Hyperfiniteness and Borel combinatorics

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Abstract

We study the relationship between hyperfiniteness and problems in Borel graph combinatorics by adapting game-theoretic techniques introduced by Marks to the hyperfinite setting. We compute the possible Borel chromatic numbers and edge chromatic numbers of bounded degree acyclic hyperfinite Borel graphs and use this to answer a question of Kechris and Marks about the relationship between Borel chromatic number and measure chromatic number. We also show that for every $d > 1$ there is a d-regular acyclic hyperfinite Borel bipartite graph with no Borel perfect matching. These techniques also give examples of hyperfinite bounded degree Borel graphs for which the Borel local lemma fails, in contrast to the recent results of Csóka, Grabowski, Máthé, Pikhurko, and Tyros.

Related to the Borel Ruziewicz problem, we show there is a continuous paradoxical action of $(\mathbb{Z}/2\mathbb{Z})^{*3}$ on a Polish space that admits a finitely additive invariant Borel probability measure, but admits no countably additive invariant Borel probability measure. In the context of studying ultrafilters on the quotient space of equivalence relations under AD, we also construct an ultrafilter U on the quotient of E_0 which has surprising complexity. In particular, Martin's measure is Rudin-Kiesler reducible to U.

We end with a problem about whether every hyperfinite bounded degree Borel graph has a witness to its hyperfiniteness which is uniformly bounded below in size.

1 Introduction

In this paper, we investigate the relationship between hyperfiniteness and problems in Borel graph combinatorics. Recall that a (simple) **Borel graph** G on a standard Borel space X is a graph whose vertex set is X and whose (symmetric irreflexive) edge relation is Borel. G is said to be **hyperfinite** if it can be written as an increasing union of Borel graphs with finite connected components. Hyperfinite graphs can be thought of as

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the simplest graphs that can display nonclassical behavior in the setting of Borel graph combinatorics. This is made precise by the Glimm-Effros dichotomy.

A fundamental theorem of Kechris, Solecki, and Todorcevic [10, Proposition 4.6] states that every Borel graph G of degree at most d has Borel chromatic number $\chi_B(G) \leq d+1$, where the **Borel chromatic number** $\chi_B(G)$ of G is the least cardinality of a Polish space Y so that there is a Borel Y-coloring of G. This bound is optimal even for acyclic graphs since for every $d \geq 1$ and $k \in \{2, \dots, d+1\}$, there is an acyclic d-regular Borel graph with $\chi_B(G) = k$ by [12]. However, the graphs used to obtain this result are not hyperfinite, and Conley and Miller have asked whether every acyclic bounded degree hyperfinite Borel graph G has $\chi_B(G) \leq 3$ [8, Problem 5.17]. We answer this question in the negative. Essentially, we reprove all of the combinatorial results from [12] about Borel colorings, edge colorings, matchings, etc. for Borel graphs with the additional property of hyperfiniteness. Hence, among bounded degree Borel graphs, even hyperfinite graphs can achieve the maximum possible combinatorial complexity as measured by how hard they are to color and match in a Borel way. This is in contrast to the measure-theoretic context, where hyperfinite bounded degree Borel graphs are known to be much simpler to measurably color than arbitrary bounded degree Borel graphs. For instance, every acyclic hyperfinite bounded degree graph on a standard probability space (X, μ) has a μ-measurable 3-coloring [3, Theorem A].

To prove these results, we associate to each countable discrete group Γ a certain hyperfinite Borel action of Γ. We then show that an analogue of the central lemma of [12] is true for these actions. Recall that if a group Γ acts on a set X, the free part of this action is $\mathrm{Free}(X) = \{x \in X : \forall \gamma \in \Gamma(\gamma \neq 1 \implies \gamma \cdot x \neq x)\}$.

Definition 1.1. *Suppose Γ is a countable discrete group. Then Γ acts on Γ^Γ by*

$$(\gamma \cdot x)(\delta) = \gamma x(\gamma^{-1}\delta)$$

for every $x \in \Gamma^\Gamma$ and $\gamma, \delta \in \Gamma$. Let $H(\Gamma^\Gamma)$ be the set of $x \in \mathrm{Free}(\Gamma^\Gamma)$ such that x is a bijection and the permutation x induces on Γ has one orbit. Let E_Γ be the orbit equivalence relation of this action of Γ on $H(\Gamma^\Gamma)$. Let $w \colon H(\Gamma^\Gamma) \to H(\Gamma^\Gamma)$ be the Borel function defined by $w(x) = (x(1))^{-1} \cdot x$.

Note that this action, which we use throughout the paper, is *not* the standard shift action. It is a combination of the shift action, and pointwise multiplication.

An easy calculation shows that $w^n(x) = (x^n(1))^{-1} \cdot x$ for every $n \in \mathbb{Z}$. Thus, since the permutation x induces on Γ has a single orbit, w generates E_Γ, which is therefore hyperfinite [9, Theorem 6.6]:

Proposition 1.2. *E_Γ is hyperfinite.* □

We prove the following version of [12, Lemma 2.1] for these hyperfinite actions:

Lemma 1.3. *Suppose Γ and Δ are countable groups and $A \subseteq H((\Gamma * \Delta)^{\Gamma * \Delta})$. Then either*

1. *There is an injective Borel Γ-equivariant $f\colon H(\Gamma^\Gamma) \to H((\Gamma * \Delta)^{\Gamma * \Delta})$ with $\operatorname{ran}(f) \subseteq A$, or*
2. *There is an injective Borel Δ-equivariant $f\colon H(\Delta^\Delta) \to H((\Gamma * \Delta)^{\Gamma * \Delta})$ with $\operatorname{ran}(f) \cap A = \emptyset$.*

By applying this lemma the same way as [12] we obtain hyperfinite versions of all the theorems in that paper, as illustrated in Theorem 1.4. Recall that a graph is said to be d**-regular** if all of its vertices have degree d.

Given a Borel graph G, we denote by $\chi'_B(G)$ its **Borel edge chromatic number** (sometimes called its chromatic index).

Theorem 1.4.

1. *For every $d \geq 1$ and every $k \in \{2, \dots, d+1\}$ there is a d-regular acyclic hyperfinite Borel graph G with $\chi_B(G) = k$.*
2. *For every $d \geq 1$ and every $k \in \{d, \dots, 2d-1\}$ there is a d-regular acyclic Borel bipartite hyperfinite graph G such that $\chi'_B(G) = k$.*
3. *For every $d > 1$ there exists a d-regular acyclic hyperfinite Borel bipartite graph with no Borel perfect matching.*

Part (1) of this theorem negatively answers a question of Conley and Miller [8, Question 5.17].

By combining part (1) of Theorem 1.4 with the result from [3] that every bounded degree acyclic hyperfinite Borel graph G has $\chi_M(G) \leq 3$ we also obtain the following, answering a question of Kechris and Marks [8, Question 6.4] (see [8] for a definition of the measure chromatic number χ_M).

