Rank-one Generated Spectral Cones Defined by Two Homogeneous Linear Matrix Inequalities

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Introduction – Motivation

Nonconvex quadratic program

$$\max_x \; x^\top Q x$$
$$\text{s.t. } x^\top M_i x \geq 0, \; i = 1, \ldots, k$$

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$$\begin{aligned} \max_x \ & \langle Q, x x^\top \rangle \\ \text{s.t. } & \langle M_i, x x^\top \rangle \geq 0, \ i = 1, \ldots, k \end{aligned}$$

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We have $x^\top A x = \mathrm{Tr}(x^\top A x) = \mathrm{Tr}(A x x^\top) = \langle A, xx^\top \rangle$.

$$\max_x \; \langle Q, xx^\top \rangle \quad \text{s.t. } \langle M_i, xx^\top \rangle \geq 0,\; i = 1,\dots,k$$

Can write $X = xx^\top$ if and only if $X \succeq 0$ and $\mathrm{rank}(X) = 1$.

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The condition $\operatorname{rank}(X) = 1$ is nonconvex.
Convex (semidefinite program) relaxation:

$$\begin{aligned} \max_X \ & \langle Q, X \rangle \\ \text{s.t. } & \langle M_i, X \rangle \geq 0, \ i = 1, \ldots, k \\ & X \succeq 0 \end{aligned}$$

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- When is this relaxation tight?
- Feasible set perspective.
- Tight for **every** objective function if and only if every extreme ray is rank one (Rank-One Generated/ROG).
- Analogous to integral polyhedra/total unimodularity.
- Burer '15, Hildebrand '16, Blekherman et al. '16.

Introduction – Our Question

Question

Let M_1, M_2 be $n \times n$ symmetric matrices.

When is

$$\mathcal{S} := \{Y \succeq 0 : \langle Y, M_1 \rangle \geq 0, \langle Y, M_2 \rangle \geq 0\}$$

an ROG cone?

Introduction – Outline

Two geometric perspectives.

Each perspective gives a sufficient condition for $\mathcal{S}$ to be ROG.

Together these conditions are also necessary.

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Introduction – Outline

Two geometric perspectives.
Each perspective gives a sufficient condition for $\mathcal{S}$ to be ROG.
Together these conditions are also necessary.

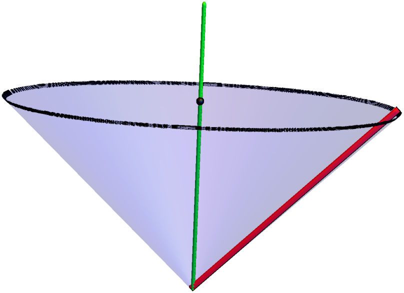
- $\langle M, Y\rangle = 0$ as a hyperplane in $\mathbb{S}^n := \{n \times n$ symmetric matrices$\}$.
- $\langle M, xx^\top\rangle = x^\top M_1 x$ as a quadratic form in $\mathbb{R}^n$.

Introduction – Recap

- A set/cone is ROG if all its extreme points/rays have rank 1.
- SDP relaxations of quadratic programs are tight for every objective function if and only if the feasible set is ROG.
- Consider $\mathcal{S} := \{Y \succeq 0 : \langle Y, M_1 \rangle \geq 0, \langle Y, M_2 \rangle \geq 0\}$ (two LMIs).
- Two geometric perspectives – $\mathbb{S}^n$ and $\mathbb{R}^n$.

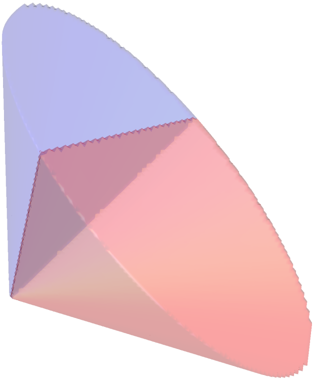
Geometry of $\mathbb{S}^n_+$ – Rank

Consider $\mathbb{S}^n_+ :=$ {positive semidefinite $n \times n$ matrices} $\subseteq \mathbb{S}^n$.



- Red ray: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
- Green ray: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- Rank 1 $\leftrightarrow$ extreme.
  Rank $\geq 2 \leftrightarrow$ not extreme.

