Some progress on the unique ergodicity problem

Colin Jahel - Soutenance de thèse

Plan of the introduction

- Graphs

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- Graphs
- Automorphisms of a graph

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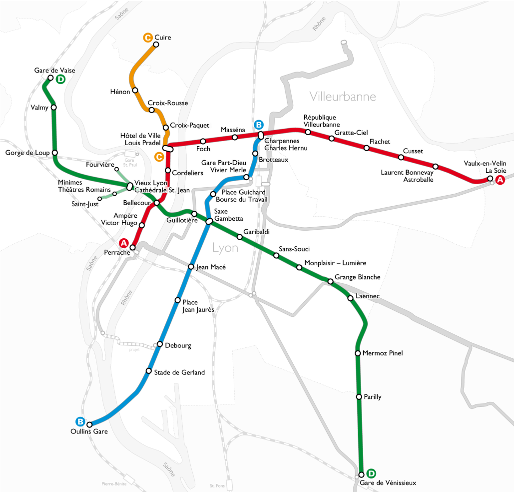
Réseaux du métro et des funiculaires de Lyon

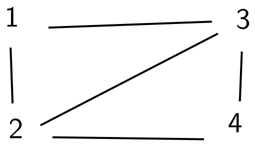
avec le réseau complémentaire du tramway

3 kilomètres



Automorphisms of a graph

An automorphism of a graph is a transformation of the vertices that preserves the presence and absence of edges.



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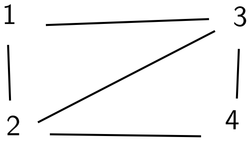
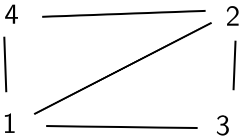
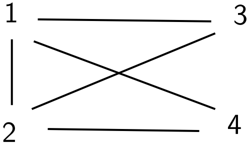


The transformation that sends $(1,2,3,4)$ to $(4,1,2,3)$ gives

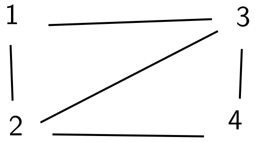


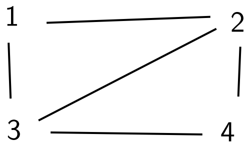
$\simeq$

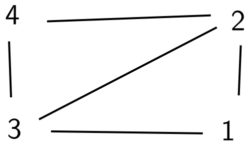


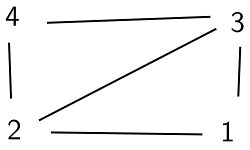
This is not an automorphism.

The automorphisms of our graph are :







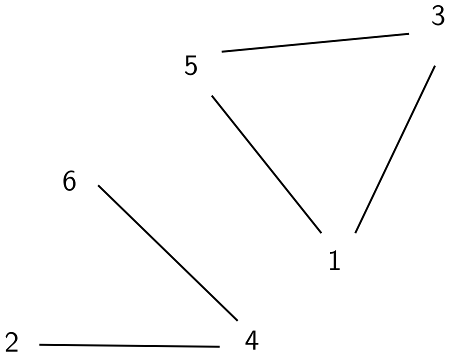


Groups

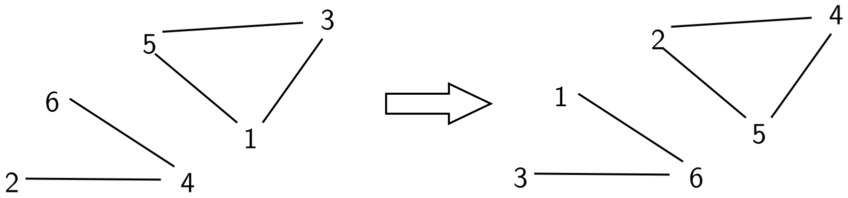
The automorphisms of a graph form what we call a group, i.e. we can compose two of them and inverse any of them.
Dynamics is (roughly) the study of groups that transform spaces.
There are actions on spaces of graphs, but also other rich examples.

We denote by S_6 the group of permutations of 6 elements. For example the map that sends $(1, 2, 3, 4, 5, 6)$ to $(2, 1, 3, 5, 4, 6)$.

