DESCRIPTIVE SET-THEORETIC DICHOTOMY THEOREMS AND LIMITS SUPERIOR

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Abstract. Suppose that X is a Hausdorff space, $\mathcal{I}$ is an ideal on X, and $(A_i)_{i \in \omega}$ is a sequence of analytic subsets of X. We investigate the circumstances under which there exists $I \in [\omega]^\omega$ with $\bigcap_{i \in I} A_i \notin \mathcal{I}$. We focus on Laczkovich-style characterizations and ideals associated with descriptive set-theoretic dichotomies.

Warning. This is a preliminary draft of our paper. Several arguments still need to be written out, there are no doubt many typos remaining, and our terminology is less than perfect. Please email corrections and suggestions to `glimmeffros@gmail.com`.

1. Introduction

Given an infinite set $I \subseteq \omega$, we use $[I]^\omega$ to denote the family of infinite subsets of I. The *limit superior* of a sequence $(A_i)_{i \in I}$ is given by $\limsup_{i \in I} A_i = \bigcap_{i \in \omega} \bigcup_{j \in I \setminus i} A_j = \{x \mid \exists^\infty i \in I \ (x \in A_i)\}$.

Suppose that X is a Hausdorff space. A set $A \subseteq X$ is *analytic* if it is the continuous image of a closed subset of ω^ω. A set $B \subseteq X$ is *Borel* if it is in the σ-algebra generated by the topology of X.

Given a pointclass Γ of subsets of Hausdorff spaces, we say that an ideal $\mathcal{I}$ on X has the Γ *limsup property* if for all sequences $(B_i)_{i \in \omega}$ of subsets of X in Γ, there exists $I \in [\omega]^\omega$ with $\limsup_{i \in I} B_i \in \mathcal{I}$ or $\bigcap_{i \in I} B_i \notin \mathcal{I}$. Note that if X is analytic, then every Borel subset of X is analytic, so if $\mathcal{I}$ has the analytic limsup property, then $\mathcal{I}$ has the Borel limsup property.

Given a sequence $(X_i)_{i \in I}$ of subsets of X, let $\mathcal{I} \upharpoonright (X_i)_{i \in I}$ denote the ideal consisting of all sets $Y \subseteq X$ with the property that $\forall J \in [I]^\omega \exists K \in [J]^\omega$ $(\limsup_{k \in K} X_k \cap Y \in \mathcal{I})$. A straightforward diagonalization shows that if $\mathcal{I}$ is a σ-ideal, then so too is $\mathcal{I} \upharpoonright (X_i)_{i \in I}$. It is easy to check that $\mathcal{I}$ has the Γ limsup property if and only if for all sequences $(B_i)_{i \in \omega}$ of subsets of X in Γ, the existence of $I \in [\omega]^\omega$ with $\bigcap_{i \in I} B_i \notin \mathcal{I}$ is governed by whether $X \notin \mathcal{I} \upharpoonright (B_i)_{i \in \omega}$.

As part of the original foray into these notions, Laczkovich showed that the ideal of countable subsets of an uncountable Polish space has the Borel limsup property [13]. Komjáth later proved that such ideals have the analytic limsup property [12]. Building on subsequent work of Balcerzak-Głąb [1], Gao-Jackson-Kieftenbeld have recently established the more general fact that for all co-analytic equivalence relations on Polish spaces, the ideal of sets which intersect only countably many equivalence classes has the analytic limsup property [4].

These are perhaps the best known examples of ideals associated with descriptive set-theoretic dichotomy theorems. It is therefore quite natural to investigate the family of descriptive set-theoretic dichotomy theorems whose associated ideals have the analytic limsup property. Of course, the sheer abundance of such theorems makes this a rather daunting task.

Fortunately, recent work [15, 16, 17, 18] indicates that many descriptive set-theoretic dichotomy theorems are consequences of a handful of dichotomy theorems concerning chromatic numbers of definable graphs. In particular, many Silver-style dichotomy theorems can be obtained from the Kechris-Solecki-Todorcevic characterization of the class of analytic graphs with countable Borel chromatic number [11].

In §2, we give a classical proof that ideals arising from a natural special case of the Kechris-Solecki-Todorcevic dichotomy theorem [11] have the analytic limsup property. Using this, we give a classical proof of the Gao-Jackson-Kieftenbeld theorem [4], answering a question of Gao. We also prove that ideals associated with Feng's special case of the open coloring axiom [3], the Friedman-Harrington-Kechris characterization of separable quasi-metric spaces [10], van Engelen-Kunen-Miller-style characterizations of vector spaces which are unions of countably many low-dimensional subspaces [2], and the Friedman-Shelah characterization of separable linear quasi-orders [19] have the analytic limsup property. Generalizing a result of Balcerzak-Głąb [1], we show that products of these ideals with analytically principal ideals have the analytic limsup property. Generalizing results of Balcerzak-Głąb [1] and Gao-Jackson-Kieftenbeld [4], we show that these ideals satisfy a parametric strengthening of the analytic limsup property. We also discuss generalizations to κ-Souslin structures.

In §3, we establish that non-trivial ideals arising from the locally countable special case of the Kechris-Solecki-Todorcevic dichotomy theorem [11] do not have the compact limsup property. Using this, we show that non-trivial ideals associated with the Harrington-Kechris-Louveau dichotomy theorem [6] do not have the compact limsup property, answering another question of Gao. We also characterize the ideals associated with the Lusin-Novikov uniformization theorem (see §18 of [9]) which have the analytic limsup property, and we show that products of ideals which have analytic perfect antichains with analytically non-principal ideals do not have the analytic limsup property. This implies that products of non-trivial ideals associated with descriptive set-theoretic dichotomy theorems do not have the analytic limsup property, and negatively answers Balcerzak-Głąb's question [1] as to whether the product of the trivial ideal on an uncountable Polish space with the ideal of countable subsets of an uncountable Polish space has the analytic limsup property, which gives rise to ideals associated with the Harrington-Marker-Shelah Borel-Dilworth theorem [7] which do not have the analytic limsup property.

2. Positive results

A *graph* on X is an irreflexive symmetric set $G \subseteq X \times X$. The *restriction* of G to a set $A \subseteq X$ is given by $G \restriction A = G \cap (A \times A)$. We say that A is *G-discrete* if $G \restriction A = \emptyset$. It is a well-known corollary of the first separation theorem that if G is analytic, then every G-discrete analytic set is contained in a G-discrete Borel set.

We say that a graph G on X has the *limsup property* if for all sequences $(A_i)_{i \in \omega}$ of subsets of X, sets S, $s \in S$, and $R \subseteq X^S$, there exists $I \in [\omega]^\omega$ such that $\mathrm{proj}_s(R \cap \mathrm{limsup}_{i \in I} A_i^S)$ is G-discrete or $\mathrm{proj}_s(R \cap \bigcap_{i \in I} A_i^S)$ is not G-discrete.

