Chasing Nested Convex Bodies

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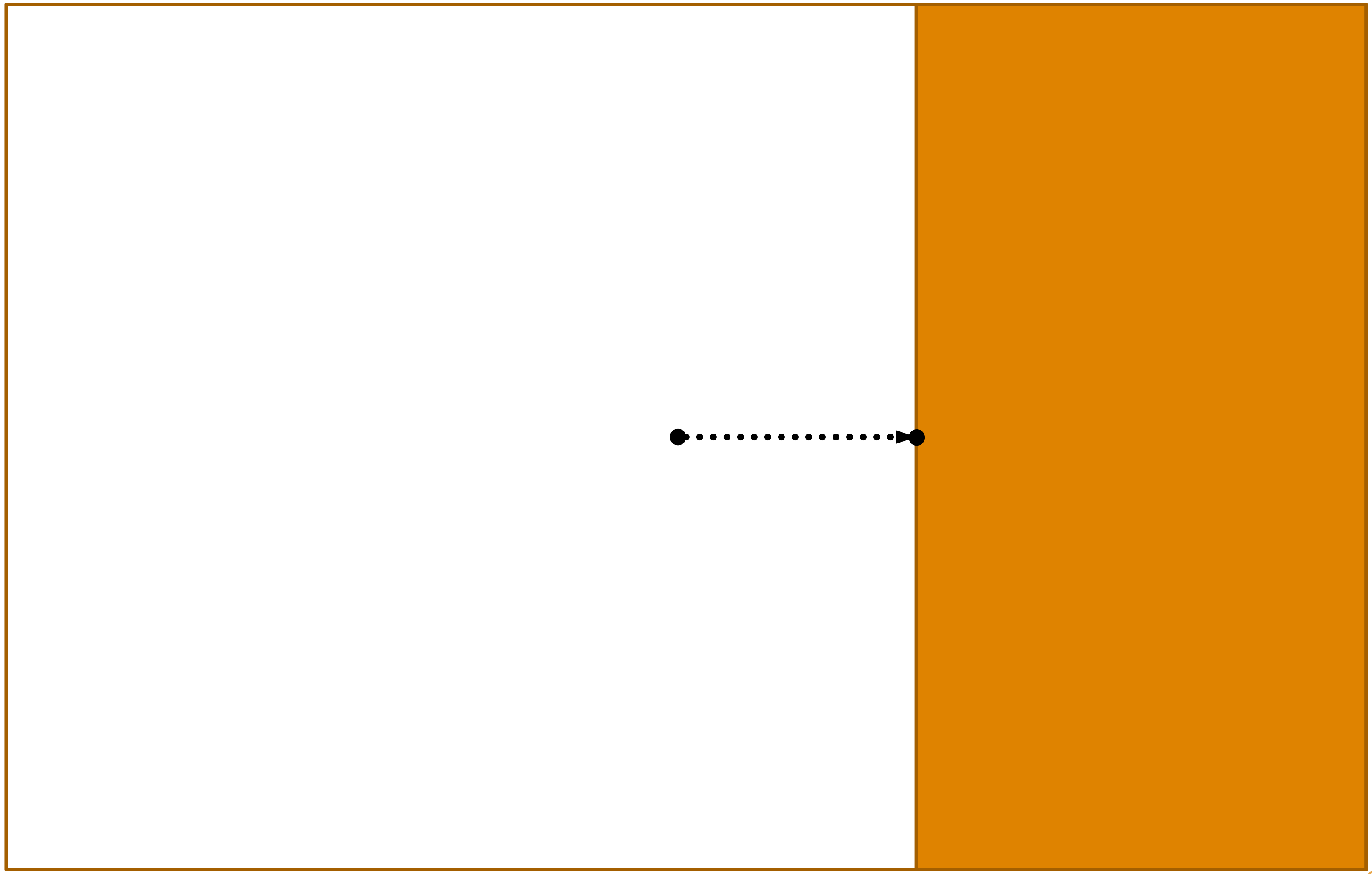
The Problem