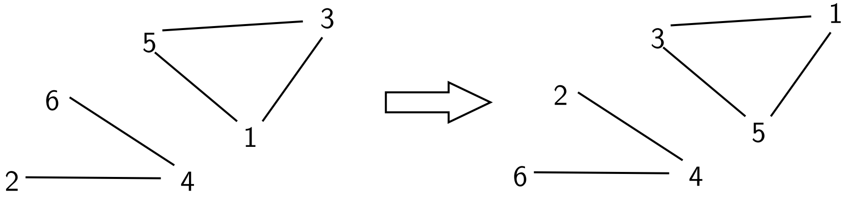
Our graph Γ is the following :



Our group S_6 acts on the space of graphs by moving the vertices around. For example the permutation that sends $(1, 2, 3, 4, 5, 6)$ to $(5, 3, 4, 6, 2, 1)$



The automorphism group of Γ, $\mathrm{Aut}(\Gamma)$, is the subset (in fact subgroup) of S_6 that sends our graph Γ to itself.



Our group S_6 acts on other spaces, for example linear orderings of 6 elements, $\mathrm{LO}(6)$.
For example the map that sends $(1, 2, 3, 4, 5, 6)$ to $(2, 1, 3, 6, 5, 4)$ transforms the ordering

$$3 < 5 < 2 < 1 < 6 < 4$$

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Our subgroup $\mathrm{Aut}(\Gamma)$ also acts on $\mathrm{LO}(6)$, the same as S_6 but with fewer elements.
We write $\mathrm{Aut}(\Gamma) \curvearrowright \mathrm{LO}(6)$.

An infinite graph with a lot of automorphisms

The random graph R.

Construction : Take $\mathbb{N}$ as a domain (vertices) and put an edge between two points with probability $1/2$. Almost surely you obtain the same structure (up to isomorphism), call it R.

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This structure has an interesting automorphism group : if A and B are two finite subgraphs of R and f an isomorphism between A and B, then there is an automorphism of R extending f. This property is called homogeneity.

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Any finite graph can be embedded in R. We say that $\mathrm{Age}(R)$, i.e. the class of finite structures embeddable in R, is the class of finite graphs.

More on Fraïssé structures

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A Fraïssé class $\mathcal{F}$ admits a Fraïssé limit $\mathbb{F}$, i.e. a Fraïssé structure such that $\mathrm{Age}(\mathbb{F})$, the class of finite structures embeddable in $\mathbb{F}$, is exactly $\mathcal{F}$. This limit is unique up to isomorphism.

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Examples :

Fraïssé class	Fraïssé limit	Aut. group
finite graphs	Random graph	$\mathrm{Aut}(R)$
finite sets	$\mathbb{N}$	S_∞
finite linear orderings	$(\mathbb{Q}, <)$	$\mathrm{Aut}(\mathbb{Q})$
finite partial orderings	The generic poset $\mathcal{PO}$	$\mathrm{Aut}(\mathcal{PO})$
finite complete partite graphs	ω-partite graph	$\mathrm{Aut}(\mathrm{Part})$

An important action

Let $\mathbb{F}$ be a Fraïssé limit. If you denote by $\mathrm{LO}(\mathbb{F})$ the space of linear orderings on $\mathbb{F}$, then there is the logic action $\mathrm{Aut}(\mathbb{F}) \curvearrowright \mathrm{LO}(\mathbb{F})$ in the following way :

$$a(g\cdot <)b \Leftrightarrow g^{-1}a < g^{-1}b.$$

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Here the invariant measure is the one such that

$$\mu(x_1 < \cdots < x_n) = \frac{1}{n!}.$$

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1) $G \curvearrowright [0,1]^{\mathbb{F}}$ by permuting the coordinates.
   This flow always admits some invariant measures of the form $\nu^{\mathbb{F}}$ for some ν measure on $[0,1]$.
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Remark : There can be more invariant measures than these.

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Proposition

Any G-flow admits a minimal subflow.

Theorem (Ellis ′69)

There exists a unique universal minimal flow (UMF) $\mathrm{M}(G)$.

This means that for any minimal G-flow $G \curvearrowright X$, there is a surjective G-map from $\mathrm{M}(G)$ to X.