Proposition 1. *Suppose that X is a set and G is a graph on X with transitive complement. Then G has the limsup property.*

Proof. Suppose that $(A_i)_{i \in \omega}$ is a sequence of subsets of X, S is a set, $s \in S$, $R \subseteq X^S$, and $\mathrm{proj}_s(R \cap \mathrm{limsup}_{i \in I} A_i^S)$ is not G-discrete for all $I \in [\omega]^\omega$. Fix $x \in R \cap \mathrm{limsup}_{i \in \omega} A_i^S$, $I \in [\omega]^\omega$ with $x \in \bigcap_{i \in I} A_i^S$, and $y, z \in R \cap \mathrm{limsup}_{i \in I} A_i^S$ with $(y(s), z(s)) \in G$. As the complement of G is transitive, it follows that $(w(s), x(s)) \in G$ for some $w \in \{y, z\}$. Fix $J \in [I]^\omega$ with $w \in \bigcap_{j \in J} A_j^S$, and note that $w, x \in R \cap \bigcap_{j \in J} A_j^S$ and $(w(s), x(s)) \in G$, so $\mathrm{proj}_s(R \cap \bigcap_{j \in J} A_j^S)$ is not G-discrete, thus G has the limsup property. ⊠

Proposition 2. *Suppose that X is a set and G is a graph on X which can be written as the union of countably many rectangles. Then G has the limsup property.*

Proof. Fix sets $B_k, C_k \subseteq X$ such that $G = \bigcup_{k \in \omega} B_k \times C_k$, and suppose that $(A_i)_{i \in \omega}$ is a sequence of subsets of X, S is a set, $s \in S$, and $R \subseteq X^S$. We will recursively construct sets $I_k \in [\omega]^\omega$ for $k \in \omega$, beginning with $I_0 = \omega$. If $B_k \cap \mathrm{proj}_s(R \cap \mathrm{limsup}_{i \in I_k} A_i^S) = \emptyset$, then we set $I_{k+1} = I_k$. Otherwise, we fix $x_k \in R \cap \mathrm{limsup}_{i \in I_k} A_i^S$ with $x_k(s) \in B_k$, as well as $I_{k+1} \in [I_k]^\omega$ such that $x_k \in \bigcap_{i \in I_{k+1}} A_i^S$.

Fix $I \in [\omega]^\omega$ with $|I \setminus I_k| < \aleph_0$ for all $k \in \omega$, and suppose that $\mathrm{proj}_s(R \cap \mathrm{limsup}_{i \in I} A_i^S)$ is not G-discrete. Then there exist $x, y \in R \cap \mathrm{limsup}_{i \in I} A_i^S$ with $(x(s), y(s)) \in G$ and $k \in \omega$ with $(x(s), y(s)) \in B_k \times C_k$, so x_k is defined. Fix $J \in [I_{k+1}]^\omega$ with $y \in \bigcap_{j \in J} A_j^S$. Then $x_k, y \in R \cap \bigcap_{j \in J} A_j^S$ and $(x_k(s), y(s)) \in G$, so $\mathrm{proj}_s(R \cap \bigcap_{j \in J} A_j^S)$ is not G-discrete, thus G has the limsup property. ⊠

A Y*-coloring* of G is a function $c \colon X \to Y$ which sends G-related points of X to distinct points of Y. More generally, a *homomorphism* from a graph G on X to a graph H on Y is a function $\pi \colon X \to Y$ which sends G-related points of X to H-related points of Y.

We use $\mathcal{I}_G$ to denote the σ-ideal generated by the family of G-discrete Borel subsets of X. It is easy to see that $X \in \mathcal{I}_G$ if and only if there is a Borel ω-coloring of G.

Theorem 3. *Suppose that X is a Hausdorff space and G is an analytic graph on X which has the limsup property. Then $\mathcal{I}_G$ has the analytic limsup property.*

Proof. Fix $s_n \in 2^n$ for $n \in \omega$ with $\forall s \in 2^{<\omega} \exists n \in \omega \ (s \sqsubseteq s_n)$, and set

$$G_0 = \{(s_n \frown i \frown x, s_n \frown \bar{\imath} \frown x) \mid i \in 2,\ n \in \omega, \text{ and } x \in 2^\omega\}.$$

As noted by Kechris-Solecki-Todorcevic, a straightforward Baire category argument shows that there is no Baire measurable ω-coloring of G_0 [11]. It is therefore sufficient to show that if $(A_i)_{i \in \omega}$ is a sequence of analytic subsets of X with $X \notin \mathcal{I}_G \restriction (A_i)_{i \in \omega}$, then for some $I \in [\omega]^\omega$ there is a continuous homomorphism from G_0 to $G \restriction \bigcap_{i \in I} A_i$.

We can assume that G and the sets along $(A_i)_{i \in \omega}$ are non-empty. Fix continuous surjections $\varphi_G \colon \omega^\omega \to G$, $\varphi_i \colon \omega^\omega \to A_i$ for $i \in \omega$, and $\varphi_X \colon \omega^\omega \to \mathrm{dom}(G)$, where $\mathrm{dom}(G) = \{x \in X \mid G_x \neq \emptyset\}$.

A *global* (n-)*approximation* is a sequence $p = (I^p, u^p, v^p, (w_i^p)_{i \in I^p})$, where $I^p \in [\omega]^n$, $u^p \colon 2^n \to \omega^n$, $v^p \colon 2^{<n} \to \omega^n$, and $w_i^p \colon 2^n \to \omega^n$ for $i \in I^p$. Fix an enumeration $(p_k)_{k \in \omega}$ of the set of all such sequences.

An *extension* of a global m-approximation p is a global n-approximation q which satisfies the following conditions:

- $I^p \subseteq I^q$.
- $\forall s_p \in 2^m \forall s_q \in 2^n \ (s_p \sqsubseteq s_q \Longrightarrow u^p(s_p) \sqsubseteq u^q(s_q))$.
- $\forall t_p \in 2^m \forall t_q \in 2^n$
  $((t_p \sqsubseteq t_q \text{ and } n - m = |t_q| - |t_p|) \Longrightarrow v^p(t_p) \sqsubseteq v^q(t_q))$.
- $\forall i \in I^p \forall s_p \in 2^m \forall s_q \in 2^n \ (s_p \sqsubseteq s_q \Longrightarrow w_i^p(s_p) \sqsubseteq w_i^q(s_q))$.

When $n = m + 1$, we say that q is a *one-step extension* of p.

A *local* (n-)*approximation* is a sequence $l = (I^l, f^l, g^l, (h^l_i)_{i \in I^l})$, where $I^l \in [\omega]^n$, $f^l \colon 2^n \to \omega^\omega$, $g^l \colon 2^{<n} \to \omega^\omega$, and $h^l_i \colon 2^n \to \omega^\omega$ for $i \in I^l$, which satisfies the following conditions:

- $\forall k \in n \forall t \in 2^{n-(k+1)}$ $(\varphi_G \circ g^l(t) = (\varphi_X \circ f^l(s_k{}^\frown 0^\frown t), \varphi_X \circ f^l(s_k{}^\frown 1^\frown t)))$.
- $\forall s \in 2^n \forall i \in I^l$ $(\varphi_X \circ f^l(s) = \varphi_i \circ h^l_i(s))$.

We say that a local n-approximation l is *compatible* with a global n-approximation p if the following conditions are satisfied:

- $I^l = I^p$.
- $\forall s \in 2^n$ $(u^p(s) \sqsubseteq f^l(s))$.
- $\forall t \in 2^{<n}$ $(v^p(t) \sqsubseteq g^l(t))$.
- $\forall s \in 2^n \forall i \in I^l$ $(w^p_i(s) \sqsubseteq h^l_i(s))$.

