mathcal{L}$)

<u>Remark</u>: Of course, we are really just using "ordinary" sequential compactness here.

(3) <u>Cor</u>: For all $z \in \mathbb{H}$, there are finitely many $(a,b) \in \mathbb{Z}^2$ with $|az+b| \leq 1$.

<u>pf</u>: Viewing $\mathbb{C}$ as $\mathbb{R}^2$, observe that z and 1 are $\mathbb{R}$-linearly independent. Lacunarity ensures that $\{az + b1 : (a,b) \in \mathbb{Z}^2\}$ intersects the unit disc in a finite set. ∎ (Cor)

With these preliminaries under our belt, we finally maximize.

<u>Prop</u>: For all $z \in \mathbb{H}$, the set

$$\{\operatorname{im}(\gamma \cdot z) : \gamma \in \Gamma\}$$

has a maximum element.

<u>pf</u>: Suppose otherwise, and fix a sequence $\gamma_n \in \Gamma$ s.t. $\gamma_0 = e$ and $\operatorname{im}(\gamma_n \cdot z)$ is strictly increasing. Write $\gamma_n = f_{A_n}$ with $A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in SL_2(\mathbb{Z})$.

Using $\operatorname{im}(\gamma_n \cdot z) = \dfrac{1}{|c_n z + d_n|^2} \operatorname{im}(z)$, we see that $\{(c_n, d_n) : n \in \mathbb{N}\}$ is an infinite subset of $\mathbb{Z}^2$ and for all $n \in \mathbb{N}$, $|c_n z + d_n| < 1$, contradicting the preceding corollary. ∎ (Prop)

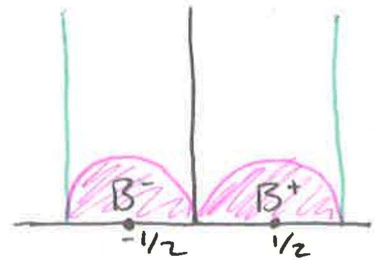
<u>Remark</u>: We actually proved that the set

$$\{\operatorname{im}(f_A(z)) : A \in SL_2(\mathbb{Z})\}$$

has order type $\omega^* =$ "$-\mathbb{N}$".

④ **Thm:** E meets every orbit of $\Gamma \curvearrowright \mathbb{H}$.

pf: Fix arbitrary $z \in \mathbb{H}$; we want $\gamma \in \Gamma$ with $\gamma \cdot z \in E$. Following our strategy, first find $\gamma_0 \in \Gamma$ maximizing $\operatorname{im}(\gamma_0 \cdot z)$. Next, find $m \in \mathbb{Z}$ s.t. $-1 \leq \operatorname{re}(\alpha^m \gamma_0 \cdot z) < 1$. Put $\gamma = \alpha^m \gamma_0$.



Claim: $\gamma \cdot z \in \overline{E}$. [closure]

pf(C): It's enough to show that $\gamma \cdot z \notin \operatorname{int}(B^-) \cup \operatorname{int}(B^+)$.

- If $\gamma \cdot z \in \operatorname{int}(B^-)$, then $|2\gamma \cdot z + 1| < 1$.
  Then $\operatorname{im}(\beta \cdot \gamma \cdot z) = \dfrac{1}{|2\gamma \cdot z + 1|^2} \operatorname{im}(\gamma \cdot z)$
  $> \operatorname{im}(\gamma \cdot z) = \operatorname{im}(\gamma_0 \cdot z)$
  This contradicts our choice of γ_0. ✓

- If $\gamma \cdot z \in \operatorname{int}(B^+)$ we get
  $\operatorname{im}(\beta^{-1} \cdot \gamma \cdot z) > \operatorname{im}(\gamma_0 \cdot z)$,
  again contradicting choice of γ_0. ✓

$\blacksquare$(C).

Finally, if $\gamma \cdot z \in \overline{E} \setminus E$, it must be on the "boundary circle" of B^+. One more application of β^{-1} solves this problem. $\blacksquare$(Thm)

1

Paradoxes

Friday, Apr 5

A Colorful approach to paradoxicality

Warm-up: If $\Gamma \curvearrowright X$ is any action of a group on a non-$\emptyset$ set X, any witness

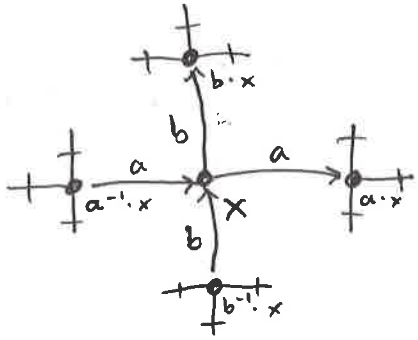
$$X = \bigsqcup_{i<m} A_i \sqcup \bigsqcup_{j<n} B_j$$

to paradoxicality must involve at least four subsets. I.e., $m+n \geq 4$.

Why? If not, $m=1$ or $n=1$. So either $A_0 = X$ or $B_0 = X$ and a contradiction quickly follows.

Today's goal: Four pieces suffice for the left-multiplication action $\mathbb{F}_2 \curvearrowright \mathbb{F}_2$ (our old argument used five). We will take the opportunity to develop a new perspective on paradoxicality.

For clarity of notation, fix a (countable) set X and an action $\mathbb{F}_2 = \langle a,b \rangle \curvearrowright X$ which is free and has one orbit:



Inspired by the ping-pong lemma, we aim to partitition $\mathbb{X} = A^+ \sqcup A^- \sqcup B^+ \sqcup B^-$ s.t.

- $a \cdot (A^+ \cup B^+ \cup B^-) = A^+$
- $a^{-1} \cdot (A^- \cup B^+ \cup B^-) = A^-$
- $b \cdot (A^+ \cup A^- \cup B^+) = B^+$
- $b^{-1} \cdot (A^+ \cup A^- \cup B^-) = B^-$

Prop: If we can partition $\mathbb{X} = A^+ \sqcup A^- \sqcup B^+ \sqcup B^-$ as above, then $A^+ \cup A^- \approx \mathbb{X}$ and $B^+ \cup B^- \approx \mathbb{X}$. In particular, we have a paradoxical decomposition using four pieces.

pf: We compute

$$a \cdot A^- = A^- \cup B^+ \cup B^- = \mathbb{X} \setminus A^+$$
$$b \cdot B^- = A^+ \cup A^- \cup B^- = \mathbb{X} \setminus B^+.$$

So $\mathbb{X} = A^+ \sqcup a \cdot A^- = B^+ \sqcup b \cdot B^-$. ■ (Prop)