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Definition

G is uniquely ergodic iff every minimal G-flows admits a unique G-invariant measure.
Equivalently : $G \curvearrowright \mathrm{M}(G)$ admits a unique invariant measure.
(Angel, Kechris, Lyons '12).

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- Angel, Kechris and Lyons prove that $\mathrm{Aut}(R)$ is uniquely ergodic.

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3) A sketch of proof of the main result.

0 - The semigeneric graph.

State of the art in 2017 (start of the thesis)

Angel, Kechris and Lyons provide a proof of unique ergodicity for automorphism groups of the Fraïssé limit of graphs, K_n-free graphs for $n \in \mathbb{N}$, metric spaces and r-uniform hypergraphs.

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Pawliuk and Sokic ('20, preprint '17), using methods from [AKL], extended the catalogue of uniquely ergodic automorphism groups with the amenable automorphism groups of homogeneous directed graphs, which were all classified by Cherlin, leaving as an open question only the case of the semigeneric directed graph.

Theorem (J.)

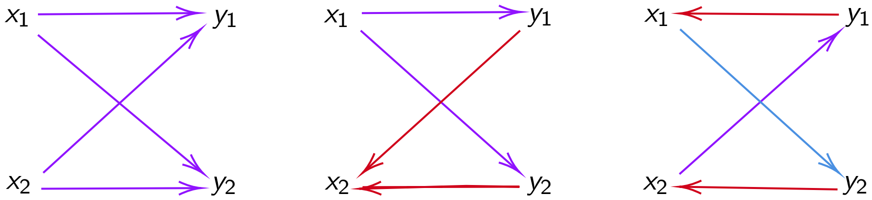
The automorphism group of the semigeneric directed graph is uniquely ergodic.

The semigeneric graph

The semigeneric graph is defined as the Fraïssé limit of finite directed graphs such that :

i) The absence of edge is an equivalence relation $\sim$.

ii) For all $x_1 \sim x_2$ and $y_1 \sim y_2$, the number of (directed) edges from $\{x_1, x_2\}$ to $\{y_1, y_2\}$ is even.



I - Stability under extension

Short exact sequences

Let G be a Polish group, H a closed normal subgroup and K such that

$$1 \to H \to G \to K \to 1$$

is an exact sequence.

Theorem (J., Zucker, $'19^+$)

If $\mathrm{M}(H)$ and $\mathrm{M}(K)$ are metrizable then $\mathrm{M}(G)$ is metrizable. Moreover, under these hypotheses, if H and K are uniquely ergodic, then G is uniquely ergodic.

$G \curvearrowright \mathrm{M}(K)$ is a minimal G-flow, so there is a G-map ϕ from $\mathrm{M}(G)$ to $\mathrm{M}(K)$.

Proposition

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If $\mathrm{M}(G)$ were non metrizable it would need to have larger cardinality.

Disintegration to get unique ergodicity

Let $\mu \in \mathrm{P}(\mathrm{M}(G))$ and $\nu = \phi_* \mu$, then there is a Borel map from $\mathrm{M}(K)$ to $P(M(G))$, $y \mapsto \mu_y$ such that :

i) $\mu_y(\phi^{-1}(\{y\})) = 1$

ii) $\mu = \int \mu_y \mathrm{d}\nu(y)$.

ν and μ_y need to be K and H invariant, therefore are uniquely determined.

	Ame.	Ext. ame.	Metr. UMF	+ unique ergo.
Grp. Ext.	✓	✓	✓	✓
Count. Prod.	✓	✓	✓	✓
Dir. lim.	✓	✓	×	×
Open subgrp	✓	✓	✓	?

Question

Let G be a uniquely ergodic group and U an open subgroup. Is U uniquely ergodic? With the extra assumption of G having metrizable UMF?

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II - Unique ergodicity of the action on linear orderings (for some groups).