Let $\mathcal{I}_n$ denote the σ-ideal of sets $R \subseteq X^{2^n}$ for which there exists $B \in \mathcal{I}_G$ such that $\forall x \in R \exists s \in 2^n$ $(x(s) \in B)$. We say that a global n-approximation p is *good* if the set

$$R^p_n = \{\varphi_X \circ f^l \mid l \text{ is compatible with } p\}$$

is not in $\mathcal{I}_n \restriction (A^{2^n}_i)_{i \in \omega}$.

Lemma 4. *Suppose that $n \in \omega$ and p is a good global n-approximation. Then p has a good one-step extension.*

Proof of lemma. As the complement of $\bigcup_{i \in \omega \setminus I^p} A^{2^n}_i$ is in $\mathcal{I}_n \restriction (A^{2^n}_i)_{i \in \omega}$, there exists $m \in \omega \setminus I^p$ with $A^{2^n}_m \cap R^p_n \notin \mathcal{I}_n \restriction (A^{2^n}_i)_{i \in \omega}$. For all $i \in 2$ and $x \in X^{2^{n+1}}$, define $x_i \in X^{2^n}$ by $x_i(s) = x(s^\frown i)$ for $s \in 2^n$, and set

$$S = \{x \in A^{2^{n+1}}_m \mid x_0, x_1 \in R^p_n \text{ and } (x_0(s_n), x_1(s_n)) \in G\}.$$

Sublemma 5. *The set S is contained in $\bigcup \{R^q_{n+1} \mid q$ is a one-step extension of $p\}$.*

Proof of sublemma. Suppose that $x \in S$. Then there are local n-approximations $l_0, l_1 \in R^p_n$ with $x_i = \varphi_X \circ f^{l_i}$ for all $i \in 2$. Fix $y \in \omega^\omega$ such that $\varphi_G(y) = (x_0(s_n), x_1(s_n))$, as well as $y_s \in \omega^\omega$ such that $\varphi_m(y_s) = x(s)$ for all $s \in 2^{n+1}$. Let l denote the local $(n+1)$-approximation given by $I^l = I^p \cup \{m\}$; $f^l(s^\frown i) = f^{l_i}(s)$ for $i \in 2$ and $s \in 2^n$; $g^l(\emptyset) = y$; $g^l(t^\frown i) = g^{l_i}(t)$ for $i \in 2$ and $t \in 2^{<n}$; $h^l_i(s^\frown j) = h^{l_j}_i(s)$ for $i \in I^p$, $j \in 2$, and $s \in 2^n$; and $h^l_m(s) = y_s$ for $s \in 2^{n+1}$. Let q denote the unique one-step extension of p with which l is compatible, and observe that $x = \varphi_X \circ l$, thus $x \in R^q_{n+1}$. ⊠

Sublemma 6. *The set S is not in $\mathcal{I}_{n+1} \restriction (A^{2^{n+1}}_i)_{i \in \omega}$.*

Proof of sublemma. Fix $I \in [\omega]^\omega$ with $A^{2^n}_m \cap R^p_n \cap \limsup_{j \in J} A^{2^n}_j \notin \mathcal{I}_n$ for all $J \in [I]^\omega$, and suppose, towards a contradiction, that $S \in \mathcal{I}_{n+1} \upharpoonright (A^{2^{n+1}}_i)_{i \in \omega}$. Fix $J \in [I]^\omega$ with $S \cap \limsup_{j \in J} A^{2^{n+1}}_j \in \mathcal{I}_{n+1}$ and a Borel set $B \in \mathcal{I}_G$ with $\forall x \in S \cap \limsup_{j \in J} A^{2^{n+1}}_j \exists s \in 2^{n+1}$ $(x(s) \in B)$. Set

$$T = \{x \in A^{2^n}_m \cap R^p_n \mid \forall s \in 2^n \ (x(s) \notin B)\},$$

and observe that $T \cap \limsup_{k \in K} A^{2^n}_k \notin \mathcal{I}_n$ for all $K \in [J]^\omega$. As G has the limsup property, there exists $K \in [J]^\omega$ such that $\mathrm{proj}_{s_n}(T \cap \bigcap_{k \in K} A^{2^n}_k)$ is not G-discrete. Fix $x \in \bigcap_{k \in K} A^{2^{n+1}}_k$ such that $x_0, x_1 \in T$ and $(x_0(s_n), x_1(s_n)) \in G$. Then $x \in S$ and $x(s) \notin B$ for all $s \in 2^{n+1}$, which contradicts the defining property of B. $\boxtimes$

Sublemmas 5 and 6 ensure the existence of a one-step extension q of p with $R^q_{n+1} \notin \mathcal{I}_{n+1} \upharpoonright (A^{2^{n+1}}_i)_{i \in \omega}$, in which case q is as desired. $\boxtimes$

Our assumption that $X \notin \mathcal{I}_G \upharpoonright (A_i)_{i \in \omega}$ ensures that the unique global 0-approximation p^0 is good, in which case Lemma 4 yields global n-approximations $p^n = (I^n, u^n, v^n, (w^n_i)_{i \in I^n})$ such that p^{n+1} is a good one-step extension of p^n for all $n \in \omega$.

Set $I = \bigcup_{n \in \omega} I^n$ and define continuous functions $\psi_i \colon 2^\omega \to \omega^\omega$ for $i \in I$ and $\psi^k_G \colon 2^\omega \to \omega^\omega$ for $k \in \omega$, as well as $\psi_X \colon 2^\omega \to \omega^\omega$, by setting $\psi_i(x) = \lim_{n \to \omega} w^n_i(x \upharpoonright n)$, $\psi^k_G(x) = \lim_{n \to \omega} v^{k+n+1}(x \upharpoonright n)$, and $\psi_X(x) = \lim_{n \to \omega} u^n(x \upharpoonright n)$. We will show that the map $\pi = \varphi_X \circ \psi_X$ is the desired homomorphism from G_0 to $G \upharpoonright \bigcap_{i \in I} A_i$.

To see that π is a homomorphism from G_0 to G, it is sufficient to show that $\varphi_G \circ \psi^k_G(x) = (\varphi_X \circ \psi_X(s_k {}^\frown 0 {}^\frown x), \varphi_X \circ \psi_X(s_k {}^\frown 1 {}^\frown x))$ for all $k \in \omega$ and $x \in 2^\omega$. By continuity of φ_G and φ_X, it is enough to show that for every open neighborhood U of $\psi^k_G(x)$ and every open neighborhood V of $(\psi_X(s_k {}^\frown 0 {}^\frown x), \psi_X(s_k {}^\frown 1 {}^\frown x))$, there exist $z \in U$ and $(z_0, z_1) \in V$ with the property that $\varphi_G(z) = (\varphi_X(z_0), \varphi_X(z_1))$. Fix $n \in \omega$ sufficiently large that $\mathcal{N}_{v^{k+n+1}(x \upharpoonright n)} \subseteq U$ and $\mathcal{N}_{u^{k+n+1}(s_k {}^\frown 0 {}^\frown (x \upharpoonright n))} \times \mathcal{N}_{u^{k+n+1}(s_k {}^\frown 1 {}^\frown (x \upharpoonright n))} \subseteq V$. Fix a local approximation l compatible with p^{k+n+1}. Then $z = g^l(x \upharpoonright n)$, $z_0 = f^l(s_k {}^\frown 0 {}^\frown (x \upharpoonright n))$, and $z_1 = f^l(s_k {}^\frown 1 {}^\frown (x \upharpoonright n))$ are as desired.