Corollary 1.5. *For every $d \geq 1$ and every $k \in \{2, \dots, d+1\}$, there is a d-regular acyclic Borel graph G with $\chi_B(G) = k$ and $\chi_M(G) = 3$.*

Csóka, Grabowski, Máthé, Pikhurko, and Tyros have recently proved a Borel version of the local lemma for bounded degree Borel graphs of uniformly subexponential growth [1]. We give the precise statement of their theorem in Section 3.2. One might hope that the Borel version of the local lemma is true for all hyperfinite bounded degree Borel graphs. (Note that every Borel graph of uniformly polynomial growth is hyperfinite by [7], and it is open whether every bounded degree Borel graph of uniformly subexponential growth is hyperfinite). We show that the Borel local lemma may fail for hyperfinite Borel graphs:

Theorem 1.6. *There is a hyperfinite bounded degree Borel graph G such that the Borel local lemma in the sense of [1] is false for G.*

The examples we give are graphs generated by free hyperfinite actions of $\mathbb{F}_{2n}$ for $n \geq 6$. The proof uses an idea of Kechris and Marks for constructing Borel graphs for which the local lemma fails using the results of [12].

Recall that if X is a Polish space and $\mathcal{B}(X)$ are the Borel subsets of X, then a **finitely additive Borel probability measure** on X is a finitely additive function $\mu\colon \mathcal{B}(X) \to [0,1]$ such that $\mu(X) = 1$. Lemma 1.3 can also be used to show the existence of certain exotic finitely additive invariant Borel probability measures. This is interesting in light of the Borel Ruziewicz problem: whether Lebesgue measure is the only finitely additive isometry-invariant probability measure defined on the Borel subsets of the n-sphere for $n \geq 2$ [17, Question 11.13]. By results of Margulis and Sullivan ($n \geq 4$) and Drinfeld ($n = 2, 3$) [5][11][15] it is known that any such measure not equal to Lebesgue measure must fail to be absolutely continuous with respect to Lebesgue measure. Furthermore, by a result of Dougherty and Foreman, it is also known that any such measure must be supported on a meager subset of X [6]. Generalizing this last result, Marks and Unger have shown that if any group Γ acts by Borel automorphisms paradoxically on a Polish space X, then any finitely additive Γ-invariant Borel measure on X must be supported on a meager subset of X [14].

It has been an open problem to find any paradoxical Borel action of a group on a standard Borel space that admits an "exotic" finitely additive invariant Borel probability measure (in particular, one that is not countably additive). We show the following, where the group $(\mathbb{Z}/2\mathbb{Z})^{*3}$ is a free product of three copies of $\mathbb{Z}/2\mathbb{Z}$:

Theorem 1.7 (AC)**.** *There is a continuous free action of $(\mathbb{Z}/2\mathbb{Z})^{*3}$ (which is hence paradoxical) on a Polish space so that this action admits a finitely additive invariant Borel probability measure, but does not admit any countably additive invariant Borel probability measure.*

Our techniques also allow us to construct interesting measures in a different context. Zapletal has suggested the problem of investigating the structure of ultrafilters on $2^{\mathbb{N}}/E_0$ under AD. Some examples of such ultrafilters are the ultrafilter U_L containing the Lebesgue conull E_0-invariant sets, and the ultrafilter U_C containing the comeager E_0-invariant sets. One can organize such ultrafilters by Rudin-Kiesler reducibility. Here, for example, it is open whether every ultrafilter on $2^{\mathbb{N}}/E_0$ is Rudin-Kiesler above U_L or U_C. (See [13, Section 4] for further discussion and a definition of Rudin-Kiesler reducibility). We show the existence of an ultrafilter on $2^{\mathbb{N}}/E_0$ which has surprising complexity:

Theorem 1.8 (AD)**.** *There is an ultrafilter U on $2^{\mathbb{N}}/E_0$ so that Martin measure on $2^{\mathbb{N}}/\equiv_T$ is Rudin-Kiesler reducible to U. In fact, the Rudin-Kiesler reduction can be chosen to be Borel.*

It is an open question whether there is a nontrivial ultrafilter on $2^{\mathbb{N}}/E_0$ that is Rudin-Kiesler reducible to Martin's ultrafilter. The existence of such an ultrafilter is equivalent to a negative answer to Thomas's question of whether Martin measure is strongly ergodic [16].

In the measure-theoretic and Baire category contexts, combinatorially simple colorings, matchings, etc. are often constructed by first finding suitably nice witnesses to hyperfiniteness (with for example points staying far from the boundaries of regions, etc.); see for example [3] for such constructions. From this perspective, one way of interpreting Theorem 1.4 is that such nice witnesses to hyperfiniteness do not generally exist in the Borel setting.

We pose an open question that is a very simple attempt to understand what global control we can exert over the witnesses to the hyperfiniteness of a bounded degree Borel graph:

Question 1.9. *Suppose G is a bounded degree hyperfinite Borel graph. Does there exist an increasing sequence $G_1 \subseteq G_2 \subseteq \ldots \subseteq G$ of Borel subgraphs of G such that*

1. *$G_1, G_2, \ldots$ witnesses that G is hyperfinite; i.e., for every n, each connected component of G_n is finite, and $\bigcup_n G_n = G$.*

2. *Every connected component of G_n has cardinality at least n.*

Let us call such a sequence of subgraphs an **everywhere-large witness to hyperfiniteness**. We find such witnesses to hyperfiniteness in some contexts:

Proposition 1.10. *Suppose G is a bounded degree hyperfinite Borel graph on a standard Borel space X. Then G admits an everywhere-large witness to hyperfiniteness modulo a nullset with respect to any Borel probability measure on X and a meager set with respect to any compatible Polish topology on X. Moreover, if G is generated by a single function, then it has an everywhere-large witness to hyperfiniteness.*

However, we conjecture that Question 1.9 has a negative answer in general.

1.1 Notation and conventions

Our notation is mostly standard, and largely follows [8]. Ideally, the reader will also have some familiarity with [12], since much of what follows builds on ideas from that paper.

2 The main lemma for $H\big((\Gamma * \Delta)^{\Gamma * \Delta}\big)$

Suppose Γ is a countable discrete group. Throughout this section we will often deal with partial functions from Γ to Γ. We may define the same action as in Definition 1.1 more generally for partial functions, and we begin by defining an associated partial order:

Definition 2.1. *Suppose x is a partial function from Γ to Γ. If $1 \in \mathrm{dom}(x)$, then define $w(x) = (x(1))^{-1} \cdot x$, otherwise $w(x)$ is undefined. Define a strict partial order $<_\Gamma$ on the space of partial functions from Γ to Γ by $x <_\Gamma y$ iff $\exists n > 0\ (w^n(x) = y)$.*

Since w generates E_Γ, if $x \in H(\Gamma^\Gamma)$ the restriction of $<_\Gamma$ to $[x]_{E_\Gamma}$ is isomorphic to $\mathbb{Z}$. More generally, if x is a partial injection from Γ to Γ, then $<_\Gamma$ is isomorphic to a subordering of $\mathbb{Z}$ on the orbit of x under the (partial) w-action defined above.