Computing UMFs - The Kechris-Pestov-Todorcevic correspondence

Theorem (Kechris-Pestov-Todorcevic, '05)

Let $\mathbb{F}$ *be a Fraïssé limit,* $\mathrm{Aut}(\mathbb{F})$ *is extremely amenable iff* $\mathrm{Age}(\mathbb{F})$ *has the Ramsey property.*

If G admits a "nice enough" extremely amenable subgroup G^*, then

$$\mathrm{M}(G) = \widehat{G/G^*}.$$

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Theorem (Ben Yaacov-Melleray-Nguyen Van Thé-Tsankov '14-'17, Zucker '14)

G *has metrizable UMF iff there exists* $G^* \leq G$ *extremely amenable such that*

$$\mathrm{M}(G) = \widehat{G/G^*}.$$

The class of finite linear orderings has the Ramsey property [Ramsey,'30], therefore [Pestov, '98] $\mathrm{Aut}((\mathbb{Q},<))$ is extremely amenable.
If $G = S_\infty$ then $G^* = \mathrm{Aut}(\mathbb{Q})$ and $\mathrm{M}(S_\infty) = \mathrm{LO}(\mathbb{N})$.

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The class of finite ordered graphs has the Ramsey property [Abramson-Harrington, Nešetřil-Rödl, '78].
If $G = \mathrm{Aut}(R)$, $G^* = \mathrm{Aut}(R_<)$ and $\mathrm{M}(\mathrm{Aut}(R)) = \mathrm{LO}(R)$.

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For the limit of the class of partite complete graphs, the UMF of the automorphism group is the space of linear orderings for which each part is an interval.

For the limit of the class of partial orderings, the UMF of the automorphism group is the space of linear orderings extending the generic poset [Nešetřil-Rödl, Paoli-Trotter-Walker, '84].

Main result

Theorem (J.)

Let $\mathbb{F}$ be a transitive, ω-categorical Fraïssé limit with no algebraicity that admits weak elimination of imaginaries. Denote $G = \mathrm{Aut}(\mathbb{F})$ and consider the action $G \curvearrowright \mathrm{LO}(\mathbb{F})$. Then exactly one of the following holds :

1. *The action $G \curvearrowright \mathrm{LO}(\mathbb{F})$ has a fixed point (i.e., there is a definable linear order on $\mathbb{F}$);*
2. *The action $G \curvearrowright \mathrm{LO}(\mathbb{F})$ is uniquely ergodic.*

Discussing the hypothesis

a) ω-categoricity : for any $n \in \mathbb{N}$ there are finitely many n-types. This is the only hypothesis we are not sure is necessary. Allows us to use a theorem of Tsankov on group representations [Tsankov '12].

Discussing the hypothesis

b) No algebraicity : fixing finitely many points in the structure fixes no other point.

Counterexample : Take $\mathbb{F}$ the countable-dimensional vector space over $\mathbf{F}_2$, the $\mathrm{M}(Aut(\mathbb{F}))$ is a proper subflow of $LO(\mathbb{F})$ [KPT] and the group is uniquely ergodic [AKL].
The group therefore admits at least two invariant measures on $\mathrm{LO}(\mathbb{F})$: the uniform and the one supported on a proper subflow.
There is also no definable ordering on $\mathbb{F}$.

Discussing the hypothesis

c) Weak elimination of imaginaries : for every proper, open subgroup $V < G$, there exists k and a tuple $\bar{a} \in \mathbb{F}^k$ such that $G_{\bar{a}} \leq V$ and $[V : G_{\bar{a}}] < \infty$.

Counterexample : ω-partite complete graph : we saw that the UMF of its automorphism group is a proper subflow of $\mathrm{LO}(\mathbb{F})$ and it also is the support for a measure.

Discussing the hypothesis

d) Transitivity : for any $a, b \in \mathbb{F}$, there is $g \in G$ such that $g(a) = b$.

Counterexample : Take $\mathbb{N}$ with two unary predicates P, Q.
Consider the measure that orders elements of P above elements of Q and orders each part uniformly.

Consequences of the result

a) Recovers a lot of known unique ergodicity results.

The random graph, the homogenenous K_n-free graph, the generic tournament...

Consequences of the result

b) Since there is always one invariant fully supported measure on $\mathrm{LO}(\mathbb{F})$, this allows us to prove non-amenability results.