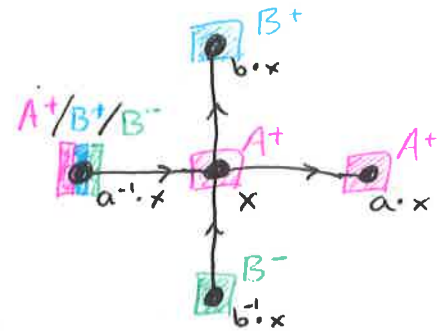
Remark: We may rewrite the four set equations above as eight containments:

$$a \cdot (A^+ \cup B^\pm) \subseteq A^+ \qquad a^{-1} \cdot (A^- \cup B^\pm) \subseteq A^-$$
$$a \cdot A^- \subseteq A^- \cup B^\pm \qquad a^{-1} \cdot A^+ \subseteq A^+ \cup B^\pm$$

$$b \cdot (A^\pm \cup B^+) \subseteq B^+ \qquad b^{-1} \cdot (A^\pm \cup B^-) \subseteq B^-$$
$$b \cdot B^- \subseteq A^\pm \cup B^- \qquad b^{-1} \cdot B^+ \subseteq A^\pm \cup B^+$$

③ We summarize these "local rules" in a big table:

x	$a\cdot x$	$a^{-1}\cdot x$	$b\cdot x$	$b^{-1}\cdot x$
A^+	A^+	not A^-	B^+	B^-
A^-	not A^+	A^-	B^+	B^-
B^+	A^+	A^-	B^+	not B^-
B^-	A^+	A^-	not B^+	B^-



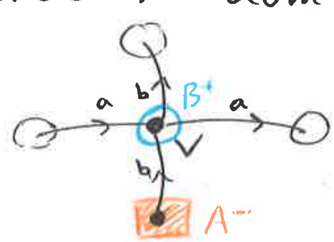
This is equivalent to solving a <u>coloring problem</u> on a labeled graph.

<u>Combinatorial Lemma</u>: If G is an acyclic 4-regular connected graph s.t. each vertex has

- one outgoing a-edge
- one incoming a-edge
- one outgoing b-edge
- one incoming b-edge

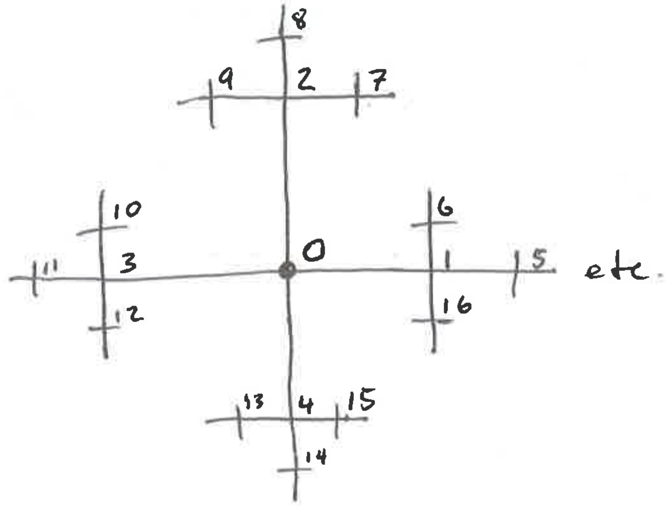
Then G admits an $\{A^\pm, B^\pm\}$-coloring satisfying the above local rules.

<u>pf (CL)</u>: First observe that whenever c is a partial coloring satisfying these rules and v is a vertex adjacent to at most one element of $\mathrm{dom}(c)$, then c may be extended to c' with $\mathrm{dom}(c') = \mathrm{dom}(c) \cup \{v\}$.



④ pf ($C\mathcal{L}$, cont.):

We now enumerate the vertices of G so that each vertex is adjacent to at most one vertex occurring earlier in the enumeration:



Following this enumeration, we iteratively extend partial colorings (per our observation) to build a valid coloring of the graph. ■ ($C\mathcal{L}$)

Cor: $\mathbb{F}_2$ is paradoxical using four pieces.

Cor [AC]: Any free action of $\mathbb{F}_2$ is paradoxical using four pieces.

Cor [ZF]: The hyperbolic plane is paradoxical via isometries using four pieces.

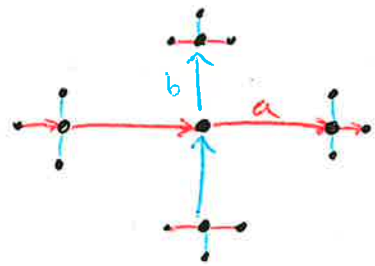
Remark: The argument actually builds such a partition with pieces of "low topological complexity." They are F_σ, hence Borel.

Paradoxes

Wednesday, Apr 17

Schreier "graphs" arising from actions of $\mathbb{F}_2$

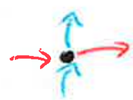
As discussed last time, with any free action $\mathbb{F}_2 \curvearrowright X$ we may associate a (labeled) graph so that each orbit looks like a 4-regular tree. How does this picture change for non-free actions of $\mathbb{F}_2$?



Def (Schreier "graph"): Given an action $\mathbb{F}_2 = \langle a, b \rangle \curvearrowright X$, for every $x \in X$ place: $x \xrightarrow{a} a\cdot x$ from x to $a \cdot x$

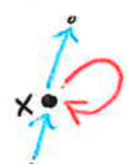
$x \xrightarrow{b} b \cdot x$ from x to $b \cdot x$

This yields a "4-regular labeled multigraph" so that locally each $x \in X$ sees:

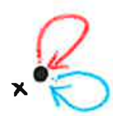


There are many potential degeneracies:

$a \cdot x = x$: • If $b \cdot x \neq x$, we get

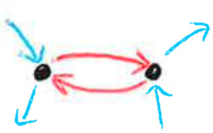


• If $b \cdot x = x$, we get

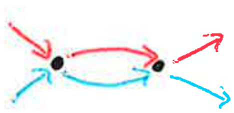


In this case, $\{x\} = \mathbb{F}_2 \cdot x$.

$a \cdot x \neq x$ but $a^2 \cdot x = x$:

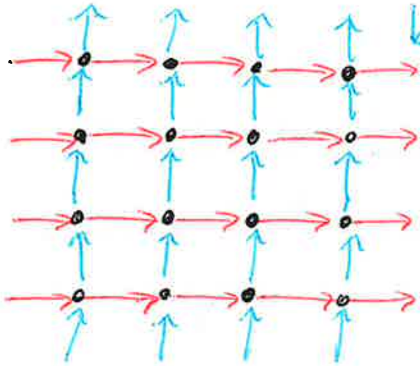


$a \cdot x = b \cdot x \neq x$:



etc...

<u>Ex</u>: $\mathbb{F}_2 \curvearrowright \mathbb{Z}^2$ via $a \cdot (m,n) = (m+1, n)$
$b \cdot (m,n) = (m, n+1)$



<u>Observation</u>: Every labeled multigraph which looks locally like  can be realized as the Schreier graph of some action of $\mathbb{F}_2$.

<u>Ex</u>: What is $aba^{-1} \cdot x$?

As usual, we will gain insight into orbits by examining stabilizers.