To see that $\pi(2^\omega) \subseteq \bigcap_{i \in I} A_i$, it is sufficient to show that $\varphi_i \circ \psi_i(x) = \varphi_X \circ \psi_X(x)$ for all $i \in I$ and $x \in 2^\omega$. By continuity of φ_i and φ_X, it is enough to show that for every open neighborhood U of $\psi_i(x)$ and every open neighborhood V of $\psi_X(x)$, there exist $z_0 \in U$ and $z_1 \in V$ with $\varphi_i(z_0) = \varphi_X(z_1)$. Fix $n \in \omega$ such that $i \in I^n$, $\mathcal{N}_{w^n_i(x \upharpoonright n)} \subseteq U$, and $\mathcal{N}_{u^n(x \upharpoonright n)} \subseteq V$. Fix a local approximation l compatible with p^n. Then $z_0 = h^l_i(x \upharpoonright n)$ and $z_1 = f^l(x \upharpoonright n)$ are as desired. $\boxtimes$

Remark 7. Our proof of Theorem 3 is based on the classical proof of the Kechris-Solecki-Todorcevic dichotomy theorem [11] appearing in [15], and yields the stronger fact that exactly one of the following holds:

(1) The set X is in $\mathcal{I}_G \restriction (A_i)_{i \in \omega}$.
(2) There is a continuous homomorphism $\varphi \colon 2^\omega \to \bigcap_{i \in I} A_i$ from G_0 to G, for some $I \in [\omega]^\omega$.

We do not emphasize this stronger form because it is a straightforward consequence of Theorem 3 and the Kechris-Solecki-Todorcevic dichotomy theorem [11]. Similar remarks apply to all of the ideals appearing in this section.

We now establish the results of Komjáth [12] and Laczkovich [13]:

Theorem 8 (Komjáth, Laczkovich)**.** *Suppose that X is an analytic Hausdorff space. Then the ideal of countable subsets of X has the analytic limsup property.*

Proof. Let $\mathcal{I}$ denote the ideal of countable subsets of X and set $G = \Delta(X)^c$. In light of Proposition 1 and Theorem 3, the desired result follows from the observation that $\mathcal{I} = \mathcal{I}_G$. ⊠

More generally, we obtain the Gao-Jackson-Kieftenbeld theorem [4]:

Theorem 9 (Gao-Jackson-Kieftenbeld)**.** *Suppose that X is a Hausdorff space and E is a co-analytic equivalence relation on X. Then the ideal of sets on which E has only countably many equivalence classes has the analytic limsup property.*

Proof. Let $\mathcal{I}$ denote the σ-ideal generated by the family of Borel sets which are contained in a single E-class, and let $\mathcal{J}$ denote the ideal of sets on which E has only countably many classes. As Silver's dichotomy theorem [20] implies that $\mathcal{I}$ and $\mathcal{J}$ agree on analytic sets, it is sufficient to show that $\mathcal{I}$ has the analytic limsup property. Set $G = E^c$. In light of Proposition 1 and Theorem 3, the desired result follows from the observation that $\mathcal{I} = \mathcal{I}_G$. ⊠

Remark 10. As noted in [15], Silver's theorem [20] follows from the Kechris-Solecki-Todorcevic dichotomy theorem [11], the observation that $\mathcal{I} = \mathcal{I}_G$, and a simple Baire category argument. In particular, despite its use of Silver's theorem [20], the above argument is classical. Similar remarks apply to all of the results of this section.

An *open coloring* on X is a function $c \colon [X]^2 \to 2$ for which $c^{-1}(\{1\})$ is open. A set $A \subseteq X$ is *i-homogeneous* if $c \restriction [A]^2$ has constant value i.

Theorem 11. *Suppose that X is an analytic Hausdorff space and c is an open coloring on X. Then the σ-ideal generated by the family of 0-homogeneous sets has the analytic limsup property.*

Proof. Let $\mathcal{I}$ denote the σ-ideal generated by the family of 0-homogeneous Borel sets, and let $\mathcal{J}$ denote the σ-ideal generated by the family of 0-homogeneous sets. As Feng's theorem [3] implies that $\mathcal{I}$ and $\mathcal{J}$ agree on analytic sets, it is sufficient to show that $\mathcal{I}$ has the analytic limsup property. Set $G = \{(x, y) \in X \times X \mid c(\{x, y\}) = 1\}$. In light of Proposition 2 and Theorem 3, the desired result follows from the observation that $\mathcal{I} = \mathcal{I}_G$. ⊠

A *quasi-metric* on X is a function $d \colon X \times X \to [0, \infty)$ with $d(x, x) = 0$, $d(x, y) = d(y, x)$, and $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 12. *Suppose that X is a Hausdorff space and d is a quasi-metric on X such that $d^{-1}[\epsilon, \infty)$ is analytic for arbitrarily small positive real numbers ϵ. Then the ideal of sets on which d is separable has the analytic limsup property.*

Proof (Sketch). Let $\mathcal{I}$ denote the σ-ideal generated by the family of Borel sets on which d is separable, and let $\mathcal{J}$ denote the ideal of sets on which d is separable. As the Friedman-Harrington-Kechris dichotomy theorem for quasi-metrics [10] implies that $\mathcal{I}$ and $\mathcal{J}$ agree on analytic sets, it is enough to show that $\mathcal{I}$ has the analytic limsup property.

Fix a sequence $(\epsilon_k)_{k\in\omega}$ of positive real numbers with the property that $\epsilon_{k+1} \leq \epsilon_k/2$ and the graph $H_k = d^{-1}[\epsilon_k, \infty)$ is analytic for all $k \in \omega$. It is straightforward to check that $\mathcal{I} = \bigcap_{k\in\omega} \mathcal{I}_{H_k}$.

By following the proof of Proposition 1, one can establish that for all $k \in \omega$, sequences $(A_i)_{i\in\omega}$ of subsets of X, sets S, $s \in S$, and $R \subseteq X^S$, there exists $I \in [\omega]^\omega$ such that $\mathrm{proj}_s(R \cap \mathrm{limsup}_{i\in I} A_i^S)$ is H_k-discrete or $\mathrm{proj}_s(R \cap \bigcap_{i\in I} A_i^S)$ is not H_{k+1}-discrete.

Suppose now that $(A_i)_{i\in\omega}$ is a sequence of analytic subsets of X. By recursively following the proof of Theorem 3, one can establish that there is a decreasing sequence of sets $I_k \in [\omega]^\omega$ for $k \in \omega$ such that $\mathrm{limsup}_{i\in I_k} A_i \in \mathcal{I}_{H_k}$ or $\bigcap_{i\in I_k} A_i \notin \mathcal{I}_{H_{k+1}}$ for all $k \in \omega$.

Clearly we can assume that $\mathrm{limsup}_{i\in I_k} A_i \in \mathcal{I}_{H_k}$ for all $k \in \omega$. Fix a set $I \in [\omega]^\omega$ such that $|I \setminus I_k| < \aleph_0$ for all $k \in \omega$, and observe that $\mathrm{limsup}_{i\in I} A_i \in \bigcap_{k\in\omega} \mathcal{I}_{H_k}$, which completes the proof of the theorem. ⊠

Suppose that $D \colon \mathcal{P}(X) \to \omega \cup \{\infty\}$. The *span* of a set $A \subseteq X$ is given by $\mathrm{span}\, A = \{x \in X \mid D(A) = D(A \cup \{x\})\}$. We say that D is a *notion of dimension* if it satisfies the following conditions:

(1) $\forall x \in X \ (D(\{x\}) \leq 1)$.