Next, we make some additional definitions related to $H(\Gamma^\Gamma)$:

Definition 2.2. *Suppose x is a finite partial injection from Γ to Γ. Say x has **one orbit** if for all $\gamma, \delta \in \mathrm{dom}(x)$ there is an $n \in \mathbb{Z}$ such that $x^n(\gamma) = \delta$. If x is nonempty, say x **begins at** γ if $\gamma \in \mathrm{dom}(x)$ but $\gamma \notin ran(x)$ and x **ends at** δ if $\delta \in \mathrm{ran}(x)$ but $\delta \notin \mathrm{dom}(x)$. If x is the empty function, then say that x begins and ends at 1.*

Note that the action of Γ on Γ^Γ in Definition 1.1 is chosen to interact well with the permutation that each bijection $x \in \Gamma^\Gamma$ induces on Γ. In particular, suppose y is a partial function from Γ to Γ and $R \subseteq \Gamma$ is an orbit of y. Then it is easy to check that for every $\gamma \in \Gamma$, γR is an orbit of the permutation induced by $\gamma \cdot y$.

We are now ready to prove our main lemma.

Proof of Lemma 1.3. We may assume that Γ and Δ are nontrivial. As in the proof of [12, Lemma 2.1], let $Y \subseteq (\Gamma * \Delta)^{\Gamma * \Delta}$ be the set of all $x \in (\Gamma * \Delta)^{\Gamma * \Delta}$ such that for all $\alpha \in \Gamma * \Delta$ and all nonidentity $\gamma \in \Gamma$ and $\delta \in \Delta$ we have $\gamma \cdot (\alpha^{-1} \cdot x) \neq (\alpha^{-1} \cdot x)$ and $\delta \cdot (\alpha^{-1} \cdot x) \neq (\alpha^{-1} \cdot x)$. Note that $\mathrm{Free}((\Gamma * \Delta)^{\Gamma * \Delta}) \subseteq Y$.

Every nonidentity word $\alpha \in \Gamma$ can be written as a reduced word of the form $\gamma_0 \delta_0 \gamma_1 \delta_1 \dots$ or $\delta_0 \gamma_0 \delta_1 \gamma_1 \dots$ where $\gamma_i \in \Gamma$ and $\delta_i \in \Delta$ are nonidentity elements. We let the **length** of $\gamma \in \Gamma * \Delta$ be its length as a reduced word. We say $\alpha \in \Gamma * \Delta$ is a **Γ-word** if it begins with an element of Γ as a reduced word, and a **Δ-word** if it begins with an element of Δ as a reduced word. So $\Gamma * \Delta$ is the disjoint union of the set of Γ-words, Δ-words, and the identity.

For each $B \subseteq Y$, define a game G_B for producing a (perhaps partial) injection y from $\Gamma * \Delta$ to $\Gamma * \Delta$ with one orbit. The players will alternate defining $y(\alpha)$ for finitely many $\alpha \in \Gamma * \Delta$ subject to the following rules:

- After each move of player I, y must be injective, have one orbit, and end at some Γ-word. After each move of player II, y must be injective, have one orbit, and end at some Δ-word.

- On each move of the game, if the current partial function y that has been defined before this move ends at $\xi \in \Gamma * \Delta$, then as part of the current move, the current player must define $y(\xi)$.

- In addition to the requirement of the previous rule, on each of their moves player I may also define $y(\alpha)$ for arbitrarily many α that are Γ-words. On each of their moves player II may also define $y(\alpha)$ for arbitrarily many nonidentity α that are Δ-words.

- At the end of the game, if y is not a total function, then II loses if and only if among the $\alpha \notin \mathrm{dom}(y)$ that are of minimal length, there is some α which is a Δ-word or the identity. If y is total but $y \notin Y$, then II loses if and only if among the α witnessing $y \notin Y$ of minimal length, there is some α which is a Δ-word, or $\alpha = 1$ witnesses $\alpha \notin Y$ via the fact that $\delta \cdot y = y$ for some nonidentity $\delta \in \Delta$. Finally, if y is total and $y \in Y$, then I wins if y is not in B.

For example, on the first turn of the game (where our current version of y is the empty function which by definition ends at 1), player I must define $y(1)$, and then may also define y on finitely many other Γ-words. The resulting finite partial function y must be injective, have one orbit, and end at some Γ-word.

Note that since y has a single orbit after each turn of the game, it will also have a single orbit at the end of the game.

Let $E_\Gamma^{\Gamma\star\Delta}$ denote the subequivalence relation of $E_{\Gamma*\Delta}$ given by the orbits of the subgroup $\Gamma \leq \Gamma*\Delta$, and likewise define $E_\Delta^{\Gamma*\Delta}$. As these equivalence relations are everywhere independent, by [12, Lemma 2.3] we may find a Borel subset C of $Y \setminus \mathrm{Free}((\Gamma * \Delta)^{\Gamma*\Delta})$ such that C meets every $E_\Delta^{\Gamma*\Delta}$-class on $Y \setminus \mathrm{Free}((\Gamma * \Delta)^{\Gamma*\Delta})$ and the complement of C meets every $E_\Gamma^{\Gamma*\Delta}$-class on $Y \setminus \mathrm{Free}((\Gamma * \Delta)^{\Gamma*\Delta})$.

By Borel determinacy, one of the two players must have a winning strategy in the game associated to the set $B = A \cup C$. Suppose player I has a winning strategy, and fix such a strategy. We will construct an injective Borel Δ-equivariant function $f\colon H(\Delta^\Delta) \to H((\Gamma * \Delta)^{\Gamma*\Delta})$ with $\mathrm{ran}(f) \cap A = \emptyset$. We will define f so that for all $x \in H(\Delta^\Delta)$, $f(x)$ is a winning outcome of player I's winning strategy in the game and so $f(x) \notin A$. We will ensure that f is injective by enforcing that $x <_\Delta y$ if and only if $f(x) <_{\Gamma*\Delta} f(y)$.

Let $E_0 \subseteq E_1 \subseteq E_2 \subseteq \ldots$ be finite Borel equivalence relations that witness the hyperfiniteness of E_Δ. We may assume that E_0 is the equality relation and also that every E_n-class is an interval in the ordering $<_\Delta$ by passing instead to the relations E_n' where $x\ E_n'\ y$ if the $<_\Delta$-interval from x to y lies inside $[x]_{E_n}$. For each $x \in H(\Delta^\Delta)$, let E_n^x be the equivalence relation on $\Gamma * \Delta$ where $\alpha_0\ E_n^x\ \alpha_1$ if and only if $\delta_0^{-1} \cdot x\ E_n\ \delta_1^{-1} \cdot x$ where δ_0, δ_1 are the unique elements of Δ such that α_0 and α_1 can be expressed as $\alpha_0 = \delta_0\beta_0$ and

$\alpha_1 = \delta_1\beta_1$ where β_0 and β_1 are Γ-words or the identity. Note that in general the classes of E_n^x will not be finite. However, for each E_n^x-class $[\alpha]_{E_n^x}$, $[\alpha]_{E_n^x} \cap \Delta$ will be finite.

Fix $x \in H(\Delta^\Delta)$. We will define $f(x)$ via a construction that takes countably many steps. In this construction, for each $\delta \in \Delta$ we will play an instance of the game whose outcome will be equal to $\delta^{-1} \cdot f(x)$. At step 0 of our construction, let player I move in the game associated to each $\delta \in \Delta$ using their winning strategy.

The only choices we will be making in our construction (other than keeping the play of the games consistent with each other) will be connecting up the orbits of the finite partial functions constructed in each of the games so that everything is eventually connected. We will do this using the witness to the hyperfiniteness of E_Δ and in our role as player II in all the games.