<u>Prop</u>: Given $\Gamma \curvearrowright X$, $\gamma \in \Gamma$, $x \in X$

$$\mathrm{Stab}(\gamma \cdot x) = \gamma \, \mathrm{Stab}(x) \, \gamma^{-1}.$$

<u>pf</u>: Check $\delta \cdot x = x$ iff $(\gamma \delta \gamma^{-1}) \cdot (\gamma \cdot x) = \gamma \cdot x$ ▨ (Prop)

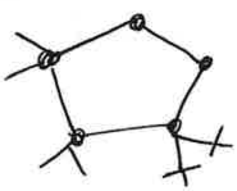
<u>Def</u>: We say that an action $\Gamma \curvearrowright X$ is <u>stabelian</u> if $\forall x \in X$ $\mathrm{Stab}(x)$ is abelian.

<u>Remark</u>: Some people call this "locally commutative" but that's no fun.

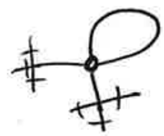
Thm: Suppose that $\mathbb{F}_2 \curvearrowright X$ is a stabelian action. Then on each orbit the associated Schreier graph is either:

- a tree, or
- an "almost-tree," with one cycle.

E.g.:



or



etc.

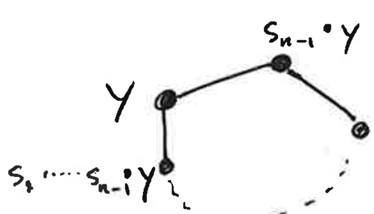
pf: Consider arbitrary $x \in X$, and examine $\mathbb{F}_2 \cdot x$. If $\mathrm{Stab}(x) = \{e\}$, then $\mathbb{F}_2 \curvearrowright \mathbb{F}_2 \cdot x$ freely, and thus the Schreier graph on this orbit is a tree. We may thus assume that $\mathrm{Stab}(x) \neq \{e\}$.

$\boxed{\text{HW}}$ Abelian subgroups of $\mathbb{F}_2$ are cyclic, so for each $y \in \mathbb{F}_2 \cdot x$ we fix $w_y \in \mathbb{F}_2$ with $\mathrm{Stab}(y) = \langle w_y \rangle$.

Fix $y \in \mathbb{F}_2 \cdot x$ so that w_y has minimal length, and write $w_y = s_0 s_1 \cdots s_{n-1}$. Note: $s_{n-1} \neq s_0^{-1}$.

Claim 1: $C = \{y, s_{n-1} \cdot y, \dots, s_1 \cdots s_{n-1} \cdot\}$ is a cycle.

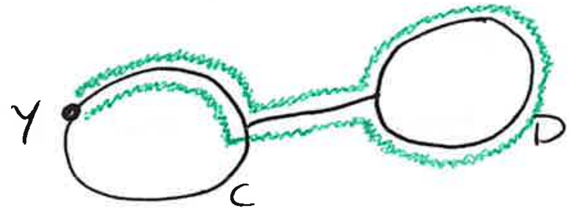
pf (C1, sketch):



and all of these vertices are distinct by minimality of w_y. ▨ (C1, sketch)

Claim 2: C is the only cycle in $\mathbb{F}_2 \cdot x$.

pf (C2, sketch): Towards a contradiction, suppose there were some other cycle D.

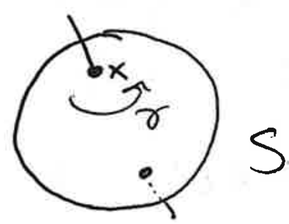


Following the green edges yields an element of $\mathrm{Stab}(y)$ not in $\langle w_y \rangle$ (since it starts with s_0 but does not end with s_{n-1}).

∎ (C2, sketch)

So the Schreier graph on $\mathbb{F}_2 \cdot x$ is an almost-tree as desired. ∎ (Thm)

Observation: The rotations of the sphere act on the sphere in a stabelian fashion. This is because $\gamma \cdot x = x$ iff x is on the axis of γ. So the subgroup of rotations fixing a given x is exactly the rotations about a prescribed axis, which is abelian (in fact isomorphic to the rotations of a circle.



This means that the action $\mathbb{F}_2 \curvearrowright S$ used in Hausdorff's paradox is stabelian as well...

Paradoxes

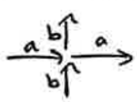
Friday, Apr 19

A Colorful approach to stabelian paradoxicality

Today's goal:

Thm [AC]: Suppose that $\mathbb{F}_2 \curvearrowright X$ is a stabelian action. Then it is paradoxical using four pieces.

Cor [AC]: The sphere is paradoxical via rotations using four pieces. [This is best possible.]

We shall use our combinatorial rephrasing of paradoxicality as a coloring problem. Given a 4-regular labeled multigraph G so that each vertex sees  , recall our local rules for detecting paradoxicality using four pieces $\{A^+, A^-, B^+, B^-\}$:

x	$a\cdot x$	$a^{-1}\cdot x$	$b\cdot x$	$b^{-1}\cdot x$
A^+	A^+	$\neg A^-$	B^+	B^-
A^-	$\neg A^+$	A^-	B^+	B^-
B^+	A^+	A^-	B^+	$\neg B^-$
B^-	A^+	A^-	$\neg B^+$	B^-

Combinatorial Lemma 2: Suppose that G is a connected 4-regular labeled multigraph as above and moreover that G is an almost-tree. Then G admits an $\{A^\pm, B^\pm\}$-coloring satisfying the above local rules.

② pf (Cl 2): Recall that an almost-tree has a single cycle and is otherwise tree-like. Since we know how to handle trees, let's just get the cycle out of the way first.

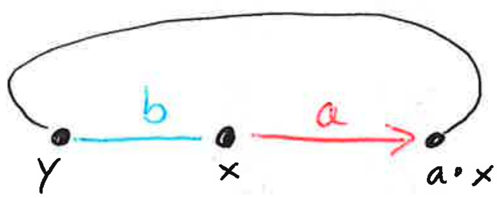
Claim: If C is a labeled cycle with at most one incoming/outgoing a/b edge at each vertex, then C admits an $\{A^\pm, B^\pm\}$-coloring following our local rules.

pf(C): We take cases on the labels appearing in our cycle C:

Case a: Every edge in the cycle has label a.
Then color all vertices A^+ (or A^-). ✓

Case b: Every edge in the cycle has label b.
Then color all vertices B^+ (or B^-). ✓

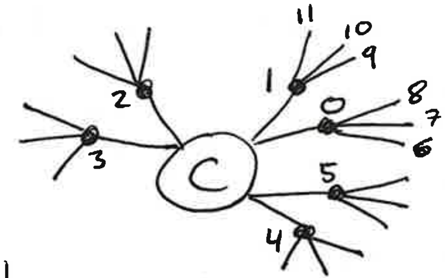
Case ab: Both labels appear. Then there is some vertex x incident to
- outgoing a
- in/out b.



We will color x last.

First, color $a \cdot x$ A^+, and use our extension property to work through $C \setminus \{x\}$. No matter what color y receives, at least one of B^+ or B^- will be a valid color for x. ✓ $\blacksquare$(C) pf ($C\mathcal{L}2$, cont.)