Corollary

Suppose that $\mathbb{F}$ *satisfies the assumptions of the Theorem and let* $G = \mathrm{Aut}(\mathbb{F})$. *If the action* $G \curvearrowright \mathrm{LO}(\mathbb{F})$ *is not minimal and has no fixed points, then* G *is not amenable.*

Applies for instance for the generic poset [Kechris-Sokić, ′12].

Consequences of the result

c) Allows us to get combinatorial results

Corollary

Suppose that $\mathbb{F}$ *satisfies the assumptions of the Theorem. If* $\mathbb{F}$ *has the Hrushovski property, then it has the ordering property, i.e. for every* $A \in \mathrm{Age}(\mathbb{F})$*, there exists* $B \in \mathrm{Age}(\mathbb{F})$ *such that for any two linear orders* $<$ *and* $<'$ *on* A *and* B *respectively, there is an embedding of* $(A, <)$ *into* $(B, <')$*.*

A very important ingredient

The proof of this relies on

Theorem (Tsankov)

Let $\mathbb{F}$ be an ω-categorical structure with no algebraicity and weak elimation of imaginaries. Then the only $\mathrm{Aut}(\mathbb{F})$-ergodic invariant measures on $[0,1]^{\mathbb{F}}$ are of the type $\nu^{\mathbb{F}}$, where ν is a Borel measure on $[0,1]$.

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Counterexample : ω-partite complete graph. Denote E the equivalence relation being in the same part. There is a map from $[0,1]^{\mathbb{F}/E}$ to $[0,1]^{\mathbb{F}}$. The pushfoward of $\nu^{\mathbb{F}/E}$ to $[0,1]^{\mathbb{F}}$ is not of the form $\nu^{\mathbb{F}}$.

III - Sketch of proof of the main result.

(Sketch of) proof

$G = \mathrm{Aut}(\mathbb{F})$.
Step 1 : An efficient way to produce measures on $\mathrm{LO}(\mathbb{F})$.

Consider the map $\rho\colon [0,1]^{\mathbb{F}} \to \mathrm{LO}(\mathbb{F})$ where
$a <_{\rho(x)} b \Leftrightarrow x(a) < x(b)$.
For any atomless measure λ on $[0,1]$, ρ is $\lambda^{\mathbb{F}}$-a.s. well-defined. We therefore have a measure $\mu_\lambda = \rho_* \lambda^{\mathbb{F}}$.

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Moreover μ_λ is S_∞-invariant, so

$$\mu_\lambda(x_1 < \ldots < x_n) = \frac{1}{n!}.$$

Therefore μ_λ does not depend on λ and we really produced just one measure.

Step 2 : Proving that all measures are produced this way or exhibiting a fixed point of the action.

Take μ a *G*-invariant ergodic measure on $\mathrm{LO}(\mathbb{F})$, i.e. an extreme point of the set of *G*-invariant measures on $\mathrm{LO}(\mathbb{F})$.
We want a map from $\mathrm{LO}(\mathbb{F})$ to $[0,1]^{\mathbb{F}}$ that reverses ρ and pushes μ to some $\lambda^{\mathbb{F}}$.

We want to associate a number to each $a \in \mathbb{F}$ and each ordering.

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First idea : associate to a, $<_x$ the number

$$\lim_{n\to\infty} \frac{\#\{b \in F_n \;:\; b <_x a\}}{\# F_n}$$

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where F_n is an enumeration of $\mathbb{F}$.
Problem : this is not well defined.

Solution :

Consider τ a 2-type and $a \in \mathbb{F}$, we call

$$D_\tau(a) = \{b \in \mathbb{F} \colon \mathrm{tp}(ab) = \tau\}.$$

Lemma

Let $a \in \mathbb{F}$ and τ a 2-type. Take $A \subset D_\tau(a)$ be a definable, infinite set. Then for μ-a.e. x,

$$\lim_{n\to\infty} \frac{\#\{b \in F_n \cap A \ : \ b <_x a\}}{\# \ F_n \cap A}$$

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Consequence of Tsankov's Theorem.