Inductively assume that after step n of our construction, for every $\delta \in \Delta$,

1. $f(x) \restriction [\delta]_{E_n^x}$ is a finite partial injection which has one orbit.
2. If $\delta_0, \dots, \delta_k$ enumerates the elements of $[\delta]_{E_n^x} \cap \Delta$ in the order so that $\delta_0^{-1} \cdot x <_\Delta \dots <_\Delta \delta_k^{-1} \cdot x$, then $f(x) \restriction [\delta]_{E_n^x}$ ends at a group element of the form $\delta_k\beta$, where β is a Γ-word.
3. For every $\delta_i, \delta_j \in [\delta]_{E_n^x} \cap \Delta$ such that $\delta_i^{-1} \cdot x \; E_n \; \delta_j^{-1} \cdot x$, we have that $\delta_i^{-1} \cdot x <_\Delta \delta_j^{-1} \cdot x$ if and only if $\delta_i^{-1} \cdot f(x) <_{\Gamma * \Delta} \delta_j^{-1} \cdot f(x)$.
4. In the game associated to δ, the last move was made by player I (using their strategy). The current finite partial function defined in the game associated to δ includes every value of $(\delta^{-1} \cdot f(x))(\alpha)$ we defined during the previous step of the construction, where α is a Γ-word or the identity.
5. If $n > 0$ and $[\delta^{-1} \cdot x]_{E_n}$ contains more than one element, then in the game associated to δ, during step n we played a move for both player II and player I (in that order), and we defined $(\delta^{-1} \cdot f(x))(\alpha)$ for every Δ-word α contained in $\delta^{-1}[\delta]_{E_{n-1}^x}$ that had been already defined in step $n-1$.

We now describe step $n+1$ of our construction. For each $\delta \in \Delta$, let $\delta_0, \dots, \delta_k$ enumerate the elements of $[\delta]_{E_{n+1}^x} \cap \Delta$ in the order so that $\delta_0^{-1} \cdot x <_\Delta \dots <_\Delta \delta_k^{-1} \cdot x$. Each E_{n+1}^x-class $[\delta]_{E_{n+1}^x}$ contains finitely many E_n^x-classes $[\beta_0]_{E_n^x}, \dots, [\beta_m]_{E_n^x}$. For every $i < k$ so that δ_i and δ_{i+1} are in different E_n^x-classes, $f(x) \restriction [\delta_i]_{E_n^x}$ ends at some group element ξ_i and $f(x) \restriction [\delta_{i+1}]_{E_n^x}$ begins at some group element α_{i+1}. In this case, define $f(x)(\xi_i) = \alpha_{i+1}$ so that

$$w(\xi_i^{-1} \cdot f(x)) = \alpha_{i+1}^{-1} \cdot f(x).$$

After doing this, note that parts (1), (2), and (3) of our induction hypothesis are true for $n+1$. However, we are not yet finished with our definition of $f(x)$ at step $n+1$ so these properties still need to be checked after we are finished.

Assume now that $k \geq 1$ so there are at least two elements of $[\delta]_{E^x_{n+1}} \cap \Delta$ (else we are finished with our definition of $f(x) \restriction [\delta]_{E^x_{n+1}}$ and part (5) of our induction hypothesis is also true). For each δ_i, one at a time and in order, we will first move for player II in the game associated to δ_i, and then let the strategy for player I move. Before we begin this process, note that by the previous paragraph, $f(x) \restriction [\delta]_{E^x_{n+1}}$ has one orbit and ends at a word of the form $\delta_k\beta$ where β is a Γ-word. Indeed, inductively, before we consider the game associated to δ_i, it will be the case that $f(x) \restriction [\delta]_{E^x_{n+1}}$ has one orbit and ends at a word of the form $\delta_{i'}\beta$ where $i' = i - 1 \bmod k+1$ and β is a Γ-word.

So for the game associated to each δ_i, we make a move for player II by playing every value of $\delta_i^{-1} \cdot (f(x) \restriction [\delta]_{E^x_{n+1}})$ that has already been defined but not yet played in the game. Playing these values will be consistent with the rules of the game by our induction hypothesis. Now we let player I's strategy move in the game to define additional values of $\delta_i^{-1} \cdot f(x)$. Note that after these two moves, $f(x) \restriction [\delta]_{E^x_{n+1}}$ will have one orbit and end at a group element of the form $\delta_i\beta$ where β is a Γ-word by the rules of the game. After doing this for each of $\delta_0, \ldots, \delta_k \in [\delta]_{E^x_{n+1}} \cap \Delta$ in order, we are finished with step $n+1$ of the construction. Verifying that our inductive hypotheses are satisfied is easy, and we are now done with the construction of $f(x)$. Verifying that f is Borel and Δ-equivariant is straightforward.

Suppose $x \in H(\Delta^\Delta)$. By part (5) of our induction hypothesis, since $[x]_{E_n}$ has at least two elements for sufficiently large n, we will play infinitely many moves in the game associated to $\delta = 1$ (so the game finishes), and the outcome of the game will be equal to the value of $f(x)$ defined by our construction by (4) and (5). The only thing that remains is to verify that $f(x)$ is total and $f(x) \in H((\Gamma * \Delta)^{\Gamma * \Delta})$ for every $x \in H(\Delta^\Delta)$.

To begin, we prove that $f(x)$ is total for every $x \in H(\Delta^\Delta)$. Note that by the definition of step 0 of our construction $f(x)(1)$ is defined for all $x \in H(\Delta^\Delta)$ (since on the first move player I must define $y(1)$). Then inductively, supposing $f(x)(\alpha)$ is defined on all words α of length n, if $\alpha = \delta\beta$ is any Δ-word of length $n+1$ where $\delta \in \Delta$ and β is a Γ-word or the identity, then $f(x)(\alpha) = \delta((\delta^{-1} \cdot f(x))(\beta)) = \delta(f(\delta^{-1} \cdot x)(\beta))$ must be defined by our induction hypothesis. Thus, $f(x)(\alpha)$ must be defined for all Δ-words of length $n+1$, and thus also defined for all Γ-words of length $n+1$ (else player I loses).

Similarly, the same inductive idea shows that $f(x) \in Y$ for every $x \in H(\Delta^\Delta)$. (Alternatively, copy the penultimate paragraph of the proof of [12, Lemma 2.1]).

Now since we have proved that $f(x) \in Y$ for every $x \in H(\Delta^\Delta)$, we must have that $f(x) \in \mathrm{Free}((\Gamma * \Delta)^{\Gamma * \Delta})$ since C meets every Δ-invariant set in $Y \setminus \mathrm{Free}((\Gamma * \Delta)^{\Gamma * \Delta})$ by the definition of C. Thus, $f(x) \in H((\Gamma * \Delta)^{\Gamma * \Delta})$ for every $x \in H(\Delta^\Delta)$. Finally, $f(x) \notin A$ since $f(x)$ is a winning outcome of player I's strategy in $G_{A \cup C}$.