(2) $\forall A \subseteq B \subseteq X \ (D(A) \leq D(B))$.
(3) $\forall A \subseteq X \ (D(A) = D(\operatorname{span} A))$.

We refer to $D(A)$ as the *dimension* of A, and we say that a finite subset of X is *dependent* if its dimension is strictly less than its cardinality.

Theorem 13. *Suppose that X is a Hausdorff space and D is a notion of dimension on X such that the family of dependent finite sets is co-analytic. Then for all $k \in \omega$, the σ-ideal generated by the family of sets of dimension at most k has the analytic limsup property.*

Proof (*Sketch*). Let $\mathcal{I}$ denote the σ-ideal generated by the family of Borel sets of dimension at most k, and let $\mathcal{J}$ denote the ideal of sets of dimension at most k. As the natural generalization of the van Engelen-Kunen-Miller style theorems [2] implies that $\mathcal{I}$ and $\mathcal{J}$ agree on analytic sets, it is sufficient to show that $\mathcal{I}$ has the analytic limsup property. Let G denote the $(k+1)$-dimensional hypergraph consisting of all independent sets of size $k+1$. It is easy to check that $\mathcal{I} = \mathcal{I}_G$.

By following the proof of Proposition 1, one can establish that for all sequences $(A_i)_{i \in \omega}$ of subsets of X, sets S, $s \in S$, and $R \subseteq X^S$, there exists $I \in [\omega]^\omega$ such that $\operatorname{proj}_s(R \cap \operatorname{limsup}_{i \in I} A_i^S)$ is G-discrete or $\operatorname{proj}_s(R \cap \bigcap_{i \in I} A_i^S)$ is not G-discrete.

Suppose now that $(A_i)_{i \in \omega}$ is a sequence of analytic subsets of X. By following the proof of Theorem 3, one can establish that there exists $I \in [\omega]^\omega$ such that $\operatorname{limsup}_{i \in I} A_i \in \mathcal{I}_G$ or $\bigcap_{i \in I} A_i \notin \mathcal{I}_G$, which completes the proof of the theorem. ⊠

Theorem 14. *Suppose that X is a Hausdorff space and D is a notion of dimension on X such that the family of dependent finite sets is co-analytic. Then the σ-ideal generated by the family of sets of finite dimension has the analytic limsup property.*

Proof (*Sketch*). Let $\mathcal{I}$ denote the σ-ideal generated by the family of finite-dimensional Borel sets, and let $\mathcal{J}$ denote the ideal of finite-dimensional sets. As the natural generalization of the van Engelen-Kunen-Miller style theorems [2] implies that $\mathcal{I}$ and $\mathcal{J}$ agree on analytic sets, it is enough to show that $\mathcal{I}$ has the analytic limsup property.

For each $k \in \omega$, let G_k denote the hypergraph on X consisting of all finite independent sets of cardinality at least k. It is easy to check that $\mathcal{I} = \bigvee_{k \in \omega} \mathcal{I}_{G_k}$.

Just as in the proof of Theorem 13, one can establish that for all $k \in \omega$, sequences $(A_i)_{i \in \omega}$ of subsets of X, sets S, $s \in S$, and $R \subseteq X^S$, there exists $I \in [\omega]^\omega$ such that $\operatorname{proj}_s(R \cap \operatorname{limsup}_{i \in I} A_i^S)$ is G_k-discrete or $\operatorname{proj}_s(R \cap \bigcap_{i \in I} A_i^S)$ is not G_k-discrete.

Suppose now that $(A_i)_{i \in \omega}$ is a sequence of analytic subsets of X. By following the proof of Theorem 3, one can establish that there exists $I \in [\omega]^\omega$ such that $\limsup_{i \in I} A_i \in \bigvee_{k \in \omega} \mathcal{I}_{G_k}$ or $\bigcap_{i \in I} A_i \notin \bigvee_{k \in \omega} \mathcal{I}_G$, which completes the proof of the theorem. ⊠

A *reduction* of a set $R \subseteq X \times X$ to a set $S \subseteq Y \times Y$ is a function $\pi \colon X \to Y$ with the property that $(x_0, x_1) \in R \iff (\pi(x_0), \pi(x_1)) \in S$ for all $x_0, x_1 \in X$. A graph has the *hereditary limsup property* if every graph which is reducible to it has the limsup property.

Proposition 15. *Suppose that X is a set and G is a graph on X with transitive complement. Then G has the hereditary limsup property.*

Proof. This follows from Proposition 1 and the observation that transitivity is closed under Borel reducibility. ⊠

Remark 16. A similar argument goes through for graphs which can be written as countable unions of rectangles.

A *reduction* of an ideal $\mathcal{I}$ on X to an ideal $\mathcal{J}$ on Y is a function $\pi \colon X \to Y$ with the property that $A \in \mathcal{I} \iff \pi(A) \in \mathcal{J}$ for all $A \subseteq X$. An ideal $\mathcal{I}$ has the *hereditary analytic limsup property* if every ideal on a Hausdorff space which is Borel reducible to it has the analytic limsup property.

Theorem 17. *Suppose that X is a Hausdorff space and G is an analytic graph on X with the hereditary limsup property. Then $\mathcal{I}_G$ has the hereditary analytic limsup property.*

Proof. Suppose that Y is a Hausdorff space, $\mathcal{I}$ is an ideal on Y, and $\pi \colon Y \to X$ is a Borel reduction of $\mathcal{I}$ to $\mathcal{I}_G$. Let H denote the graph on X given by $H = \{(y_0, y_1) \in Y \times Y \mid (\pi(y_0), \pi(y_1)) \in G\}$. Then π is a reduction of H to G, thus a reduction of $\mathcal{I}_H$ to $\mathcal{I}_G$, hence $\mathcal{I} = \mathcal{I}_H$. As H has the limsup property, Theorem 3 ensures that $\mathcal{I}$ has the analytic limsup property. ⊠

In particular, we obtain the following strengthening of the Gao-Jackson-Kieftenbeld theorem [4]:

Theorem 18. *Suppose that X is a Hausdorff space and E is a co-analytic equivalence relation on X. Then the ideal of sets on which E has only countably many equivalence classes has the hereditary analytic limsup property.*

Proof. Simply repeat the proof of Theorem 9 with Proposition 15 and Theorem 17 in place of Proposition 1 and Theorem 3. ⊠

Remark 19. Similar arguments yield analogous results for all of the ideals we have discussed thus far.

The *product* of an ideal $\mathcal{I}$ on X with an ideal $\mathcal{J}$ on Y is given by

$$\mathcal{I} * \mathcal{J} = \{A \subseteq X \times Y \mid \{x \in X \mid A_x \notin \mathcal{J}\} \in \mathcal{I}\}.$$

We say that $\mathcal{J}$ is *analytically principal* if there is a non-empty analytic set $A \subseteq Y$ such that $\mathcal{J} = \mathcal{P}(A^c)$.