We can now define for almost all $x \in \mathrm{LO}(\mathbb{F})$

$$\eta_a^\tau(x) = \lim_{n\to\infty} \frac{\#\{b \in F_n \cap D_\tau(a) \ : \ \ b <_x a\}}{\# \ F_n \cap D_\tau(a)}.$$

Lemma

If we denote λ the distribution of η_a^τ, then the family $(\eta_a^\tau)_{a\in\mathbb{F}}$ has distribution $\lambda^{\mathbb{F}}$.

This is again a consequence of Tsankov's Theorem.

We want to prove (if possible) that a.s.

1) λ is atomless.

2) For all $a, b \in \mathbb{F}$, we have

$$a < b \Leftrightarrow \eta_a^\tau < \eta_b^\tau.$$

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1) λ is atomless.

2) For all $a, b \in \mathbb{F}$, we have

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1) is not always true, we will have to assume it (for now), and prove 2).

Lemma

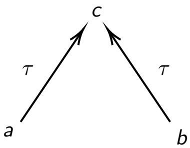
If the distribution of η_a^τ *is atomless, then we have a.s. for all* $a, b \in \mathbb{F}$ *:*

$$a < b \Leftrightarrow \eta_a^\tau < \eta_b^\tau.$$

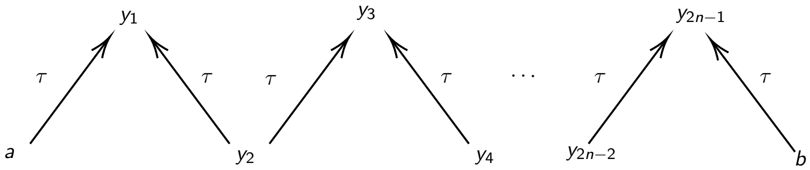
Remark

If $D_\tau(a) \cap D_\tau(b) \neq \varnothing$*, then*

$$
\begin{aligned}
a < b \Rightarrow & \{c \in D_\tau(a) \cap D_\tau(b) \cap F_n \, : \, c < a\} \\
& \subset \{c \in D_\tau(a) \cap D_\tau(b) \cap F_n \, : \, c < b\} \\
\Rightarrow & \frac{\#\{c \in D_\tau(a) \cap D_\tau(b) \cap F_n \, : \, c < a\}}{\# \, D_\tau(a) \cap D_\tau(b) \cap F_n} \\
& \leq \frac{\#\{c \in D_\tau(a) \cap D_\tau(b) \cap F_n \, : \, c < b\}}{\# \, D_\tau(a) \cap D_\tau(b) \cap F_n} \\
\Rightarrow & \eta_a^\tau \leq \eta_b^\tau \\
\Rightarrow & \eta_a^\tau < \eta_b^\tau \text{ (by the atomless assumption).}
\end{aligned}
$$



If $D_\tau(a) \cap D_\tau(b) = \varnothing$, our hypothesis imply that there are infinitely many "alternating τ-paths" between a and b.



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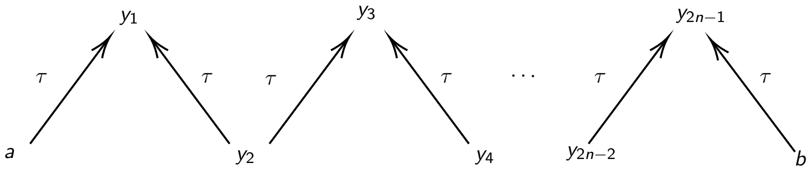

Since $\eta_a^\tau < \eta_b^\tau$ and the $(\eta_c^\tau)_{c\in\mathbb{F}}$ are i.i.d., there must be a path such that

$$\eta_a^\tau < \eta_{y_2}^\tau < \cdots < \eta_{y_{2n-2}}^\tau < \eta_b^\tau$$

which implies

$$a < y_2 < \cdots < y_{2n-2} < b.$$

We proved

Lemma

If the distribution of η_a^τ is atomless, then we have a.s. for all $a, b \in \mathbb{F}$:

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Lemma

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Denote λ the distribution of η_a^τ and assume it is atomless. The hypothesis of the Lemma are verified, and the map ϕ

$$\begin{aligned} \mathrm{LO}(\mathbb{F}) &\to [0,1]^{\mathbb{F}} \\ <_x &\mapsto (\eta_a^\tau(x))_{a\in\mathbb{F}} \end{aligned}$$

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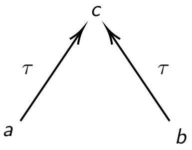
is the converse of ρ and $\phi_*\mu$ is of the form $\lambda^{\mathbb{F}}$.
By step 1, μ is the uniform measure!