The claim furnishes us with a valid coloring of the cycle C. Enumerate the vertices of $G \setminus C$ so that each vertex is adjacent to at most one vertex in C OR enumerated earlier. Extend. ■($C\mathcal{L}2$)



pf (Thm): Given a stabelian action $\mathbb{F}_2 \curvearrowright X$, we saw last time that on each orbit the associated Schreier graph is either a tree or an almost-tree. Our two combinatorial lemmas ensure that each orbit admits a non-empty set of valid colorings. Using AC, choose a valid coloring on each orbit, obtaining a valid coloring of the entire Schreier graph. We did it! ■(Thm)

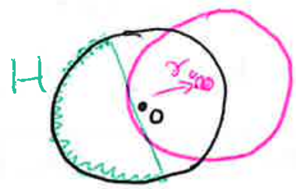
Remark: While this theorem applies to paradoxicality of the sphere, it somewhat annoyingly does not apply to the ball

$$\mathbb{B} = \{x \in \mathbb{R}^3 : d(x, 0) \le 1\}$$

as the natural action $\mathbb{F}_2 \curvearrowright \mathbb{B}$ is not stabelian at 0. This is not an accident.

④ **Prop:** The ball $B = \{x \in \mathbb{R}^3 : d(x,0) \leq 1\}$ does not admit a paradoxical decomposition via isometries using four pieces.

pf (sketch): Towards a contradiction, suppose it did. Say $B = C_0 \sqcup C_1 \sqcup D_0 \sqcup D_1$ with
$$B = \gamma_0 \cdot C_0 \sqcup \gamma_1 \cdot C_1$$
$$= \delta_0 \cdot D_0 \sqcup \delta_1 \cdot D_1$$
WLOG $0 \in C_0$, so $0 \notin D_i$.
WLOG $0 \in \delta_0 \cdot D_0$, so γ_0 MOVES 0.



This yields a closed hemisphere $H \subseteq B \setminus \delta_0 \cdot B$. We must have $H \subseteq \delta_1 \cdot D_1$

So D_1 contains a closed hemisphere $H' = \delta_1^{-1} \cdot H$
Then none of C_0, C_1, D_0 contains a closed hemisphere. Hence, by the above reasoning, neither γ_0 nor γ_1 moves 0. Thus, γ_0 and γ_1 act on the sphere S by isometries. But $C_0 \cap S$ and $C_1 \cap S$ are both contained in the open hemisphere $S \setminus H'$, so $\gamma_0 \cdot C_0 \cap S$, $\gamma_1 \cdot C_1 \cap S$ cannot cover S, a contradiction. ∎ (sketch)

Remark: It's not too hard to argue that five pieces suffice for the ball, given that four suffice for the sphere.

Paradoxes
Monday, Apr 22

Divisibility, pt I

Def: Suppose that $\Gamma \curvearrowright X$. For $k \in \mathbb{N}$, we say that the action is <u>k-divisible</u> if there is a partition $X = A_0 \sqcup \cdots \sqcup A_{k-1}$ so that $\forall i, j < k \; \exists \gamma_{ij} \in \Gamma$ with $\gamma_{ij} \cdot A_i = A_j$.

Remark: It's enough to show $\forall i < k \; \exists \gamma_i \in \Gamma$ with $\gamma_i \cdot A_0 = A_i$, as then $(\gamma_j \gamma_i^{-1}) \cdot A_i = A_j$.

Idea: We want to partition X into k-many pieces that are "congruent" via the Γ-action.

Let's try to k-divide the sphere via rotations!

Observation 1: S is 1-divisible. $A_0 =$ 

Observation 2: S is NOT 2-divisible via rotations.

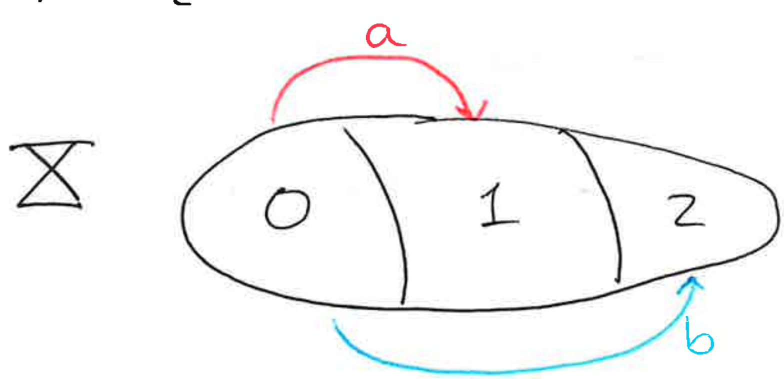
Towards a contradiction, suppose we had a 2-division $S = A_0 \sqcup A_1$ and some γ with $\gamma \cdot A_0 = A_1$. Then, necessarily, $\gamma \cdot A_1 = A_0$ as well. In particular, γ would have no fixed points. This contradicts the fact that every rotation of S has fixed pts.

Thm (Mycielski '55) [AC]: The sphere is 3-divisible via rotations.

We shall establish this in the more general context of stabelian actions of $\mathbb{F}_2$.

Thm [AC]: Suppose that $\mathbb{F}_2 = \langle a, b \rangle \curvearrowright X$ is a stabelian action. Then there is a partition $X = A_0 \sqcup A_1 \sqcup A_2$ such that: $a \cdot A_0 = A_1$
$b \cdot A_0 = A_2$.

pf: We shall rephrase this as a coloring problem on the associated Schreier graph G. To ease notation, we say x receives color i if $x \in A_i$. Let's understand how $a, b \in \mathbb{F}_2$ must behave on these color sets.



Since a bijects color 0 with color 1, a must also map $1 \cup 2$ to $0 \cup 2$.

Similarly, as b bijects 0 with 2, it also must map $1 \cup 2$ to $0 \cup 1$.

③ <u>pf</u> (Thm, cont.)

We may now summarize this in a table:

x	$a\cdot x$	$a^{-1}\cdot x$	$b\cdot x$	$b^{-1}\cdot x$
0	1	$\neg 0$	2	$\neg 0$
1	$\neg 1$	0	$\neg 2$	$\neg 0$
2	$\neg 1$	$\neg 0$	$\neg 2$	0

Any $\{0,1,2\}$-coloring satisfying these local rules witnesses our desired 3-divisibility.

<u>Lemma 1:</u> (Extension) If c is a valid partial coloring of G, and v is a vertex of G with at most one G-edge between $\{v\}$ and $\mathrm{dom}(c)$, then there is a valid coloring c' extending c with $\mathrm{dom}(c') = \mathrm{dom}(c) \cup \{v\}$.

<u>pf($\mathcal{L}1$):</u> Inspect the table. ■($\mathcal{L}1$)

<u>Lemma 2:</u> (Cycle) Suppose that C is a cycle in the Schreier graph G. Then C admits a valid coloring.