Proposition 20. *Suppose that X and Y are Hausdorff spaces, $\mathcal{I}$ is an ideal on X which has the hereditary analytic limsup property, and $\mathcal{J}$ is an analytically principal ideal on Y. Then $\mathcal{I} * \mathcal{J}$ has the hereditary analytic limsup property.*

Proof. Suppose that Z is a Hausdorff space, $\mathcal{K}$ is an ideal on Z, and $\pi \colon Z \to X \times Y$ is a Borel reduction of $\mathcal{K}$ to $\mathcal{I} * \mathcal{J}$. Fix a non-empty analytic set $A \subseteq Y$ with $\mathcal{J} = \mathcal{P}(A^c)$. Set $B = \pi^{-1}(X \times A)$. Then $\mathcal{K}$ has the analytic limsup property if and only if $\mathcal{K} \upharpoonright B$ has the analytic limsup property. As $\pi \upharpoonright B$ is a reduction of $\mathcal{K} \upharpoonright B$ to $\mathcal{I} * (\mathcal{J} \upharpoonright A)$ and proj_X is a reduction of $\mathcal{I} * (\mathcal{J} \upharpoonright A)$ to $\mathcal{I}$, it follows that $\mathcal{K}$ has the analytic limsup property, thus the product $\mathcal{I} * \mathcal{J}$ has the hereditary analytic limsup property. ⊠

In particular, we obtain the following generalization of Example 13 of Balcerzak-Głąb [1]:

Theorem 21. *Suppose that X and Y are Hausdorff spaces, E is a co-analytic equivalence relation on X, $\mathcal{I}$ is the ideal of sets which intersect only countably many E-classes, and $\mathcal{J}$ is an analytically principal ideal on Y. Then $\mathcal{I} * \mathcal{J}$ has the hereditary analytic limsup property.*

Proof. This follows from Theorem 18 and Proposition 20. ⊠

Remark 22. Similar arguments yield analogous results for all of the ideals we have discussed thus far.

Suppose that R is a linear quasi-order on X. The *open interval* determined by x and y is given by $(x, y)_R = \{z \in X \mid x <_R z <_R y\}$. Define $I_R = \{(x, y) \in X \times X \mid (x, y)_R \neq \emptyset\}$. We say that a set $A \subseteq X$ is *dense* in a set $I \subseteq I_R$ if $A \cap (x, y)_R \neq \emptyset$ for all $(x, y) \in I$.

Theorem 23. *Suppose that X is a Hausdorff space and R is a linear co-analytic quasi-order on X. Then the ideal of sets $I \subseteq I_R$ which have countable dense sets has the hereditary analytic limsup property.*

Proof. By the Harrington-Marker-Shelah characterization of linear quasi-orders [7] and the fact that the ideal of countable subsets of an analytic Hausdorff space has the hereditary analytic limsup property, it is enough to show that if $\alpha \in \omega_1$, $X = 2^\alpha$, and R is the lexicographic ordering, then there is a Borel reduction of the ideal in question to the countable ideal on $2^{<\alpha}$. Towards this end, let $\delta(x, y)$ denote the least $\beta \in \alpha$ such that $x(\beta) \neq y(\beta)$, and note that the function $\pi \colon I_R \to 2^{<\alpha}$ given by $\pi(x, y) = x \restriction \delta(x, y) = y \restriction \delta(x, y)$ is as desired. ⊠

Following Balcerzak-Głąb [1], we say that an ideal $\mathcal{I}$ on X has the *parametric analytic limsup property* if for all Hausdorff spaces Y and all sequences $(A_i)_{i \in \omega}$ of analytic subsets of $Y \times X$, there exists $I \in [\omega]^\omega$ such that $\limsup_{i \in I} (A_i)_y \in \mathcal{I}$ for some $y \in Y$ or $\bigcap_{i \in I} (A_i)_y \notin \mathcal{I}$ for perfectly many $y \in Y$.

Proposition 24. *Suppose that X is a Hausdorff space and G is an analytic graph on X for which $\mathcal{I}_G$ has the analytic limsup property. Then $\mathcal{I}_G$ has the parametric analytic limsup property.*

Proof. An ideal $\mathcal{I}$ on X is *co-analytic on analytic* if for every Hausdorff space Y and analytic set $R \subseteq Y \times X$, the set $C = \{y \in Y \mid R_y \in \mathcal{I}\}$ is co-analytic. The proof of the Kechris-Solecki-Todorcevic dichotomy theorem [11] easily implies that $\mathcal{I}_G$ is co-analytic on analytic. As Proposition 9 of Balcerzak-Głąb [1] ensures that every co-analytic on analytic ideal with the analytic limsup property has the parametric analytic limsup property, the proposition follows. ⊠

In particular, we obtain a classical proof of the following:

Theorem 25 (Gao-Jackson-Kieftenbeld)**.** *Suppose that X is a Hausdorff space and E is a co-analytic equivalence relation on X. Then the ideal of sets on which E has only countably many equivalence classes has the parametric analytic limsup property.*

Proof. This follows from the proof of Theorem 9 and Proposition 24. ⊠

Remark 26. Similar arguments yield analogous results for all of the ideals we have discussed thus far.

Remark 27. Given an ideal $\mathcal{J}$ on a Hausdorff space Y, we say that an ideal $\mathcal{I}$ on X has the *$\mathcal{J}$-parametric analytic limsup property* if for all sequences $(A_i)_{i \in \omega}$ of analytic subsets of $Y \times X$, there exists $I \in [\omega]^\omega$ such that $\limsup_{i \in I} (A_i)_y \in \mathcal{I}$ for some $y \in Y$ or $\bigcap_{i \in I} (A_i)_y \notin \mathcal{I}$ for a $\mathcal{J}$-positive set of $y \in Y$. We seem to have an argument establishing the analog of Proposition 24 for the $\mathcal{J}$-parametric analytic limsup property, where $\mathcal{J}$ is any σ-ideal associated with a descriptive set-theoretic dichotomy theorem, but it must still be checked.

Throughout this section, we have assumed $\mathtt{AC}_\omega$. Even this small fragment of choice can typically be avoided by replacing the sort of argument we used in the proof of Theorem 3 with one based on derivatives. Strangely enough, in our context the derivative argument seems to require even more choice: the existence of a function $\varphi\colon ([\omega]^\omega)^{<\omega_1} \to [\omega]^\omega$ with the property that $\varphi((I_\beta)_{\beta\in\alpha}) \subseteq^* I_\beta$ for all $\beta \in \alpha \in \omega_1$ and $\subseteq^*$-decreasing sequences $(I_\beta)_{\beta\in\alpha} \in ([\omega]^\omega)^\alpha$.

Although few would worry about our need for $\mathtt{AC}_\omega$, the real difficulty becomes apparent when one tries to use our arguments to establish analogous results for κ-Souslin structures, in which case $\mathtt{AC}_\kappa$ is required. As $\mathtt{AC}_{\omega_1}$ is already inconsistent with $\mathtt{AD}$, this rules out the possibility of using our arguments to establish natural analogs of our results in models of determinacy. While it seems likely that Kanovei-style arguments [8] can be used to obtain such results, we have yet to verify this.

On the positive side, our results do generalize to κ-Souslin structures in models of $\mathtt{ZFC}$. By placing appropriate restrictions on κ, we obtain particularly natural generalizations. The *tower number* is the least cardinal $\mathfrak{t}$ for which there is a $\subseteq^*$-decreasing sequence in $([\omega]^\omega)^{\mathfrak{t}}$ with no $\subseteq^*$-lower bound.

Theorem 28. *Work in* $\mathtt{ZFC}$. *Suppose that $\kappa < \mathfrak{t}$, X is a Hausdorff space, and E is a co-κ-Souslin equivalence relation on X. Then the ideal of sets on which E has at most κ-many equivalence classes has the κ-Souslin limsup property.*

Remark 29. Similar results go through for all of the ideals we have discussed thus far.