Geometry of $\mathbb{S}^n_+$ – One LMI

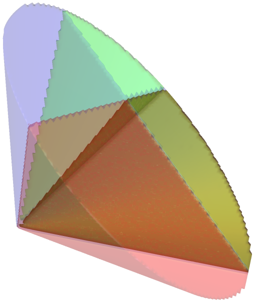


Fact [Ye, Zhang '03]

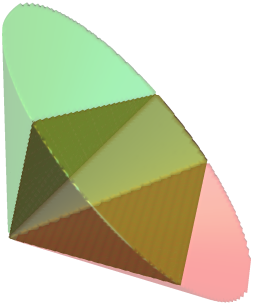
$\mathcal{S} := \{Y \succeq 0 : \langle M, Y \rangle \geq 0\}$ is ROG for any $M \in \mathbb{S}^3$.

Geometry of $\mathbb{S}^n_+$ – Two LMIs

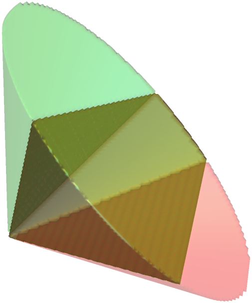
Interacting inside $\mathbb{S}^n_+$.



Non-interacting inside $\mathbb{S}^n_+$.



Geometry of $\mathbb{S}^n_+$ – Non-interacting LMIs



If M_1 and M_2 are non-interacting, then every extreme ray of

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is an extreme ray of either

$$\{Y \succeq 0 : \langle Y, M_1 \rangle \geq 0\}$$

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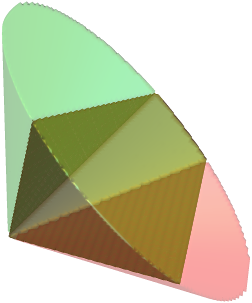

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$\Rightarrow \mathcal{S}$ is ROG.

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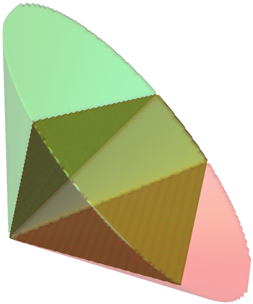

Non-interacting inside $\mathbb{S}^n_+$ when:

- One LMI does not intersect $\mathbb{S}^n_+$, i.e. when $\pm M_i \succeq 0$.

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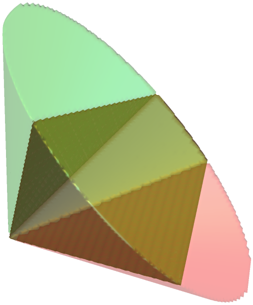

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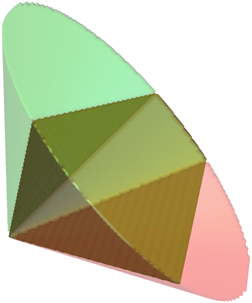

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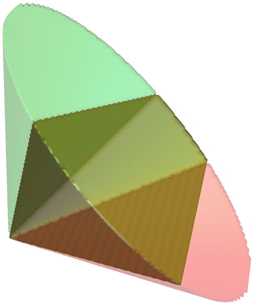
- Using S-lemma, this is true when $\lambda(\pm M_1) - (\pm M_2) \succeq 0$ for some $\lambda \geq 0$.

Geometry of $\mathbb{S}^n_+$ – Non-interacting LMIs



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In sum, non-interacting when $\alpha M_1 + \beta M_2 \succeq 0$ for some $(\alpha, \beta) \neq (0, 0)$.

Geometry of $\mathbb{S}^n_+$ – Recap

- Non-interacting LMIs yield ROG cones.
- M_1, M_2 are non-interacting if $\langle \pm M_2, Y \rangle \geq 0$ along with $Y \succeq 0$ implies $\langle \pm M_1, Y \rangle \geq 0$.

Proposition 1

If $\alpha M_1 + \beta M_2 \succeq 0$ has a nontrivial solution, i.e. $(\alpha, \beta) \neq (0, 0)$ then $\mathcal{S}$ is ROG.

Showing $Y \notin \text{Ext}(\mathcal{S})$

Question

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In general, how do we show that a cone is ROG?

