A DE FINETTI THEOREM FOR THE RATIONAL URYSOHN SPACE (AND OTHER STRUCTURES)

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Abstract. The classical De Finetti Theorem classifies the S_∞-invariant probability measure on $[0,1]^{\mathbb{N}}$. More precisely it states that those invariant measures are mixes of measures of the form $\nu^{\mathbb{N}}$ where ν is a measure on $[0,1]$. Recently, Tsankov generalized this theorem showing that under condition that $\mathbf{M}$ is an $\aleph_0$-categorical structure with no algebraicity and that weakly eliminates imaginaries, then the Aut($\mathbf{M}$)-invariant measures on $[0,1]^{\mathbf{M}}$ are mixes of measures of the form $\nu^{\mathbf{M}}$ where ν is a measure on $[0,1]$.

In this note, we use the result of Tsankov to generalize it to a wider class of structures, that includes the rational Urysohn space.

1. Preliminaries

1.1. **Model Theory.** A relational countable language $\mathcal{L}$ is a countable collection of symbols (relations), to each of which is associated a positive natural number, that we call its arity. A structure $\mathbf{M}$ in a language $\mathcal{L}$ is a domain, that we denote by $\mathrm{Dom}(\mathbf{M})$, and an interpretation of the symbols in $\mathcal{L}$, i.e. to each relation $R \in \mathcal{L}$ of arity r is associated a subset of $\mathrm{Dom}(\mathbf{M})^r$, that corresponds to the elements verifying the relation. For a structure $\mathbf{M}$ and R a symbol of arity r in its language, we write $R^{\mathbf{M}}(x_1, \ldots, x_r)$ to mean that $(x_1, \ldots, x_r)$ verifies R in $\mathbf{M}$.

A substructure of a given structure $\mathbf{A}$ is a structure whose domain is included in $\mathrm{Dom}(\mathbf{A})$ and the relations are the relations induced by restriction. An embedding from a structure $\mathbf{A}$ into a structure $\mathbf{B}$ in the same language $\mathcal{L}$ is a map f from $\mathrm{Dom}(\mathbf{A})$ to $\mathrm{Dom}(\mathbf{B})$ such that for any $R \in \mathcal{L}$ with arity r and $x_1, \ldots, x_r \in \mathrm{Dom}(\mathbf{A})$, we have $R^{\mathbf{A}}(x_1, \ldots, x_r) \Leftrightarrow R^{\mathbf{B}}(f(x_1), \ldots, f(x_r))$. If there is such an f, it needs to be injective. If there is such a map that is bijective, we say that $\mathbf{A}$ and $\mathbf{B}$ are isomorphic. If it is a bijection and $\mathbf{A} = \mathbf{B}$, we call it an automorphism of $\mathbf{A}$.

A class $\mathcal{F}$ of finite structures is a Fraïssé class if it contains structures of arbitrarily large (finite) cardinality and satisfies the following:

i) (Hereditary Property) If $A \in \mathcal{F}$ and B is a substructure of A, then $B \in \mathcal{F}$.

ii) (Joint Embedding Property) If $A, B \in \mathcal{F}$ then there exists $C \in \mathcal{F}$ such that A and B can be embedded in C.

iii) (Amalgamation Property) If $A, B, C \in \mathcal{F}$ and $f\colon A \to B$, $g\colon A \to C$ are embeddings, then there exists $D \in \mathcal{F}$ and $h\colon B \to D$, $l\colon C \to D$ embeddings such that $h \circ f = l \circ g$.

A Fraïssé class $\mathcal{F}$ admits a Fraïssé limit which is a countable structure whose age, i.e. the set of its finite substructures up to isomorphism, is $\mathcal{F}$. Fraïssé limits are *homogeneous*, i.e. any isomorphism between two finite substructures of the structure can be extended to an automorphism of the structure. The Fraïssé limit of a Fraïssé class is unique up to isomorphism. For more details on Fraïssé classes see [H].

An example of a Fraïssé class is the class of finite metric space with rational distances. The limit of this class is called the rational Urysohn space, it is the main example to which the main theorem of this note applies. More generally, if $D \subset \mathbb{Q}_+$ is such that the finite metric spaces with distances in D is a Fraïssé class, then we call its limit the Urysohn space over D and denote it by $\mathcal{U}_D$.