3. Negative results

We say that $(G, \mathcal{I})$ has the Γ *anti-limsup property* if there is a sequence $(B_i)_{i\in\omega}$ of subsets of X in Γ such that $\limsup_{i\in I} B_i \notin \mathcal{I}$ and $\bigcap_{i\in I} B_i$ is G-discrete for all $I \in [\omega]^\omega$. Let G_{fin} denote the graph on $\mathcal{P}(\omega)$ given by $G_{\mathrm{fin}} = \{(x,y) \in \mathcal{P}(\omega) \times \mathcal{P}(\omega) \mid 0 < |x \cap y| < \aleph_0\}$.

Proposition 30. *Suppose that $\mathcal{I}$ is the meager ideal on $\mathcal{P}(\omega)$. Then $(G_{\mathit{fin}}, \mathcal{I})$ has the clopen anti-limsup property.*

Proof. For each $i \in \omega$, set $U_i = \{x \in [\omega]^\omega \mid i \in x\}$. A straightforward Baire category argument shows that if $I \in [\omega]^\omega$, then $\limsup_{i\in I} U_i$ is comeager. As $\bigcap_{i\in I} U_i$ is clearly G_{fin}-discrete, the proposition follows. ⊠

Proposition 31. *Suppose that $\mathcal{I}$ is the meager ideal on 2^ω. Then $(E_0 \setminus \Delta(2^\omega), \mathcal{I})$ has the clopen anti-limsup property.*

Proof. For each $i \in \omega$, set $U_i = \{x \in 2^\omega \mid \exists s \in 2^i \ (s^\frown s \sqsubseteq x)\}$. A straightforward Baire category argument shows that if $I \in [\omega]^\omega$, then $\limsup_{i \in I} U_i$ is comeager. As $\bigcap_{i \in I} U_i$ is clearly a partial transversal of E_0, the proposition follows. ⊠

Remark 32. Let $\mathcal{I}$ denote the family of subsets of 2^ω which are null with respect to the probability measure on 2^ω given by $\mu(\mathcal{N}_s) = 1/2^{|s|}$ for $s \in 2^{<\omega}$. Then $(E_0 \setminus \Delta(X), \mathcal{I})$ does not have the clopen anti-limsup property. Moreover, it appears to be the case that for every sequence $(B_i)_{i \in \omega}$ of μ-measurable subsets of 2^ω there exists $I \in [\omega]^\omega$ such that $\limsup_{i \in I} B_i \in \mathcal{I}$ or $E_0 \upharpoonright \bigcap_{i \in I} B_i$ is non-smooth, although our argument must still be checked.

We say that G has the Γ *anti-limsup property* if there is a sequence $(B_i)_{i \in \omega}$ of subsets of X in Γ such that $\limsup_{i \in I} B_i \notin \mathcal{I}_G$ and $\bigcap_{i \in I} B_i$ is G-discrete for all $I \in [\omega]^\omega$. Note that if G has the Γ anti-limsup property, then $\mathcal{I}_G$ does not have the Γ limsup property.

Proposition 33. *The graph G_{fin} has the clopen anti-limsup property.*

Proof. As a straightforward Baire category argument shows that every G_{fin}-discrete set with the Baire property is meager, the desired result follows from Proposition 30. ⊠

Given graphs $G \subseteq H$ on X, we say that the pair (G, H) has the Γ *anti-limsup property* if there is a sequence $(B_i)_{i \in \omega}$ of subsets of X in Γ such that $\limsup_{i \in I} B_i \notin \mathcal{I}_G$ and $\bigcap_{i \in I} B_i$ is H-discrete for all $I \in [\omega]^\omega$. Note that if (G, H) has the Γ anti-limsup property, then so too does every graph which lies between G and H. Recall that E_0 is the equivalence relation on 2^ω given by

$$x E_0 y \iff \exists m \in \omega \forall n \in \omega \setminus m \ (x(m) = y(m)).$$

Proposition 34. *The pair of graphs $(G_0, E_0 \setminus \Delta(2^\omega))$ has the clopen anti-limsup property.*

Proof. As a straightforward Baire category argument shows that every G_0-discrete set with the Baire property is meager, the desired result follows from Proposition 31. ⊠

In particular, we obtain the following:

Theorem 35. *Suppose that X is a Hausdorff space and G is a locally countable analytic graph on X which does not have a Borel ω-coloring. Then G has the compact anti-limsup property.*

Proof. By Theorem 4.1 of Lecomte-Miller [14], there is a locally countable Borel graph H on 2^ω, with $G_0 \subseteq H \subseteq E_0$, for which there is a continuous embedding of H into G. As Proposition 34 ensures that H has the compact anti-limsup property, so too does G. ⊠

A bi-analytic equivalence relation E on X is *smooth* if it is Borel reducible to $\Delta(2^\omega)$. The following fact answers a question of Gao:

Theorem 36. *Suppose that X is a Hausdorff space and E is a non-smooth bi-analytic equivalence relation on X. Then the σ-ideal generated by the family of Borel sets on which E is smooth does not have the compact limsup property.*

Proof. By the Harrington-Kechris-Louveau dichotomy theorem [6], we can assume that $E = E_0$. Set $G = E_0 \setminus \Delta(2^\omega)$ and observe that $\mathcal{I}_G$ is the σ-ideal generated by the family of Borel sets on which E is smooth. As Proposition 34 ensures that G has the compact anti-limsup property, the theorem follows. ⊠

Along similar lines, we have the following:

Theorem 37. *Suppose that X is a Hausdorff space, E is an analytic equivalence relation on X, F is a relatively co-analytic subequivalence relation of E of index 2, and there is no Borel E-complete set on which E and F agree. Then the σ-ideal generated by the family of Borel sets on which E and F agree does not have the compact limsup property.*

Proof. Define $\varphi\colon 2^\omega \to 2^\omega$ by $\varphi(x)(n) = \sum_{m\in n} x(m) \pmod 2$, and let F_0 be the equivalence relation on 2^ω given by $x F_0 y \iff \varphi(x) E_0 \varphi(y)$. By an unpublished result of Louveau, there is a continuous embedding of (E_0, F_0) into (E, F), so we can assume that $(E_0, F_0) = (E, F)$. Set $G = E_0 \setminus F_0$, and observe that $\mathcal{I}_G$ is the σ-ideal generated by the family of Borel sets on which E and F agree. As Proposition 34 ensures that G has the compact anti-limsup property, the theorem follows. ⊠

The following simple observation will allow us to show that a number of other ideals do not have the analytic limsup property:

Proposition 38. *Suppose that X and Y are analytic Hausdorff spaces and X is uncountable. Then there is a sequence $(A_i)_{i\in\omega}$ of analytic subsets of $X \times Y$ such that if $I \in [\omega]^\omega$, then $Y = \operatorname{limsup}_{i\in I}(A_i)_x$ for perfectly many $x \in X$ and $|\bigcap_{i\in I}(A_i)_x| \leq 1$ for all $x \in X$.*

Proof. Fix Borel injections $\varphi\colon S_\infty \to X$ and $\psi\colon Y \to 2^\omega$, as well as an enumeration $(s_i)_{i\in\omega}$ of $2^{<\omega}$. For all $i \in \omega$, define $A_i \subseteq X \times Y$ by

$$A_i = \{(x, y) \in X \times Y \mid \exists \tau \in S_\infty\ (x = \varphi(\tau) \text{ and } \psi(y) \in \mathcal{N}_{s_{\tau(i)}})\}.$$