This completes the proof in the case that player I has a winning strategy in $G_{A \cup C}$. The proof in the case that player II has a winning strategy is very similar. $\square$

Various bells and whistles can be added onto the above lemma. For example, the generalization of this lemma to countable free products is also true:

Lemma 2.3. *Suppose $I \in \{2, 3, \dots, \mathbb{N}\}$, $\{\Gamma_i\}_{i\in I}$ is a collection of countably many countable discrete groups, and $\{A_i\}_{i\in I}$ is a Borel partition of $H((*_i \Gamma_i)^{*_i \Gamma_i})$. Then there exists some $j \in I$ and an injective Borel Γ_j-equivariant function $f\colon H({\Gamma_j}^{\Gamma_j}) \to H((*_i \Gamma_i)^{*_i \Gamma_i})$ so that $\mathrm{ran}(f) \subseteq A_j$.*

Proof Sketch. This lemma can be proved in a roughly identical way to the way [13, Theorem 1.2] generalizes [12, Lemma 2.1]. Similarly to that proof, either player I has a winning strategy in the game above associated to the complement of A_0, viewing $*_i \Gamma_i$ as a free product of the two groups $\Gamma = *_{i\neq 0} \Gamma_i$ and $\Delta = \Gamma_0$, or else there is some $j > 0$ so that player II has a winning strategy in the game associated to A_j, viewing $*_i \Gamma_i$ as a free product of the two groups $\Gamma = \Gamma_j$ and $\Delta = *_{i\neq j} \Gamma_i$. (This is because if not, playing winning strategies for the other players in all these games simultaneously would yield some $y \in H((*_i \Gamma_i)^{*_i \Gamma_i})$ not in any A_i, contradicting the fact that $\{A_i\}_{i\in I}$ partitions this set). One then copies the construction from the proof of Lemma 1.3 above. □

In a different direction, we could work instead with a universal free hyperfinite action of Γ (in the sense of [7, Section 2.5] and [2]) instead of the action we have used on $H(\Gamma^\Gamma)$. Using this universal action, Lemma 1.3 would remain true using a very similar proof.

There is a different way of viewing the action of Γ on $H(\Gamma^\Gamma)$:

Remark 2.4. *Suppose Γ and Δ are countable discrete groups. Then Γ acts on Δ^Γ via*

$$\gamma \cdot x(\gamma') = x(\gamma^{-1})^{-1} x(\gamma^{-1}\gamma').$$

Let $H'(\Delta^\Gamma)$ be the set of $x \in \Delta^\Gamma$ such that $x(1_\Gamma) = 1_\Delta$, x is a bijection, and $x \in \mathrm{Free}(\Delta^\Gamma)$ and $x^{-1} \in \mathrm{Free}(\Gamma^\Delta)$. Let $E_{\Gamma,\Delta}$ be the orbit equivalence relation of the action of Γ on $H'(\Delta^\Gamma)$. Then it is easy to see that $E_{\Gamma,\Delta}$ and $E_{\Delta,\Gamma}$ are Borel isomorphic via the map sending $x \in H'(\Delta^\Gamma)$ to $x^{-1} \in H'(\Gamma^\Delta)$. Hence, $E_{\Gamma,\Delta}$ is generated by free actions of both Γ and Δ. If Γ is a countably infinite group, the action of Γ on $H(\Gamma^\Gamma)$ is Borel isomorphic to the action of Γ on $H'(\mathbb{Z}^\Gamma)$ via the equivariant map sending $x \in H(\Gamma^\Gamma)$ to $f(x) \in H'(\mathbb{Z}^\Gamma)$ where $f(x)(\gamma)$ is the unique $n \in \mathbb{Z}$ such that $(x^n(1))^{-1} = \gamma^{-1}$.

3 Corollaries

3.1 Colorings and matchings

The proof of Theorem 1.4 is identical to the proofs in [12], simply replacing $\mathrm{Free}(\mathbb{N}^\Gamma)$ with $H(\Gamma^\Gamma)$ in the definition of $G(\Gamma, \mathbb{N})$ in that paper, and the proofs of Theorems 1.3, 1.4, and 1.5 of [12].

For instance, let us see how to use Lemma 1.3 to construct acyclic d-regular hyperfinite graphs with Borel chromatic number $d+1$ in analogy with [12, Theorem 1.2]. For a group Γ with fixed finite generating set, let G_Γ be the graph on $H(\Gamma^\Gamma)$ rendering two points adjacent if a generator sends one to the other.

Theorem 3.1. *Suppose that Γ and Δ are groups with fixed finite generating sets, and equip $\Gamma * \Delta$ with the union of these generating sets. Then*

$$\chi_B(G_{\Gamma*\Delta}) \geq \chi_B(G_\Gamma) + \chi_B(G_\Delta) - 1.$$

Proof. It suffices to show that if G_Γ has no Borel m-coloring and if G_Δ has no Borel n-coloring, then $G_{\Gamma*\Delta}$ has no Borel $(m+n)$-coloring. Towards a contradiction, suppose such a Borel coloring $c\colon H((\Gamma*\Delta)^{\Gamma*\Delta}) \to \{0,\dots,m+n-1\}$ exists, and put $A = \{x \in H((\Gamma*\Delta)^{\Gamma*\Delta}) : c(x) < m\}$. Applying Lemma 1.3 to the set A, we obtain either a Γ-equivariant Borel $f\colon H(\Gamma^\Gamma) \to A$ or a Δ-equivariant Borel $f\colon H(\Delta^\Delta) \to H((\Gamma*\Delta)^{\Gamma*\Delta}) \setminus A$. In the former case, $c \circ f$ is a Borel m-coloring of G_Γ, while in the latter case $c \circ f$ is a Borel n-coloring of G_Δ. Both are contradictions. □

An easy induction now shows that $G_{\mathbb{F}_d}$ has Borel chromatic number $2d+1$.

3.2 The local lemma

Let us begin by recalling the Borel version of the Lovász local lemma in [1]. Suppose G is a Borel graph on X, but where we allow loops so that we do not assume G is irreflexive. We use the notation $G(x) = \{y \in X : x\ G\ y\}$ to denote the neighborhood of x. We also let $G^{\leq 2}$ be the Borel graph on X where $x\ G^{\leq 2}\ y$ if $d_G(x,y) \leq 2$. (This graph is called $\mathrm{Rel}(G)$ in [1]).

Suppose $b \geq 1$. Then a **Borel b-local rule** R for G is a Borel function whose domain is X and where for each $x \in X$, $R(x)$ is a set of functions from $G(x)$ to b. Say that $f\colon X \to b$ **satisfies** R if $f \restriction G(x) \in R(x)$ for every $x \in X$. Define $p_R(x)$ to be the probability that a random function from $G(x) \to b$ is not in $R(x)$. So $p_R(x) = 1 - \frac{|R(x)|}{b^{|G(x)|}}$.

Theorem 3.2 ([1, Theorem 1.3])**.** *Suppose G is a Borel graph on X so that $G^{\leq 2}$ has uniformly subexponential growth and degree bounded by Δ. If R is a Borel b-local rule for G such that $p_R(x) < \frac{1}{e\Delta}$ for all $x \in X$, then there exists a Borel function $f\colon X \to b$ which satisfies R.*

Theorem 1.6 clearly follows from the following:

Lemma 3.3. *Suppose $n \geq 6$, and let S be a free symmetric generating set for $\mathbb{F}_{2n}$, which acts on the space $H(\mathbb{F}_{2n}^{\mathbb{F}_{2n}})$ via Definition 1.1. Let G be the graph on $H(\mathbb{F}_{2n}^{\mathbb{F}_{2n}})$ where $x\ G\ y$ if there exists $\gamma \in S \cup \{1\}$ such that $\gamma \cdot x = y$. Then there exists a Borel 2-local rule R for G such that $p_R(x) < \frac{1}{e\Delta}$ for all x, however there is no Borel function f which satisfies R.*

Proof. Partition the generating set S into two symmetric sets S_0 and S_1 so that S_0 and S_1 generate two isomorphic copies Γ_0 and Γ_1 of $\mathbb{F}_n$, where $\Gamma_0 * \Gamma_1 = \mathbb{F}_{2n}$.