We define here the model-theoretic assumptions that we will need to express the theorems. The definitions here are given from a permutation group perspective and may require some work to prove that they are equivalent their original formulation. Let $\mathbf{M}$ be a Fraïssé limit, we say that:

1) $\mathbf{M}$ has no algebraicity if for any tuple $\bar{a} \in \mathbf{M}$, for any $x \notin \bar{a}$, $G_{\bar{a}} \cdot x$ is infinite, where $G_{\bar{a}}$ denotes the stabilizer of $\bar{a}$ for the action of $\mathrm{Aut}(\mathbf{M})$ on $\mathbf{M}$.
2) $\mathbf{M}$ is $\aleph_0$-categorical if for all $n \in \mathbb{N}$, $G \curvearrowright M^n$ has finitely many orbits.
3) $\mathbf{M}$ has weak elimination of imaginaries if for every proper, open subgroup $V < \mathrm{Aut}(\mathbf{M})$, there exists k and a tuple $\bar{a} \in \mathbf{M}^k$ such that $G_{\bar{a}} \leq V$ and $[V : G_{\bar{a}}] < \infty$.
4) $\mathbf{M}$ is said to be transitive if for any $a, b \in \mathbf{M}$, there is $g \in \mathrm{Aut}(\mathbf{M})$ such that $g(a) = b$.

1.2. **Dynamics.** Let G be a Polish group. A G-flow is a continuous action of a topological group G on a compact space, we write $G \curvearrowright X$. For example, if $\mathbf{M}$ is a Fraïssé limit, then $\mathrm{Aut}(\mathbf{M})$ acts on $[0,1]^M$ as follows. If $x \in [0,1]^M$, $g \in G$ and $a \in M$, then $(g \cdot x)(a) = x(g^{-1}a)$.

An invariant measure on a flow $G \curvearrowright X$ is a Borel measure μ on X such that for all $g \in G$ and $A \subset X$ measurable, $\mu(g \cdot A) = \mu(A)$. The following definition is useful when describing measures.

Definition 1.1. Let G be a Polish group acting continuously on a compact space X. A G-invariant measure ν is said to be *G-ergodic* if for all $A \subset X$ measurable such that

$$\forall g \in G,\ \nu(A \triangle g \cdot A) = 0,$$

we have $\nu(A) \in \{0, 1\}$.

We can now state the the theorem describing the extreme point of the convex set of invariant measures (see [P1] Proposition 12.4):

Theorem 1.2. *Let G be a Polish group acting continuously on a compact space X. Let $P_G(X)$ denote the convex compact space of G-invariant measures on X. Then the extreme points of $P_G(X)$ are the G-ergodic invariant measures.*

2. Main result

Definition 2.1. A Fraïssé limit $\mathbf{M}$ is said to be locally well behaved if every finite substructure A can be embedded in a Fraïssé limit $\mathbf{M}'$ such that $\mathrm{Age}(\mathbf{M}') \subset \mathrm{Age}(\mathbf{M})$ and $\mathbf{M}'$ is ω-categorical, has no algebraicity and weakly eliminates imaginaries.

Definition 2.2. A Fraïssé limit is said to be countably-homogeneous if for all Fraïssé limit $\mathbf{M}'$ such that $\mathrm{Age}(\mathbf{M}') \subset \mathrm{Age}(\mathbf{M})$, there is a copy $\mathbf{N} \subset \mathbf{M}$ of $\mathbf{M}'$ such that any automorphism of $\mathbf{N}$ can be extended into an automorphism of $\mathbf{M}$.

We will see in the next two sections that the rational Urysohn space is countably homogeneous and locally well behaved and it therefore we can apply to it the following theorem:

Theorem 2.3. *Let $\mathbf{M}$ be a countably-homogeneous locally well behaved Fraïssé limit. Let μ be an $\mathrm{Aut}(\mathbf{M})$ invariant ergodic measure on $[0,1]^{\mathbf{M}}$. Then μ has to be of the form $\nu^{\otimes \mathbf{M}}$ for some ν measure on $[0,1]$.*

Before starting the proof, we recall this following theorem from [JT]:

Theorem 2.4. *Let $\mathbf{M}$ be an $\aleph_0$-categorical structure with no algebraicity that admits weak elimination of imaginaries. Then the only invariant, ergodic probability measures on $[0,1]^{\mathbf{M}}$ are product measures of the form $\lambda^{\otimes \mathbf{M}}$, where λ is a probability measure on $[0,1]$.*

Proof. Let us take a family $(\eta_a)_{a\in \mathbf{M}}$ with distribution μ. We need to show that for any two disjoint finite $A, B \subset \mathbf{M}$, then $(\eta_a)_{a\in A} \perp\!\!\!\perp (\eta_b)_{b\in B}$. Let $\mathbf{N}$ be a Fraïssé limit such that $\mathbf{N} \subset \mathbf{M}$ that is ω-categorical, has no algebraicity and weakly eliminates imaginaries. We also assume that any automorphism of $\mathbf{N}$ can be extended into an automorphism of $\mathbf{M}$ and $A, B \subset \mathbf{N}$.