Suppose that $I \in [\omega]^\omega$. To see that $Y = \operatorname{limsup}_{i\in I}(A_i)_x$ for perfectly many $x \in X$, observe that if $\tau \in S_\infty$ and there exists $J \in [\omega]^\omega$ with

$\bigcup_{j \in J} 2^j \subseteq \tau(I)$, then $Y = \limsup_{i \in I}(A_i)_{\varphi(\tau)}$. To see that $|\bigcap_{i \in I}(A_i)_x| \leq 1$ for all $x \in X$, observe that if $y \in \bigcap_{i \in I}(A_i)_x$, then $\{s_{\tau(i)} \mid i \in I\} \subseteq \{x \restriction n \mid n \in \omega\}$, so $\bigcap_{i \in I}(A_i)_x \subseteq \{y\}$. ⊠

Remark 39. A well-known theorem of Lusin asserts that if X and Y are Polish spaces and $R \subseteq X \times Y$ is Borel, then so too is the set $U = \{x \in X \mid |R_x| = 1\}$ (see §18 of [9]), thus in this case the sets A_i defined in the proof of Proposition 38 are Borel.

As a corollary, we obtain the following:

Theorem 40. *Suppose that X and Y are Hausdorff spaces and $R \subseteq X \times Y$ is analytic. Then exactly one of the following holds:*

1. *The set R has uncountably many uncountable vertical sections.*
2. *The ideal of sets $A \subseteq R$ all of whose vertical sections are countable has the analytic limsup property.*

Proof. To see (1) $\Longrightarrow$ ¬(2), set $X' = \{x \in X \mid |R_x| > \aleph_0\}$. By Proposition 38 there is a sequence $(A_i)_{i \in \omega}$ of analytic subsets of $X' \times Y$ such that if $I \in [\omega]^\omega$, then $Y = \limsup_{i \in I}(A_i)_x$ for perfectly many $x \in X'$ and $|\bigcap_{i \in I}(A_i)_x| \leq 1$ for all $x \in X'$. Set $B_i = A_i \cap R$ for each $i \in \omega$, and observe that if $I \in [\omega]^\omega$, then $\limsup_{i \in I} B_i$ has uncountably many uncountable vertical sections and every vertical section of $\bigcap_{i \in I} B_i$ has cardinality at most one, thus the ideal in question does not have the analytic limsup property.

To see ¬(1) $\Longrightarrow$ (2), suppose that $(A_i)_{i \in \omega}$ is a sequence of analytic subsets of R with the property that for all $I \in [\omega]^\omega$, every vertical section of $\bigcap_{i \in I} A_i$ is countable. Fix an enumeration $(x_i)_{i \in \omega}$ of the set of $x \in X$ for which R_x is uncountable and set $I_0 = \omega$. Given a set $I_n \in [\omega]^\omega$, appeal to Komjáth's theorem [12] to obtain a set $I_{n+1} \in [I_n]^\omega$ for which $\limsup_{i \in I_{n+1}}(A_i)_{x_i}$ is countable. Fix $I \in [\omega]^\omega$ such that $|I \setminus I_n| < \aleph_0$ for all $n \in \omega$, and observe that every vertical section of $\limsup_{i \in I} A_i$ is countable, thus the ideal in question has the analytic limsup property. ⊠

Remark 41. If X and Y are Polish spaces, then a similar argument shows that the analogous result goes through with the Borel limsup property in place of the analytic limsup property.

A *perfect antichain* for $\mathcal{I}$ is a set $R \subseteq 2^\omega \times X$ such that $R_x \notin \mathcal{I}$ and $R_x \cap R_y = \emptyset$ for all $x, y \in 2^\omega$. We say that $\mathcal{I}$ is Γ *non-principal* if there is a subset of X in $\Gamma \setminus \mathcal{I}$ whose singletons are all in $\mathcal{I}$.

Proposition 42. *Suppose that X and Y are analytic Hausdorff spaces, $\mathcal{I}$ is an ideal on X which has an analytic perfect antichain, and $\mathcal{J}$ is* *an analytically non-principal ideal on Y. Then the ideal $\mathcal{I} * \mathcal{J}$ does not have the analytic limsup property.*

Proof. Fix an analytic set $A \subseteq Y$ with $A \notin \mathcal{J}$ and $\{y\} \in \mathcal{J}$ for all $y \in A$. By Proposition 38, there is a sequence $(A_i)_{i \in \omega}$ of analytic subsets of $2^\omega \times Y$ such that if $I \in [\omega]^\omega$, then $\limsup_{i \in I} A_i$ has perfectly many $\mathcal{J}$-positive vertical sections and every vertical section of $\bigcap_{i \in I} A_i$ is in $\mathcal{J}$. Fix an analytic perfect antichain R for $\mathcal{I}$. For all $i \in \omega$, define $B_i \subseteq X \times Y$ by

$$B_i = \{(x, y) \in X \times A \mid \exists w \in 2^\omega \ (x \in R_w \text{ and } y \in (A_i)_w)\}.$$

Suppose that $I \in [\omega]^\omega$. Then $\limsup_{i \in I} B_i$ has an $\mathcal{I}$-positive set of $\mathcal{J}$-positive vertical sections and every vertical section of $\bigcap_{i \in I} B_i$ is in $\mathcal{J}$, so $\mathcal{I} * \mathcal{J}$ does not have the analytic limsup property. ⊠

Remark 43. As ideals associated with descriptive set-theoretic dichotomy theorems always have compact perfect antichains, neither their products with analytically non-principal ideals nor their products with each other have the analytic limsup property.

Remark 44. If X and Y are Polish spaces and the hypotheses on $\mathcal{I}$ and $\mathcal{J}$ are replaced with their Borel analogs, then a similar argument shows that the analogous result goes through with the Borel limsup property in place of the analytic limsup property.

The following corollary answers a question of Balcerzak-Głąb [1]:

Proposition 45. *Suppose that X and Y are uncountable analytic Hausdorff spaces, $\mathcal{I}$ is the trivial ideal on X, and $\mathcal{J}$ is the ideal of countable subsets of Y. Then $\mathcal{I} * \mathcal{J}$ does not have the analytic limsup property.*

Proof. As $\mathcal{I}$ has a compact perfect antichain and $\mathcal{J}$ is compactly non-principal, this follows from Proposition 42. ⊠

Remark 46. If X and Y are Polish spaces and the hypotheses on $\mathcal{I}$ and $\mathcal{J}$ are replaced with their Borel analogs, then a similar argument shows that the analogous result goes through with the Borel limsup property in place of the analytic limsup property.

Theorem 9 yields many examples of Borel quasi-orders for which the σ-ideal generated by the family of Borel chains has the analytic limsup property. On the other hand, we have the following:

Proposition 47. *There is a Borel quasi-order on a Polish space such that the σ-ideal generated by the family of Borel chains does not have the Borel limsup property.*

Proof. Let R denote the quasi-order on $2^\omega \times 2^\omega$ whose corresponding strict quasi-order is given by $(x_0, y_0) <_R (x_1, y_1) \iff x_0 <_{\text{lex}} x_1$. As the Lusin-Novikov uniformization theorem (see §18 of [9]) implies that the σ-ideal generated by the family of Borel chains is the product of the trivial ideal on X with the ideal of countable subsets of Y, the desired result follows from Remark 46. ⊠

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