Now let R be the local rule where $f \in R(x)$ if $f(x) = 0$ implies there is a $\gamma \in S_0$ such that $f(\gamma \cdot x) = 1$ and $f(x) = 1$ implies there is a $\gamma \in S_1$ such that $f(\gamma \cdot x) = 0$. By Lemma 1.3, for every Borel function $f \colon H(\mathbb{F}_{2n}^{\mathbb{F}_{2n}}) \to 2$, by viewing f as the characteristic function of some set, there is either an entire Γ_0-orbit whose image is $\{0\}$ or a Γ_1-orbit whose image is $\{1\}$. Hence, there can be no Borel function f satisfying R.

However, for every x, we have $p_R(x) = 1/2^{2n}$ and the graph $G^{\leq 2}$ has degree $1+(4n)^2$. To finish, note that

$$\frac{1}{2^{2n}} < \frac{1}{e(1+(4n)^2)}$$

for $n \geq 6$. □

3.3 An exotic finitely additive invariant Borel measure

Proof of Theorem 1.7. Consider the action of $\Gamma = (\mathbb{Z}/2\mathbb{Z})^{*3} = \langle a, b, c : a^2 = b^2 = c^2 = 1 \rangle$ on $X = H(\Gamma^\Gamma)$. X is a Borel subset of the Polish space Γ^Γ, and so by changing topology, we may give a Polish topology to X that has the same Borel sets but so that the action of Γ on X is continuous. Since Γ is nonamenable and a free probability measure preserving action of a nonamenable group on a standard probability space (X, μ) cannot be μ-hyperfinite, this action does not admit any countably additive invariant Borel probability measure.

Let $\mathcal{B}(X)$ be the σ-algebra of Borel subsets of X. Now $\mathcal{B}(X)$ is invariant under the action of Γ and hence by [17, Theorem 9.1] there is a finitely additive Γ-invariant probability measure $\nu \colon \mathcal{B}(X) \to [0, 1]$ with $\mu(X) = 1$ if and only if for all $n \in \mathbb{N}$, $n+1$ copies of X are not Borel equidecomposable with a subset of n copies of X. So it suffices to show that for all $n \in \mathbb{N}$ and finite sets $S \subseteq \Gamma$ there do not exist $n+1$ Borel functions $f_0, \ldots, f_n$ such that for all $x \in X$ and $i \leq n$, $f_i(x) = \gamma \cdot x$ for some $\gamma \in S$ and for every $y \in X$, $\{(z, i) : f_i(z) = y\}$ has at most n elements. For notational convenience we will assume that $1 \notin S$.

Suppose for a contradiction that there did exist such a finite set $S \subseteq \Gamma$ and Borel functions $f_0, \ldots, f_n$ as above. Let G be the Borel graph on X where $x \; G \; y$ if there is a generator $\gamma \in \{a, b, c\}$ such that $\gamma \cdot x = y$. Note that G is acyclic (since the action of Γ is free, and the Cayley graph of Γ with respect to its generators is acyclic), so there is a unique path between any two points in G in the same connected component.

We say that a function $h \colon X \to X$ is an **antimatching** of G if for all $x \in X$, $x \; G \; h(x)$ and $h^2(x) \neq x$. We will construct a Borel antimatching assuming the existence of these functions $f_0, \ldots, f_n$. Then we obtain the desired contradiction by showing Lemma 1.3 precludes the existence of such an antimatching.

More precisely, let g be the Borel function which associates to each directed edge (x, y) of G the number of pairs of the form (z, i) where $z \in X$ and $i \leq n$ and the unique G-path from z to $f_i(z)$ includes the directed edge (x, y). Note that since S is finite and G has bounded degree, g is bounded above. Now we claim that for every $x \in X$, there is some neighbor y of x such that $g((x, y)) > g((y, x))$. To see this, consider the quantity

$$\sum_{\{y: yGx\}} g((x, y)) - g((y, x)).$$

Take a pair (z, i) that contributes to this sum because the path from z to $f_i(z)$ includes x. If $x \neq z$ and $x \neq f_i(z)$, then this path has one edge directed towards x and one away from x, so the net contribution to the sum is zero. If $z = x$, then there are exactly $n + 1$ pairs of the form (x, i), and so $n + 1$ edges directed away from x. However, if $f_i(z) = x$, then by assumption there are at most n pairs of the form (z, i) such that $f_i(z) = x$. Hence the total sum is positive, and so there must be some y such that $g((x, y)) - g((y, x))$ is positive.

Let $<$ be a Borel linear ordering of X. We now define a Borel function $h\colon X \to X$ by setting $h(x) = y$ where y is the $<$-least neighbor of x such that $g((x, y)) - g((y, x)) > 0$. Note that $h^2(x) \neq x$ for every x. Now let $A_\gamma = \{x : h(x) = \gamma \cdot x\}$ for $\gamma \in \{a, b, c\}$ so these sets partition $H(\Gamma^\Gamma)$. Finally, by applying Lemma 1.3 twice (or Lemma 2.3 once), there must be some $\gamma \in \{a, b, c\}$ so that if $\langle\gamma\rangle$ is the subgroup generated by γ, there is a Borel injective $\langle\gamma\rangle$-equivariant function $f\colon H(\langle\gamma\rangle^{\langle\gamma\rangle}) \to A_\gamma \subseteq H(\Gamma^\Gamma)$. But any $y \in \mathrm{ran}(f)$ has $h(y) = \gamma \cdot y$ and $h(\gamma \cdot y) = y$, since both y and $\gamma \cdot y$ are in A_γ. This contradicts the fact that $h^2(x) \neq x$ for all $x \in X$. □

3.4 An ultrafilter on $\mathbb{R}/E_0$

Proof of Theorem 1.8. Instead of E_0, we will construct the ultrafilter U on the equivalence relation $E_{\mathbb{F}_2}$ on $H(\mathbb{F}_2^{\mathbb{F}_2})$. Since $E_{\mathbb{F}_2}$ is hyperfinite, by [4] it is Borel bireducible with E_0 restricted to some Borel subset of $2^{\mathbb{N}}$. Hence our construction will also yields an ultrafilter on the quotient of E_0.

Fix C as in the definition of the proof of Lemma 1.3 where $\Gamma = \Delta = \mathbb{Z}$ so $\Gamma * \Delta = \mathbb{F}_2$. Given an $E_{\mathbb{F}_2}$-invariant subset $A \subseteq H(\mathbb{F}_2^{\mathbb{F}_2})$, we define $A \in U$ if and only if player II wins the game $G_{A \cup C}$ defined in the proof of Lemma 1.3. The proof that this defines an ultrafilter is identical to the proof of [13, Lemma 4.9].