We consider $\lambda = \phi_* \mu$ where ϕ is the restriction map from $[0,1]^{\mathbf{M}}$ to $[0,1]^{\mathbf{N}}$. We want to show that λ is Aut($\mathbf{N}$)-invariant.

For $u \in$ Aut($\mathbf{N}$), we denote by g_u one of its extensions in Aut($\mathbf{M}$)

Fact 2.5. *For any event $A \subset [0,1]^{\mathbf{N}}$, $\phi^{-1}(u \cdot A) = g_u \cdot \phi^{-1}(A)$.*

Proof. Consider $\omega \in g_u \cdot \phi^{-1}(A)$, then $g_u^{-1} \cdot \omega \in \phi^{-1}(A)$, hence $g_u^{-1}\omega_{\restriction \mathbf{N}} \in A$, and since $(g_u)_{\restriction \mathbf{N}} = u$, we have $\omega_{\restriction \mathbf{N}} \in u \cdot A$, we have the first inclusion.

For the second one, let us take $\omega \in \phi^{-1}(u \cdot A)$, we construct α so that $\alpha \in \phi^{-1}(A)$ and $g_u \sigma = \omega$. We take $\sigma = g_u^{-1}\omega$ which obviously satisfies the conditions. □

This implies that λ is Aut($\mathbf{N}$)-invariant. We can apply theorem 2.4 in $\mathbf{N}$, we have $\lambda = \int \nu^{\otimes \mathbf{N}} \mathrm{d}\alpha$ for some measure α on P([0,1]). In particular, we have $(\eta_a)_{a\in A} \perp\!\!\!\perp (\eta_b)_{b\in B}$ conditionally on α. It is now easy to conclude that $\mu = \int \nu^{\otimes \mathbf{M}} \mathrm{d}\alpha$, simply by equality of the finite dimensional marginals. By ergodicity of μ, α needs to be a Dirac mass and we have the result. □

3. Locally well-behaved structures

An important remark regarding locally well-behaved structures is that for $\aleph_0$ categoric structures with algebraicity, weak elimination of imaginaries has a somewhat simple caracterization. We use a criterion from Poizat, stated as Lemma 16.17 in [P2]:

Proposition 3.1. *Suppose $\mathbf{M}$ is $\aleph_0$-categorical with no algebraicity. Let $H = Aut(\mathbf{M})$. Then weak elimination of imaginaries is implied by $\langle H_B, H_C \rangle = H_{B\cap C}$ for all finite $B, C \subseteq M$, where H_B is the pointwise stabiliser of B.*

I am thankful to David Evans for pointing me towards this proposition.

Fact 3.2. *Let D be a finite subset of the positive rationals. There is $D \subset D'$ such that the metric spaces over D' amalgamate and the Fraïssé limit $\mathcal{U}_{D'}$ weakly eliminates imaginaries and has no algebraicity.*

Up to multiplying by an integer, we can assume that D is a subset of $\mathbb{N}$ and denote by d its biggest element, then $D' = \{0, 1, \ldots, d\}$ satisfies the above conditions.

4. Countably-homogeneous structures

Being countably homogeneous for a Fraïssé limit $\mathbf{M}$ is the same as saying that Aut($\mathbf{M}$) is universal in the class of Aut($\mathbf{N}$) where $\mathbf{N}$ ranges over the class Fraïssé limits embeddable in $\mathbf{M}$.

This property has been researshed in the past, Uspenskij proving that the rational Urysohn space is countably homogeneous [U], Bilge and Melleray generalized this construction to free-amalgamation classes [BM] and Müller extending it to structures with stationnary independence relation [M]. The proofs of those results relies on the same idea of starting with $\mathbf{N}$ an embeddable structure in $\mathbf{M}$ and an automorphism ϕ of $\mathbf{N}$ and extending ϕ to the one point-extension of $\mathbf{N}$ and iterating this process. This construction is generalizable to other structure that have a "canonical" way of amalgamating, but it does not work for tournaments for example.