Show that $Y \notin \text{Ext}(\mathcal{S})$ when:

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- For $Y - xx^\top \succeq 0$, need $x \in \text{Range}(Y)$.

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Find $x \in \mathbb{R}^n$ such that $Y \pm xx^\top \in \mathcal{S}$.

- For $Y - xx^\top \succeq 0$, need $x \in \mathrm{Range}(Y)$.
- Since $\langle Y, M_i \rangle = 0$, need $0 = \langle xx^\top, M_i \rangle = x^\top M_i x$.

Quadratic Forms – Zero Sets

Fix a candidate extreme ray Y. Define

$$\mathcal{N}_1 := \{x \in \mathbb{R}^n : x^\top M_1 x = 0\}.$$

$$\mathcal{N}_2 := \{x \in \mathbb{R}^n : x^\top M_2 x = 0\}.$$

When is $\mathrm{Range}(Y) \cap \mathcal{N}_1 \cap \mathcal{N}_2 \neq \{0\}$?

Quadratic Forms

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- If $\mathcal{N}_1 \cap \mathcal{N}_2 \subseteq \mathbb{R}^3$ contains a plane, then it intersects every plane nontrivially.

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Observation

$\mathcal{S}$ is ROG when $\mathcal{N}_1 \cap \mathcal{N}_2$ contains a plane.

Quadratic Forms – $\mathcal{N}_1 \cap \mathcal{N}_2$ contains a plane

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- $\{x \in \mathbb{R}^3 : x^\top M x = 0\}$ contains a plane when $\operatorname{rank}(M) \leq 2$ and M is indefinite.

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- $\{x \in \mathbb{R}^3 : x^\top M x = 0\}$ contains a plane when $\text{rank}(M) \le 2$ and M is indefinite.
- For any (α, β),

$$\mathcal{N}_{\alpha,\beta} := \{x \in \mathbb{R}^3 : x^\top(\alpha M_1 + \beta M_2)x = 0\} \supseteq \mathcal{N}_1 \cap \mathcal{N}_2.$$

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Answer*

When $\operatorname{rank}(\alpha M_1 + \beta M_2) \leq 2$ for all (α, β).

Geometry of Quadratic Forms – Recap

- Y is not an extreme ray when $\mathrm{Range}(Y) \cap \mathcal{N}_1 \cap \mathcal{N}_2$ has a nonzero element.

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Proposition 2

$\mathcal{S}$ is ROG when $\mathrm{rank}(\alpha M_1 + \beta M_2) \leq 2$ for all (α, β), $\mathrm{Span}\{\mathrm{Range}(M_1) \cup \mathrm{Range}(M_2)\}$ has dimension 3, and $\alpha M_1 + \beta M_2 \succeq 0$ has only the trivial solution $(\alpha, \beta) = (0, 0)$.

Main Result

Theorem 3 (A, Kılınç-Karzan, '17)

$\{Y \succeq 0 : \langle M_1, Y \rangle \geq 0, \langle M_2, Y \rangle \geq 0\}$ *is ROG iff one of the following holds*

(i) $\alpha M_1 + \beta M_2 \succeq 0$ *for some* $(\alpha, \beta) \neq (0, 0)$.

(ii) $\operatorname{rank}(\alpha M_1 + \beta M_2) \leq 2$ *for all* (α, β) *and* $\operatorname{Span}\{\operatorname{Range}(M_1) \cup \operatorname{Range}(M_2)\}$ *has dimension* 3.

Proving Necessity (Sketch)

First consider the case of $\mathbb{S}^3$.
Suppose that:

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We reduce the general case of $\mathbb{S}^n$ to the case of $\mathbb{S}^3$.

Extensions/Questions

- Necessary and sufficient conditions for more than 2 LMIs.
- Use results to analyze conic constraints.
  - Alternate analysis of Burer's work on extensions of the Trust Region Subproblem.
  - Necessary and sufficient conditions for more general conic constraints.

Thank you!

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Further Reading

Samuel Burer (2015).
A Gentle Geometric Introduction to Copositive Optimization.
Mathematical Programming, June 2015, Volume 151, Issue 1, pp 89-116.

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Do Sums of Squares Dream of Free Resolutions?
SIAM J. Appl. Algebra Geometry, 1(1), 175-199.