<u>pf($\mathcal{L}2$):</u> As before, we case on the edge-labels appearing in C.

<u>Case a:</u> Every edge in C has label a.

Use color 2. ✓

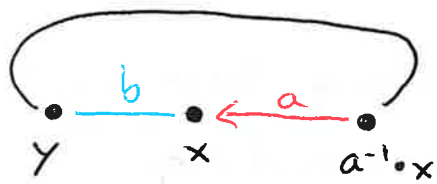
④ pf (Thm, cont.)

pf($\mathcal{L}2$, cont.)

Case b: Every edge in C has label b.

Use color 1. ✓

Case ab: Both labels appear. Then there is some vertex x incident to
- incoming a
- in/out b



We will color x last.

First, color $a^{-1}\cdot x$ with 1, then extend around C. No matter what color y receives, either 0 or 2 works for x. ✓

∎ ($\mathcal{L}2$).

The proof of the theorem now follows the strategy from last week. On each orbit of $\mathbb{F}_2 \curvearrowright X$, the Schreier graph is either a tree or an almost-tree. Thus, each orbit admits a valid coloring: first color the cycle using $\mathcal{L}2$ (if it exists), then iteratively extend using $\mathcal{L}1$. Finally, use AC to choose one coloring per orbit to color all of X. ∎ (Thm)

Paradoxes
Wednesday, Apr 24

Divisibility, pt II

Today's goal:

Thm (Mycielski) [AC]: For all $k \geq 3$, the sphere S is k-divisible via rotations. I.e., there is a partition $S = A_0 \sqcup \cdots \sqcup A_{k-1}$ s.t. $\forall i < k$ there is a rotation γ_i with $\gamma_i \cdot A_0 = A_i$.

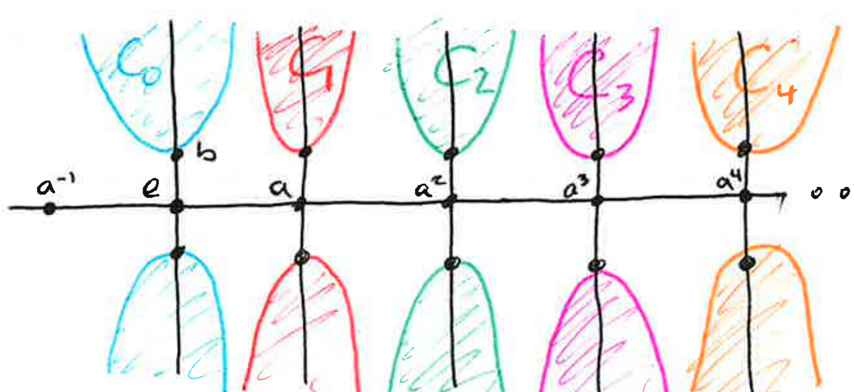
We will mimic our prior argument for 3-divisibility, increasing the rank of our free group.

Prop [HW]: For all $m \in \mathbb{N}$, the group $\mathbb{F}_2$ has a subgroup isomorphic to $\mathbb{F}_m$.

pf: Consider $\alpha_i = a^i b a^{-i}$, so for $z \in \mathbb{Z}$ we have $\alpha_i^z = a^i b^z a^{-i}$. We build a multiplayer ping-pong family for the action $\mathbb{F}_2 \curvearrowright \mathbb{F}_2$. Put $C_i = \{w \in \mathbb{F}_2 : w \text{ starts with } a^i b^n, n \neq 0\}$

Observe that for $\gamma \in \langle \alpha_i \rangle \setminus \{e\}$, $\gamma \cdot (\mathbb{F}_2 \setminus C_i) \subseteq C_i$.

A straightforward modification of the ping-pong lemma grants $\langle \alpha_i : i < m \rangle \cong \mathbb{F}_m$. ∎ (Prop)



Since we know that the rotations of S have a subgroup isomorphic to $\mathbb{F}_2$, it also has for all $m \in \mathbb{N}$ a subgroup isomorphic to $\mathbb{F}_m$. We will examine the action $\mathbb{F}_m \curvearrowright S$.

<u>Facts:</u>

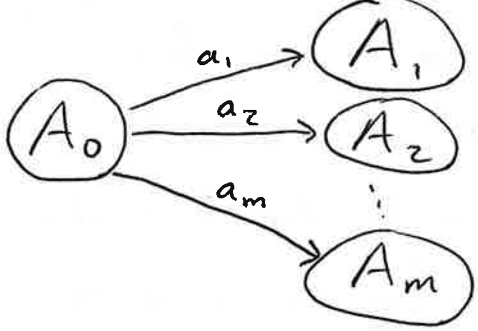
ⓐ The action $\mathbb{F}_m \curvearrowright S$ is stabelian

ⓑ Abelian subgroups of $\mathbb{F}_m$ are cyclic.

ⓒ Schreier graphs for stabelian actions of $\mathbb{F}_m$ have at most one cycle per orbit.

<u>pf</u> (of Mycielski's thm):

Fix $k \geq 3$, and put $m = k-1$. We want a partition $S = A_0 \sqcup \cdots \sqcup A_m$ s.t.



We shall analyze the (stabelian) action of $\mathbb{F}_m = \langle a_1, \dots a_m \rangle$ on S discussed above. Coloring a point i to denote membership in A_i, we recover the following big table of local rules for coloring the resulting Schreier graph G:

x	$a_1 \cdot x$	$a_1^{-1} \cdot x$	$\cdots$	$a_m \cdot x$	$a_m^{-1} \cdot x$
0	1	$\neg 0$	$\cdots$	m	$\neg 0$
1	$\neg 1$	0	$\cdots$	$\neg m$	$\neg 0$
2	$\neg 1$	$\neg 0$	$\cdots$	$\neg m$	$\neg 0$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
m	$\neg 1$	$\neg 0$	$\cdots$	$\neg m$	0

Lemma 1 (Extension): If c is a valid partial coloring of G, and v is a vertex of G with at most one G-edge between $\{v\}$ and $\mathrm{dom}(c)$, then there is a valid coloring c' extending c with $\mathrm{dom}(c') = \mathrm{dom}(c) \cup \{v\}$.

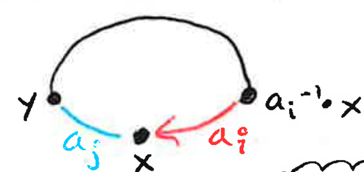
pf(L1): Inspection. ■ (L1)

Lemma 2 (Cycle): Any cycle C in G admits a valid coloring.

pf(L2):

Case 0: All edges labeled a_j. Use any color EXCEPT 0 and j. ✓

Case 1: Two labels appear:



$i \neq j$

Color $a_i^{-1} \cdot x$ with 1, ruling out color i for x.
Work around to y, the a_j-edge forces one of $0, j, \neg 0, \neg j$. All work ✓ ■ (L2)

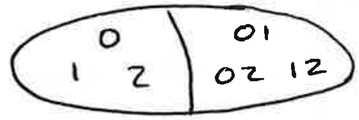
Finish off the proof as usual. ■ (Thm)

④ Let's see how far we can push this!