There remains the case when $\mu(\eta_a^\tau = \eta_b^\tau = p) > 0$ for some p. This is the case when there will be a definable ordering.

The important remark is that if



then

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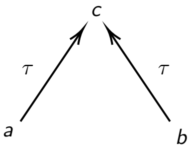

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Indeed,

$$\begin{aligned}\mu(a < c < b | \eta_a^\tau = \eta_b^\tau = p) \\ &= \mathbb{E}\left[\frac{\#\{c' \in F_n \cap (G_{a,b} \cdot c) \; : \; a < c' < b\}}{\# F_n \cap (G_{a,b} \cdot c)} | \eta_a^\tau = \eta_b^\tau = p\right] \\ &\to \mathbb{E}\left[\eta_b^\tau - \eta_a^\tau | \eta_a^\tau = \eta_b^\tau = p\right] = 0.\end{aligned}$$

In particular :

$$\mu(a < c < b | \eta_a^\tau = \eta_b^\tau = \eta_c^\tau = p) = 0$$

for all $c \in D_\tau(a) \cap D_\tau(b)$.

In particular :

$$\mu(a < c < b | \eta^\tau_a = \eta^\tau_b = \eta^\tau_c = p) = 0$$

for all $c \in D_\tau(a) \cap D_\tau(b)$.
We define a new measure ν by taking

$$\nu(x_1 < \cdots < x_n) = \mu(x_1 < \cdots < x_n | \eta^\tau_{x_1} = \ldots = \eta^\tau_{x_n} = p).$$

ν is supported on a proper subflow of $G \curvearrowright \mathrm{LO}(\mathbb{F})$.

Under ν, one can again define $\eta_a^{\tau^{-1}}$ for all $a \in \mathbb{F}$.
Necessarily, this $\nu(\eta_a^{\tau^{-1}} = q) > 0$ for some $q \in [0, 1]$.
We define ν' as

$$\nu'(x_1 < \cdots < x_n) = \nu(x_1 < \cdots < x_n | \eta_{x_1}^{\tau^{-1}} = \ldots = \eta_{x_n}^{\tau^{-1}} = q).$$

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For all $a, b, c \in \mathbb{F}$ such that $c \in D_{\tau^{-1}}(a) \cap D_{\tau^{-1}}(b)$

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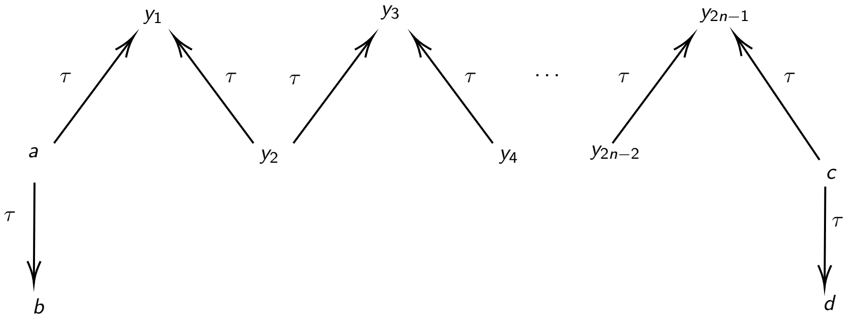
For all $a, b, c \in \mathbb{F}$ such that $c \in D_{\tau^{-1}}(a) \cap D_{\tau^{-1}}(b)$

$$\nu'(a < c < b) = 0.$$

Take $a, b, c, d \in \mathbb{F}$ such that $\mathrm{tp}(ab) = \mathrm{tp}(cd) = \tau$, then ν'-as $a < b$ iff $c < d$.



y1
y3
y2n−1
τ
τ
τ
τ
· · ·
τ
τ
a
y2
y4
y2n−2
c
τ
τ
b
d

By iterating this process for all 2-types, we get a measure that is a Dirac mass. Therefore we have a fixed point !

Thank you !