It is trivial to see that given a winning strategy for player II in the game G, then there are plays of the game using this winning strategy of every Turing degree above the Turing degree of this strategy. Hence, given any subset of $A \subseteq H(\mathbb{F}_2^{\mathbb{F}_2})$ which is Turing invariant, A is in the ultrafilter U if and only if A contains a Turing cone. Thus, U is Rudin-Kiesler above Martin's measure, as witnessed by the identity function (which is a homomorphism from $E_{\mathbb{F}_2} \restriction H({\mathbb{F}_2}^{\mathbb{F}_2})$ to $\equiv_T$ on ${\mathbb{F}_2}^{\mathbb{F}_2}$, which we can identify with $\mathbb{N}^{\mathbb{N}}$). □

4 Lower bounds on component size in witnesses to hyperfiniteness

In this section, we address Question 1.9: Given a bounded-degree hyperfinite Borel graph G, does there exist an increasing sequence $G_1 \subseteq G_2 \subseteq \ldots \subseteq G$ of Borel subgraphs of G witnessing its hyperfiniteness such that every connected component of G_n has cardinality at least n. Such a sequence is called an everywhere-large witness to hyperfiniteness. We provide a positive answer in a handful of contexts.

We begin with a lemma about forward recurrent sets for bounded-to-one Borel functions. Given a function $f\colon X \to X$, recall that the graph G_f on X renders distinct points x and y adjacent if $y = f(x)$ or vice-versa.

Lemma 4.1. *Suppose $f\colon X \to X$ is a bounded-to-one Borel function. Then there is an G_f-independent Borel set $A \subseteq X$ such that for all $x \in X$, one of $x, f(x), f^2(x), f^3(x)$ is in A.*

Proof. By [10, Corollary 5.3], there is a Borel 3-coloring $c\colon X \to 3$ of G_f. Let A be the set of $x \in X$ such that either $c(x) = 0 \wedge c(f(x)) = 2$ or $c(x) = 1 \wedge c(f(x)) = 2$ or $c(x) = 0 \wedge c(f(x)) = 1 \wedge c(f^2(x)) = 0$. It is easy to see that A is G_f-independent. Given $x \in X$, the sequence $c(x), c(f(x)), c(f^2(x)), \ldots$ begins with either

- 01, which continues 010 so $x \in A$ or 012 so $f(x) \in A$.
- 02, so $x \in A$.
- 10, which continues 102 so $f(x) \in A$ or 1012 so $f^2(x) \in A$ or 1010 so $f(x) \in A$.
- 12, so $x \in A$.
- 20, which continues 202 so $f(x) \in A$ or 2010 so $f(x) \in A$ or 2012 $f^2(x) \in A$.
- 21, which continues 212 so $f(x) \in A$ or 2102 so $f^2(x) \in A$ or 21010 so $f^2(x) \in A$ or 21012 so $f^3(x) \in A$.

Thus A is as desired. □

Lemma 4.2. *Suppose G is a bounded degree Borel graph on X, and $A \subseteq X$ is a Borel set such that every connected component of G contains exactly one connected component of $G \restriction A$. If $G \restriction A$ admits an everywhere-large witness to hyperfiniteness, so does G.*

Proof. Fix a Borel linear ordering of X. We will begin by defining a graph H with the same connectedness relation as G. Let $H' \subseteq G$ be the graph on X where $x\ H'\ y$ if $x\ G\ y$ and the edge $\{x, y\}$ is contained in the lex-least path from either x to A or y to A. Using

properties of the lex-least ordering, it is easy to see that H' is acyclic, and each connected component of H' contains exactly one element of A. Let H be the union of H' and $G \restriction A$.

Let $H'' \subseteq H'$ be the graph where $x \ H'' \ y$ if $x \ H' \ y$ and there are only finitely many z so that the lex-least path from z to A includes the edge $\{x, y\}$. By König's lemma, all the connected components of H'' are finite. Let $m(\{x, y\}) = \max(d(x, A), d(y, A))$.

Now let $G'_0 \subseteq G'_1 \subseteq \dots$ be the hypothesized witness to the hyperfiniteness of $G \restriction A$. We can define a witness $H_0 \subseteq H_1 \subseteq \dots$ to the hyperfiniteness of H by setting $x \ H_i \ y$ if (i) $x \ G'_i \ y$, or (ii) $x \ H' \ y$ and $2^i \nmid m(\{x, y\})$, or (iii) $x \ H'' \ y$. Clearly $H_0 \subseteq H_1 \subseteq \dots$. We will check that each connected component C of H_i is finite and contains at least i elements.

First, suppose C contains no element of A. Then C contains a unique element x_0 that is closest to A since H' is acyclic. Let x_1 be the unique neighbor of x_0 such that $d(x_1, A) < d(x_0, A)$. Then x_0 is not H_i-adjacent to x_1 by definition, and so x_0 and x_1 are not H''-adjacent by (iii). By (ii), we therefore must have that $d(x_0, A) = k2^i$ for some k. By (iii) there must be infinitely many z such that the lex-least path from z to A includes the edge $\{x_0, x_1\}$. So by König's lemma, there is some H' path of length $2^i - 1$ from x_0 to some point z where $d(z, A) > d(x_0, A)$ so that this path does not use any H''-edges. Thus, this path lies inside H_i, which therefore has at least 2^i many elements. This suffices since $2^i \geq i$. Now if $x, y \in C$ and $x \ H_i \ y$ but x and y are not H''-adjacent, then we see that $d(x, x_0) < 2^i$ and $d(y, x_0) < 2^i$ by (ii). Thus, there are finitely many edges in $H_i \restriction C$ coming from condition (ii), and so $H_i \restriction C$ is the union of these edges with the finitely many H''-components that are incident to them by condition (iii). So C is finite.

Second, suppose C does contain an element of A. Then $C \cap A$ is a connected component of G'_i since each H'-component contains only one element of A. Thus, since each G'_i component has at least i many elements, C also has at least i many elements. Now if $x, y \in C$ and $x \ H_i \ y$ but x and y are not H''-adjacent, then $m(\{x, y\}) < 2^i$ by (ii). Hence, $H_i \restriction C$ contains finitely many edges coming from condition (ii) and also from (i) by above, and so $H_i \restriction C$ is the union of these edges with the finitely many H''-components that are incident to them by condition (iii). So C is finite.

Finally, we can define the desired witness $G_0 \subseteq G_1 \subseteq \dots$ to the hyperfiniteness of G by setting $x \ G_i \ y$ if $x \ G \ y$ and x and y are in the same connected component of H_i. □

Now given a bounded degree Borel graph G on a standard Borel space X, if $G' \subseteq G$ is a subgraph of G with finite connected components, then we can form the **graph minor** G/G' of G by the connectedness relation of G'. That is, the vertex set of this minor is the standard Borel space X/G' of connected components of G', and the edge relation of G/G' is defined by $[x]_{G'} \ G/G' \ [y]_{G'}$ if $[x]_{G'} \neq [y]_{G'}$ and there exists $x' \in [x]$ and $y' \in [y]_{G'}$ such that $x \ G \ y$. Let $H = G/G'$ and suppose now that $H' \subseteq H$ is a subgraph of H with finite connected components. Then H' naturally lifts to a subgraph of G with finite connected

components that contains G'. That is, there is an edge in this lifted graph between x and y if $x\ G'\ y$, or $x\ G\ y$ and $[x]'_G\ H'\ [y]'_G$. In several of our proofs below, we will define iterated sequences of graph minors in this way, which will naturally lift to witnesses of the hyperfiniteness of the original graph.