<u>Def</u>: Given a finite set $\mathcal{C}$, its <u>proper power set</u> is $\mathcal{P}_{pr}(\mathcal{C}) = \mathcal{P}(\mathcal{C}) \setminus \{\emptyset, \mathcal{C}\}$.

<u>Def</u> (Wagon): An <u>abstract system of congruence</u> (ASC) is an equivalence relation $\approx$ on $\mathcal{P}_{pr}(\mathcal{C})$ s.t. $\mathcal{U} \approx \mathcal{V} \Rightarrow \mathcal{U}^c \approx \mathcal{V}^c$.

<u>Ex</u>: Consider $\mathcal{C} = \{0, 1, 2\}$ so $\mathcal{P}_{pr}(\mathcal{C}) = \{0, 1, 2, 01, 02, 12\}$ and $\approx$ with two classes:



<u>Def</u>: Given an action $\Gamma \curvearrowright \mathbb{X}$ and an ASC $\approx$ on $\mathcal{P}_{pr}(\mathcal{C})$, a <u>realization</u> of the ASC in the action is a partition $\mathbb{X} = \bigsqcup_{c \in \mathcal{C}} A_c$ s.t. $\forall\, \mathcal{U}, \mathcal{V} \in \mathcal{P}_{pr}(\mathcal{C})$ if $\mathcal{U} \approx \mathcal{V}$ then there is $\gamma \in \Gamma$ with $\gamma \cdot \left(\bigcup_{c \in \mathcal{U}} A_c\right) = \bigcup_{c \in \mathcal{V}} A_c$.

<u>Ex</u>: A realization of 

is exactly a 3-division of the action.

<u>Ex</u>: If $\mathcal{C} = \{a^+, a^-, b^+, b^-\}$, the ASC

$$\{a^+\} \approx \{a^+, b^+, b^-\} \qquad \{b^+\} \approx \{a^+, a^-, b^+\}$$
$$\{a^-\} \approx \{a^-, b^+, b^-\} \qquad \{b^-\} \approx \{a^+, a^-, b^-\}$$

encodes 4-piece paradoxicality.

1

Paradoxes

Friday, Apr 26

Abstract systems of congruence

We fix a finite set $\mathcal{C}$ of labels and work with its proper power set $\mathcal{P}_{pr}(\mathcal{C}) = \mathcal{P}(\mathcal{C}) \setminus \{\emptyset, \mathcal{C}\}$.

Def (Wagon): ⓐ An ASC is an equivalence relation $\approxeq$ on $\mathcal{P}_{pr}(\mathcal{C})$ s.t. $\mathcal{U} \approxeq V \Rightarrow \mathcal{U}^c \approxeq V^c$.

ⓑ Such an ASC is noncomplementing if $\forall \mathcal{U}$ $\mathcal{U} \not\approxeq \mathcal{U}^c$.

ⓒ Given an action $\Gamma \curvearrowright X$ and an ASC $\approxeq$ on $\mathcal{P}_{pr}(\mathcal{C})$, a realization of the ASC in the action is a partition $X = \bigsqcup_{c \in \mathcal{C}} A_c$ s.t. $\mathcal{U} \approxeq V \Rightarrow \exists \gamma \in \Gamma$ $\gamma \cdot A_{\mathcal{U}} = A_V$, where $A_{\mathcal{U}} = \bigcup_{c \in \mathcal{U}} A_c$.

Today's goal:

Thm (Wagon) [AC]: Suppose that $\mathbb{F}_2 \curvearrowright X$.

ⓐ If the action is free, then every ASC can be realized in the action

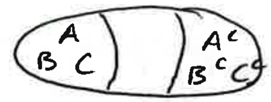
ⓑ If the action is stabelian, then every non-complementing ASC can be realized.

Cor: An ASC can be realized in rotations of the sphere $\boxed{\text{iff}}$ it is noncomplementing.

② We focus on proving ⓑ, as a (much easier) variation of the argument establishes ⓐ.

pf (Thm ⓑ): Each pair $(\mathcal{U}, V) \in \approxeq$ comes with a complementary pair $(\mathcal{U}^c, V^c) \in \approxeq$.

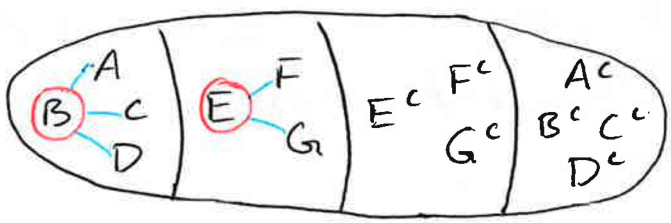
This allows us to pair off equivalence classes into complementary pairs:



Choose one equivalence class from each such pair, and moreover choose one element from each such class.

Enumerate pairs $(\mathcal{U}_i, V_i)_{i<m}$, where $\mathcal{U}_i$ is one of these chosen elements, and $V_i \neq \mathcal{U}_i$ is another element of its equivalence class.

Ex:



$i=0$	(B, A)
1	(B, C)
2	(B, D)
3	(E, F)
4	(E, G)

Our goal is to cook up an action of $\mathbb{F}_m = \langle a_i : i<m \rangle$ and a partition $X = \bigsqcup_{c \in \mathcal{C}} A_c$ such that $a_i \cdot A_{\mathcal{U}_i} = A_{V_i}$. This automatically mandates

$$\begin{cases} a_i \cdot A_{\mathcal{U}_i^c} = A_{V_i^c} \quad \text{as well.} \\ a_i^{-1} \cdot A_{V_i} = A_{\mathcal{U}_i} \\ a_i^{-1} \cdot A_{V_i^c} = A_{\mathcal{U}_i^c} \end{cases}$$

pf (Thm ⓑ, cont.):

The (stabelian) action of any subgroup of $\mathbb{F}_2$ isomorphic to $\mathbb{F}_m$ will do. We recast the search for the partition $X = \bigsqcup_{c \in \mathcal{C}} A_c$ as a coloring problem $X \to \mathcal{C}$ satisfying local rules

x	$a_i \cdot x$
$\mathcal{U}_i$	V_i
$\mathcal{U}_i^c$	V_i^c

x	$a_i^{-1} \cdot x$
V_i	$\mathcal{U}_i$
V_i^c	$\mathcal{U}_i^c$

for $i < m$.

Ex: $\mathcal{C} = \{0, 1, 2, 3, 4\}$
$\mathcal{U}_i = \{0, 3\}$
$V_i = \{0, 1, 2\}$

x	$a_i \cdot x$	$a_i^{-1} \cdot x$
0	V_i	$\mathcal{U}_i$
1	V_i^c	$\mathcal{U}_i$
2	V_i^c	$\mathcal{U}_i$
	V_i	$\mathcal{U}_i^c$
4	V_i^c	$\mathcal{U}_i^c$

Lemma 1 (Extension): The usual thing

pf ($\mathcal{L}$1): Inspection. ▨ ($\mathcal{L}$1)

Lemma 2 (Cycle): Any cycle in the Schreier graph associated with $\mathbb{F}_m \curvearrowright X$ admits a coloring satisfying the local rules.

pf ($\mathcal{L}$2): We take cases on the number of edge labels.