5. Consequences of this theorem

One can use Theorem 2.3 to prove:

Theorem 5.1. *Let $\mathbf{M}$ be a transitive, locally well-behaved and countably homogeneous structure. Consider the action $\mathrm{Aut}(\mathbf{M}) \curvearrowright \mathrm{LO}(\mathbf{M})$. Then exactly one of the following holds:*

(i) *The action $\mathrm{Aut}(\mathbf{M}) \curvearrowright \mathrm{LO}(\mathbf{M})$ has a fixed point (i.e., there is a definable linear order on M);*
(ii) *The action $\mathrm{Aut}(\mathbf{M}) \curvearrowright \mathrm{LO}(\mathbf{M})$ is uniquely ergodic.*

This is a generalization of Theorem 1.2 in [JT]; the proof of this theorem closely follows the proof in [JT]. The only modification necessary for the proof to work is to check that alternating τ-path do exist and that there are infinitely many of them, which can be done using the locally well-behaved assumption.

The remaining of the paper is a description of some aspects of Kingman's theory, taken from [B]. The classical Kingman theorem being a consequence of the classical De Finetti's theorem, our point is to show that this correlation still holds in a more general context.

Consider a Fraïssé limit $\mathbf{M}$. We denote by $\mathrm{Part}(\mathbf{M})$ the space of equivalence relations on the domain of $\mathbf{M}$. This is a compact space on which $\mathrm{Aut}(\mathbf{M})$ acts. A random partition is a measure on $\mathrm{Part}(\mathbf{M})$, it is said to be $\mathrm{Aut}(\mathbf{M})$-ergodic invariant iff the associated measure is.

An example of such a random partition is the so-called *Paint Box partition*. Consider ρ a partitions of $[0,1]$ in intervals and $(U_i)_{i\in\mathbf{M}}$ an i.i.d. family of uniform random variables on $[0,1]$, we define the random partition π as

$$i \sim_\pi j \Leftrightarrow U_i \text{ and } U_j \text{ are in the same interval of } \rho.$$

Kingman's theorem says that the $S_{<\infty}$-ergodic invariant permutations of $\mathbb{N}$ are precisely the Paint Box partitions.

We say that $\mathbf{M}$ satisfies a De Finetti's theorem iff the only $\mathrm{Aut}(M)$-ergodic invariant measures on $[0,1]^{\mathbf{M}}$ are of the form $\nu^{\otimes\mathbf{M}}$ for some ν measure on $[0,1]$.

Theorem 5.2 (Kingman's Theorem)**.** *If a Fraïssé limit $\mathbf{M}$ satisfies a De Finetti's theorem, then the only $\mathrm{Aut}(\mathbf{M})$-ergodic invariant random partitions of $\mathbf{M}$ are Paint Box partitions.*

This proof follows closely the proof of Theorem 2.1 in [B].

Proof. For a given partition π, we call $b\colon \mathbf{M} \to \mathbf{M}$ a selection if for all $i \in \mathbf{M}$, $b(i) \sim_\pi i$ and if $i \sim_\pi j$, then $b(i) = b(j)$. For example, if one orders discretely the domain of $\mathbf{M}$, a selection can be the minimum of the equivalence class for this ordering.

Let $\tilde{\pi}$ be an $\mathrm{Aut}(\mathbf{M})$-ergodic invariant random partition of $\mathbf{M}$, and b a selection for $\tilde{\pi}$. Consider $(U_i)_{i\in\mathbf{M}}$ a family of i.i.d. uniform random variable on $[0,1]$ that is independent from $\tilde{\pi}$ and b. We define $\xi_i = U_{b(i)}$. Observe that the distribution of $(\xi_i)_{i\in M}$ does not depend on b. Moreover, $(\xi_i)_{i\in\mathbf{M}}$ is $\mathrm{Aut}(\mathbf{M})$-ergodic invariant.

Let ν be the distribution of ξ_i and q its quantile, i.e.

$$q(p) = \inf\{x \in [0,1] \colon \nu([0,x]) \geq p\}.$$

We can define the family of flat points of q,

$$F = \{x \in [0,1] \colon \exists \varepsilon > 0 \text{ such that } q(x) = q(y) \text{ whenever } |x-y| < \varepsilon\}.$$

Let ρ be the partition of $[0,1]$ in intervals induced by F, i.e. the family of interval in F and in $[0,1]\setminus F$.

Let $(V_i)_{i\in\mathbf{M}}$ be an i.i.d. family of uniform random variables on $[0,1]$. In particular $(q(V_i))_{i\in\mathbf{M}}$ has the same distribution as $(\xi_i)_{i\in\mathbf{M}}$. The Paint Box partition induced by ρ and V_i has same distribution as $\tilde{\pi}$. □

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