Lemma 4.3. *Suppose $f\colon X \to X$ is bounded-to-one Borel function that induces the graph G_f. Then G_f admits an everywhere-large witness to hyperfiniteness.*

Proof. Let $f_0 = f$ and $X_0 = X$. Given the bounded-to-one function f_i on X_i, let $A_i \subseteq X_i$ be as in Proposition 4.1, and let $G'_i \subseteq G_{f_i}$ be the graph on X_i with finite connected components where $x\ G'_i\ f(x)$ if $x \notin A_i$. Note that every connected component of G_f has size at least 2, and size at most $1 + d + d^2 + d^3$, if f_i is $\leq d$-to-one.

Let $X_{i+1} = X_i/G'_i$ and for each $x \in X$, let $[x]_{i+1} \in X_{i+1}$ be the representative of x in X_{i+1}, so $[x]_{i+1}$ is a finite set of elements of X_i, one of which is $[x]_i$. Let f_{i+1} be the bounded-to-one Borel function on X_{i+1} where $f_{i+1}([x]_{i+1}) = [y]_{i+1}$ if $[x]_{i+1} \neq [y]_{i+1}$ and there are $[x']_i \in [x]_{i+1}$ and $[y']_i \in [y]_{i+1}$ such that $f_i([x']_i) = [y']_i$. Note that $G_{f_{i+1}}$ is equal to the graph minor G_{f_i}/G'_i. The sequence $G'_0, G'_1, \ldots$ lifts to an increasing union $G''_0 \subseteq G''_1 \subseteq \ldots$ of Borel graphs on X. By induction, the connected components of each G''_i are finite, and have size at least 2^i.

Let $H = G_f \setminus \bigcup_i G''_i$, so that $x\ H\ f(x)$ if and only if $[x]_i \in A_i$ for every i. Let G_i be the graph on X where $x\ G_i\ y$ if $x\ G''_i\ y$ or $x\ H\ y$. Then clearly $G_0 \subseteq G_1 \ldots$, every connected component of G_i has at least $2^i \geq i$ elements (since this is true of G''_i), and $\bigcup_i G_i = G_f$. We just need to show that every connected component of G_i is finite.

Let H_i be the graph on X_i where $[x]_i\ H_i\ [y]_i$ if $[x]_i \in A_i$ and $f_i([x]_i) = [y]_i$ or $[y]_i \in A_i$ and $f_i([y]_i) = [x]_i$. Since f_i is finite-to-one, by the definition of A_i, it is easy to see that every H_i class is finite. Now if $x \in X$, then the G''_i-class of x is $\{y \in X\colon [y]_{i+1} = [x]_{i+1}\}$ by the definition of X_{i+1}. Thus, the G_i-class of x is a subset of $\{y \in X : [y]_{i+1}$ is in the same H_{i+1}-class as $[x]_{i+1}\}$, which is clearly finite since H_{i+1} has finite connected components. □

Lemma 4.4. *Suppose G is a Borel graph on X where every connected component of G has two ends. Then G admits an everywhere-large witness to hyperfiniteness.*

Proof. Let $Y \subseteq [X]^{<\infty}$ be the collection of finite connected sets $C \subseteq X$ such that removing C from G disconnects the connected component containing C into exactly two infinite pieces. Using a countable Borel coloring of the intersection graph on Y (see [9, Lemma 7.3] and [3, Proposition 2]), we may find a Borel set $Z \subseteq Y$ of pairwise disjoint subsets of X which meets every connected component of G. Let G' be the graph on Z where $C_0\ G'\ C_1$ if C_0 and C_1 are in the same connected component of G and there is no $D \in Z$ such that removing D from G places C_0 and C_1 in different connected compo-

nents. The graph G' has degree at most 2, and it clearly suffices to build everywhere large witnesses for G' instead of G.

Thus, we may restrict our attention just to 2-regular acyclic Borel graphs. However, this is trivial by using the existence of maximal Borel independent sets [10, Proposition 4.6] for such graphs, and the same idea as Lemma 4.3. □

We are now ready to prove Proposition 1.10.

Proof of Proposition 1.10. Suppose G is a hyperfinite bounded degree Borel graph on a standard Borel space X, and μ is a Borel probability measure on X. We may then build an everywhere-large witness to hyperfiniteness off a μ-nullset, by using Adams' end selection theorem [7, Lemma 3.21], and Lemmas 4.2, 4.3, and 4.4.

A straightforward Kuratowski-Ulam argument yields everywhere-large witnesses to hyperfiniteness modulo a meager set. □

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References

[1] E. Csóka, Ł. Grabowski, A. Máthé, O. Pikhurko, and K. Tyros, Borel version of the Local Lemma, *arXiv:* 1605.04877.

[2] R. Chen and A.S. Kechris, Structurable equivalence relations, *arXiv:*1606.01995.

[3] C.T. Conley and B.D. Miller, A bound on measurable chromatic numbers of locally finite Borel graphs, to appear in *Math. Res. Lett.*.

[4] R. Dougherty, S. Jackson, and A.S. Kechris, The structure of hyperfinite Borel equivalence relations, *Trans. Amer. Math. Soc.*, **341** (1994) No. 1, 193–225.

[5] V. G. Drinfeld, Finitely-additive measures on S^2 and S^3, invariant with respect to rotations, *Funktsional. Anal. Prilozhen.* **18** (1984), no. 3, 77

[6] R. Dougherty and M. Foreman, Banach-Tarski paradox using pieces with the property of Baire, *Proc. Nat. Acad. Sci.* **89** (1992), no. 22, 10726–10728.

[7] S. Jackson, A. Kechris, and A. Louveau, Countable Borel equivalence relations, *J. Math. Log.* **2** (2002), no. 1, 1–80.

[8] A.S. Kechris and A. Marks, Descriptive graph combinatorics, preprint 2016.

[9] A.S. Kechris and B.D. Miller, *Topics in Orbit Equivalence*, Springer, 2004.

[10] A.S. Kechris, S. Solecki and S. Todorcevic, Borel chromatic numbers, *Adv. Math.*, **141** (1999), 1–44.

[11] G. A. Margulis, Some remarks on invariant means, *Monatsh. Math.* **90** (1980), no. 3, 233–235.

[12] A. Marks, A determinacy approach to Borel combinatorics, *J. Amer. Math. Soc.* 29 (2016), 579-600.

[13] A. Marks, Uniformity, universality, and computability theory. *arXiv:* 1606.01976.

[14] A. Marks and S. Unger, Baire measurable paradoxical decompositions via matchings, *Adv. Math.*, **289** (2016), 397–410.

[15] D. Sullivan, For $n > 3$ there is only one finitely additive rotationally invariant measure on the n-sphere defined on all Lebesgue measurable subsets, *Bull. Amer. Math. Soc.* **4** (1981), no. 1, 121–123.

[16] S. Thomas, Martin's conjecture and strong ergodicity, *Arch. Math. Logic.* **48** (2009), no. 8, 749–759.

[17] S. Wagon, *The Banach-Tarski paradox*, Cambridge University Press, Cambridge, 1993.