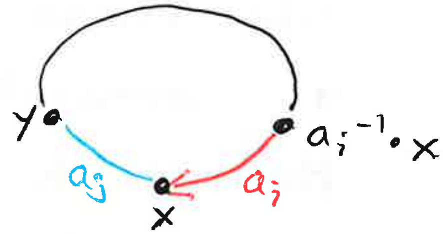
Case 1: All edges are labeled a_i. If $\mathcal{U}_i \cap V_i \neq \emptyset$, use any color in that intersection. If $\mathcal{U}_i \cap V_i = \emptyset$, then $V_i \subsetneq \mathcal{U}_i^c$ and hence $\mathcal{U}_i^c \cap V_i^c \neq \emptyset$ and any color in that intersection works. ✓

non-complementing

④ pf (Thm ⓑ, cont.)

pf ($\mathcal{L}2$, cont.)

Case 2: Two labels appear:



We will first color $a_i^{-1} \circ x$, which will force V_i or V_i^c on x.

Subcase 2.1: $y \bullet \xrightarrow{a_j} \bullet x$. If $V_i \cap V_j \neq \emptyset$ and $V_i \cap V_j^c \neq \emptyset$, first color $a_i^{-1} \circ x$ any color in $\mathcal{U}_i$. Work around to y, and we are guaranteed a color left for x.

If one of $V_i \cap V_j = \emptyset$ or $V_i \cap V_j^c = \emptyset$, it follows that $V_i^c \cap V_j \neq \emptyset$ and $V_i^c \cap V_j^c \neq \emptyset$. First color $a_i^{-1} \circ x$ any color in $\mathcal{U}_i^c$ and work around as before.

Subcase 2.2: $y \bullet \xleftarrow{a_j} \bullet x$. Repeat the prior argument with $\mathcal{U}_j$ in place of V_j.

[Note, this uses $\mathcal{U}_j \neq V_j$, which we set up.]

So Case 2 works out, too! ✓ ∎ ($\mathcal{L}2$)

By now we all know how to finish off the proof of Thm ⓑ. ∎ (Thm ⓑ).

Wednesday, April 10th

LCLs

Idea: A large amount of graph combinatorics is about tasks of the following sort: label the vertices, edges, etc. of a graph following some "rule" which can be checked locally

Eg:
- Proper coloring
- Perfect matching
- The problem we saw last time corresponding to a paradoxical decomp. of $\mathbb{F}_2$ with 4 pieces.

- We want a framework for discussing these sorts of tasks generally

- Let's say a graph is given by a pair (V, E), where $E \subseteq V \times V$ is
  - symmetric
  - irreflexive

$\uparrow$ vertices $\nwarrow$ edges

Def: (Σ)- • A labeled graph is a tuple (V, E, f) where (V, E) is a graph & $f : V \sqcup E \to \Sigma$ is a partial function.

- Isomorphisms of labeled graphs are required to preserve the labeling. I.e, we say $(V, E, f) \cong (V', E', f')$ if there is a graph isomorphism $\phi : V \to V'$ between (V, E) & (V', E') such that
  - $\forall v \in V$, $f(v) = f'(\phi(v))$
  - $\forall (u, v) \in E$, $f(u, v) = f'(\phi(u), \phi(v))$

$\uparrow$

Note $(\phi(u), \phi(v)) \in E'$ since ϕ is a graph iso.

Def: A *rooted graph* is a graph with a distinguished vertex. I.e., a tuple (V, E, x) where (V, E) is a graph and $x \in V$. Isomorphisms of rooted graphs are
↑ the "root"
required to preserve the root.

Def: A *rooted labeled graph* is ... you get the idea.

Def A *locally checkable labeling problem* (LCL) consists of the following data

- A finite set Σ ("labels")
- Some $r \in \mathbb{N}^+$
- A collection $\mathbb{P}$ of (isomorphism classes of) finite rooted Σ-labeled graphs.

Write $\Pi = (\Sigma, r, \mathbb{P})$.

Let $G = (V, E)$ be a locally finite graph. $f : V \cup E \to \Sigma$ is called a *Π-coloring* of G or a *solution* to Π if $\forall x \in V$,

$$\left(B_G(x,r),\ E \upharpoonright B_G(x,r),\ x,\ f \upharpoonright B_G(x,r) \right) \in \mathbb{P}.$$

radius r nbhd of x — induced subgraph — restriction of f to induced subgraph

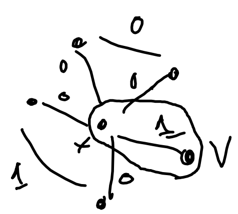
• So $\mathcal{P}$ lists the configurations we are "allowed" to see locally. It is a "rule" for Σ-labelings that can be checked by considering radius r neighborhoods.

<u>E.g</u>: • Proper q-coloring. $\Sigma = [q] := \{1, \dots, q\}$.

$r = 1$, $\mathcal{P} = \{ (V, E, x, f) \mid f : V \to q \ \& \ \forall (u,v) \in E, \ f(u) \neq f(v) \}$

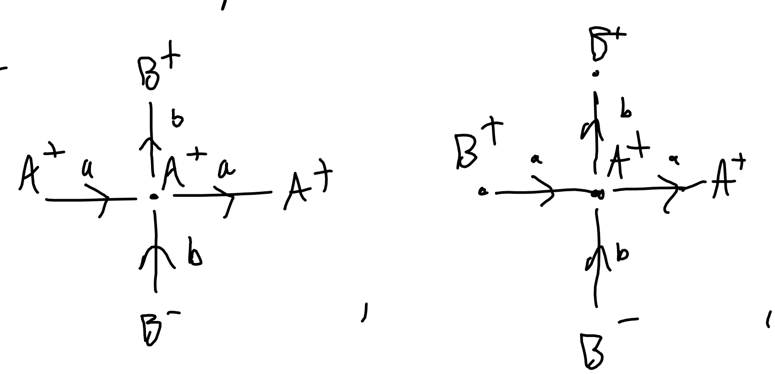
• Perfect matching. $\Sigma = \{0, 1\}$. $r = 1$.

$\mathcal{P} = \{ (V, E, x, f) \mid f : E \to \{0,1\}$ is symmetric & $\exists!\, v \in V$ s.t. $(x, v) \in E$ & $f(x, v) = 1 \}$



• Paradoxical decomp of $\mathbb{F}_2$ w/ 4 pieces: $\Sigma = \{A^+, A^-, B^+, B^-\}$.

$r = 1$

$\mathcal{P} = \Bigg\{$  , ...

⚠ • This last example was a slight lie, the objects we are labeling are not just graphs, they are graphs with extra structure, in this case corresponding to the generators of $\mathbb{F}_2$.

• We can extend the previous definitions to <u>situations like this</u> mutatis mutandis.

↳ <u>E.g.</u>: • hypergraphs • multigraphs
• weighted graphs, ...

Questions we like to ask about LCLs

- **Existence**: Does a graph G (maybe satisfying some hypotheses) have a Π-coloring?

- **Algorithms**: Is there a "fast" algorithm which, given a graph G (maybe satisfying some hypotheses) computes a Π-coloring?

- **Complexity**: How "hard" is it to decide whether a given graph G (maybe satisfying some hypotheses) has a Π-coloring.

... and more!

Compactness

Def: Let $\Pi = (\Sigma, r, P)$ be an LCL, $G = (V, E)$ a locally finite graph, $X \subseteq V$. We say a Σ labeling $f : V \sqcup E \to \Sigma$ is a *Π-coloring on X* if $\forall x \in X$,

$$(B_G(x,r),\ E \restriction B_G(x,r),\ x,\ f \restriction B_G(x,r)) \in P.$$

- So, a Π-coloring is the same as a Π-coloring on V.

Thm [AC]: Let Π, G be as above. TFAE.

1) G has a Π-coloring
2) For every finite $X \subseteq V$, $G \restriction B_G(X, r)$ has a Π-coloring on X.

pf (1) $\Rightarrow$ (2): Given a Π-coloring $f : V \sqcup E \to \Sigma$ of G & $X \subseteq V$, $f \restriction B_G(X, r)$ is a Π-coloring on X of $G \restriction B_G(X, r)$. $\square$

Remark: What is wrong with

(2'): For every finite $X \subseteq V$, $G \restriction X$ has a Π-coloring?

- Counterexample: Π = Perfect matching. Then (2') cannot hold for $|X|$ odd!

- In this case, (2) says: $\forall$ finite $X \subseteq V$, $G \restriction B(X,1)$ has a matching covering X.

- However, if Π has the property that $\forall$ Π-colorings f of G & $X \subseteq V$, $f \restriction X$ is a Π-coloring of $G \restriction X$, then

  (2) $\Leftrightarrow$ (2')

  E.g: Proper coloring.

The proof of (2) $\Rightarrow$ (1) will be like our proof of Hall's Theorem for infinite graphs
This shows the utility of "LCL" as a definition.
Without it, we'd have to repeat the proof for every problem we wanted to study on infinite graphs.

pf of (2) => (1):

- let $\mathcal{U}$ be an u.f. on $\mathrm{FIN}(V)$ extending the cone filter.
- $\forall X \in \mathrm{FIN}(V)$, choose a Π-coloring of $B_G(X, r)$ on X, say f_X.
- let $f = \lim_{X \to \mathcal{U}} f_X$. I.e. (since Σ is finite)

$\forall v \in V$, $f(v) = a \iff \underbrace{\{X \mid f_X(v) = a\}}_{=: S_v} \in \mathcal{U}$.

$\forall (u,v) \in E$, $f(u,v) = a \iff \underbrace{\{X \mid f_X(u,v) = a\}}_{=: S_{(u,v)}} \in \mathcal{U}$

- let $x \in V$. We want to check

$(*)$ $(B_G(x,r), E \restriction B_G(x,r), x, f \restriction B_G(x,r)) \in P$.

- f_X has this property $\forall X \ni x$

Now, $\{X \mid x \in X\} \cap \bigcap_{v \in B_G(x,r)} S_v \cap \bigcap_{(u,v) \in E \restriction B_G(x,r)} S_{(u,v)} \in \mathcal{U}$.

If X is in this set, $f = f_X$ on $B_G(x,r)$, & f_X satisfies $(*)$ since $x \in X$. $\square$

Some well known instances of this are ...

- M. Hall's Theorem (matchings in bipartite graphs)

- de Bruijn - Erdös Theorem (proper coloring):

The Lovasz Local Lemma

- This is another example of how having a general framework like "LCL" is useful.

<u>Idea</u>: Suppose we try to build a Π-coloring of a graph $G=(V,E)$ by assigning labels "randomly".

$\Pi = (\Sigma, r, \mathbb{P})$

↳ We'll probably want these quantities to be small.

- For $v \in V$, $p(v) := \mathbb{P}\big[$a uniform random Σ-labeling of $B_G(v,r)$ "violates $\mathbb{P}$"$\big]$

$p := \sup_v p(v)$.

- For $v \in V$, $d(v) := |B_G(v, 2r)|$. $d = \sup_v d(v)$.

Idea: p small $\rightsquigarrow$ high probability of local success

d small $\rightsquigarrow$ more independence.

__Thm__ (Erdős-Lovasz): If

$$e \cdot p \cdot d \leq 1,$$

Then G has a Π-coloring

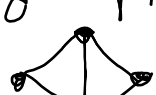
- This is nontrivial, but you can see it in any class on probabilistic combinatorics.

- To apply it, we often introduce an auxiliary LCL on on auxiliary graph.

__Eg__:

- let $G = (V,E)$ be a graph of max. degree Δ. $q \in \mathbb{N}$
- Want $f: V \to q$ with no monochromatic $\triangle$

  clearly an LCL

- Interested in optimal q as a fn of Δ
- If $G = K_{\Delta+1}$, need $q \sim \frac{\Delta}{2}$. On the other hand this clearly suffices by a greedy algorithm
- let's add the hypothesis that G has no 

<u>Remark</u>: This is secretly about coloring 3-uniform (linear) hypergraphs

- Auxiliary graph: let $\mathcal{T}$ be the set of triangles of G.

- let H be the bipartite graph on $V \sqcup \mathcal{T}$ with edge set $E \sqcup \mathcal{E}$.

(i.e., $(v, T) \in E_H$, where $T \in \mathcal{T}$, $v \in V$, iff $v \in T$).

- Note each $T \in \mathcal{T}$ has degree 3.

- Want $f: V \longrightarrow q$ s.t. $\forall T \in \mathcal{T}$, $N_H(T)$ is not monochromatic

  ← clearly an LCL with $r = 1$.

- For $T \in \mathcal{T}$, $p(T) = \frac{1}{q^2}$. For $v \in V$, $p(v) = 1$.

  So $p = \frac{1}{q^2}$.

- By our assumption, each $v \in V$ is in at most $\frac{\Delta}{2}$ triangles. So $d \leq \frac{3\Delta}{2}$.

- LLL will apply if $e \cdot \frac{1}{q^2} \cdot \frac{3\Delta}{2} \leq 1$.

$\Rightarrow$ get $q = O(\sqrt{\Delta})$.

- Even with our assumption on G, it is not clear how to improve on our greedy $O(\Delta)$ bound without the LLL.

↳ Indeed it is possible to formalize the idea that $O(\Delta)$ cannot be improved using "deterministic algorithms".