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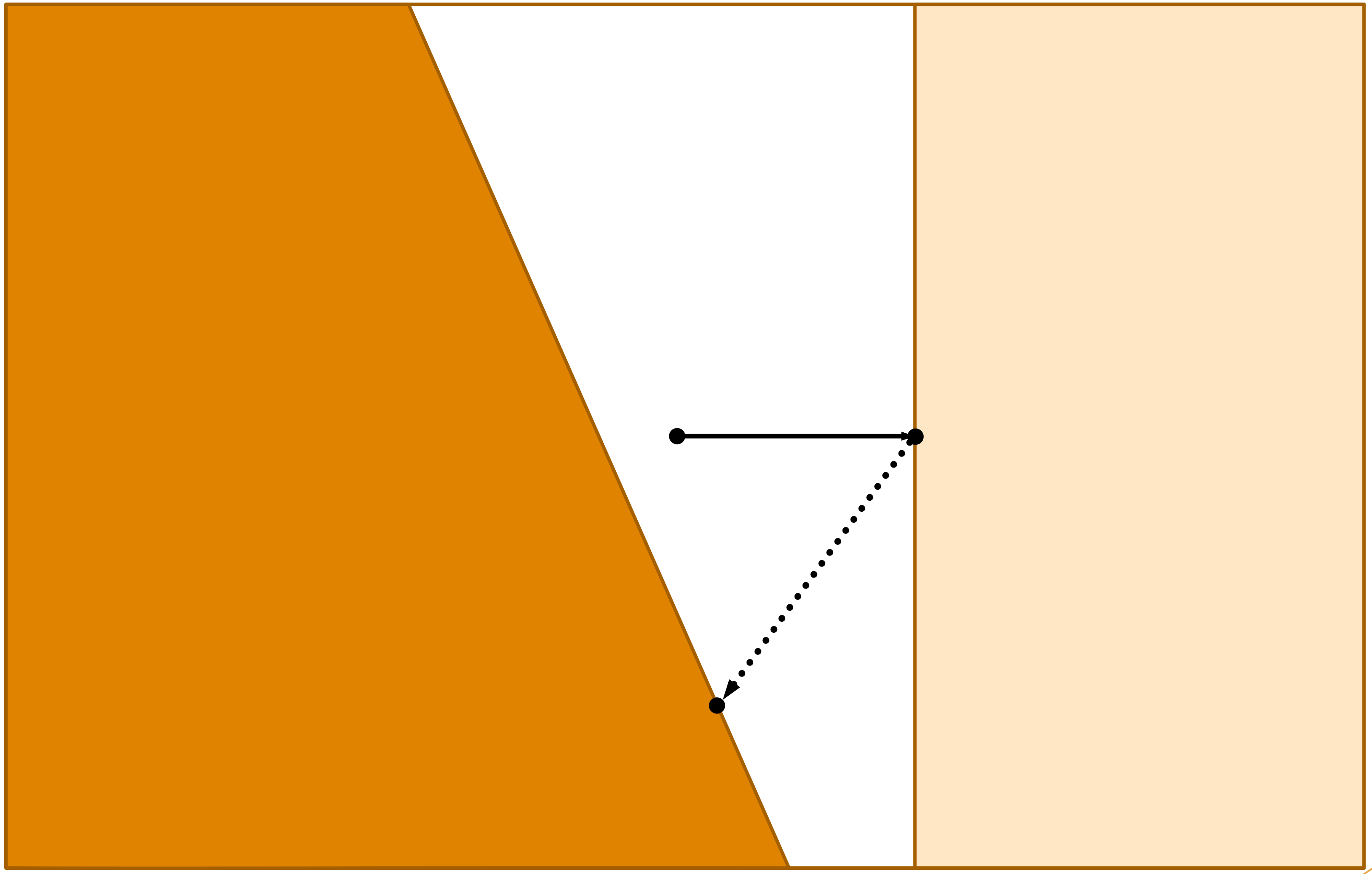

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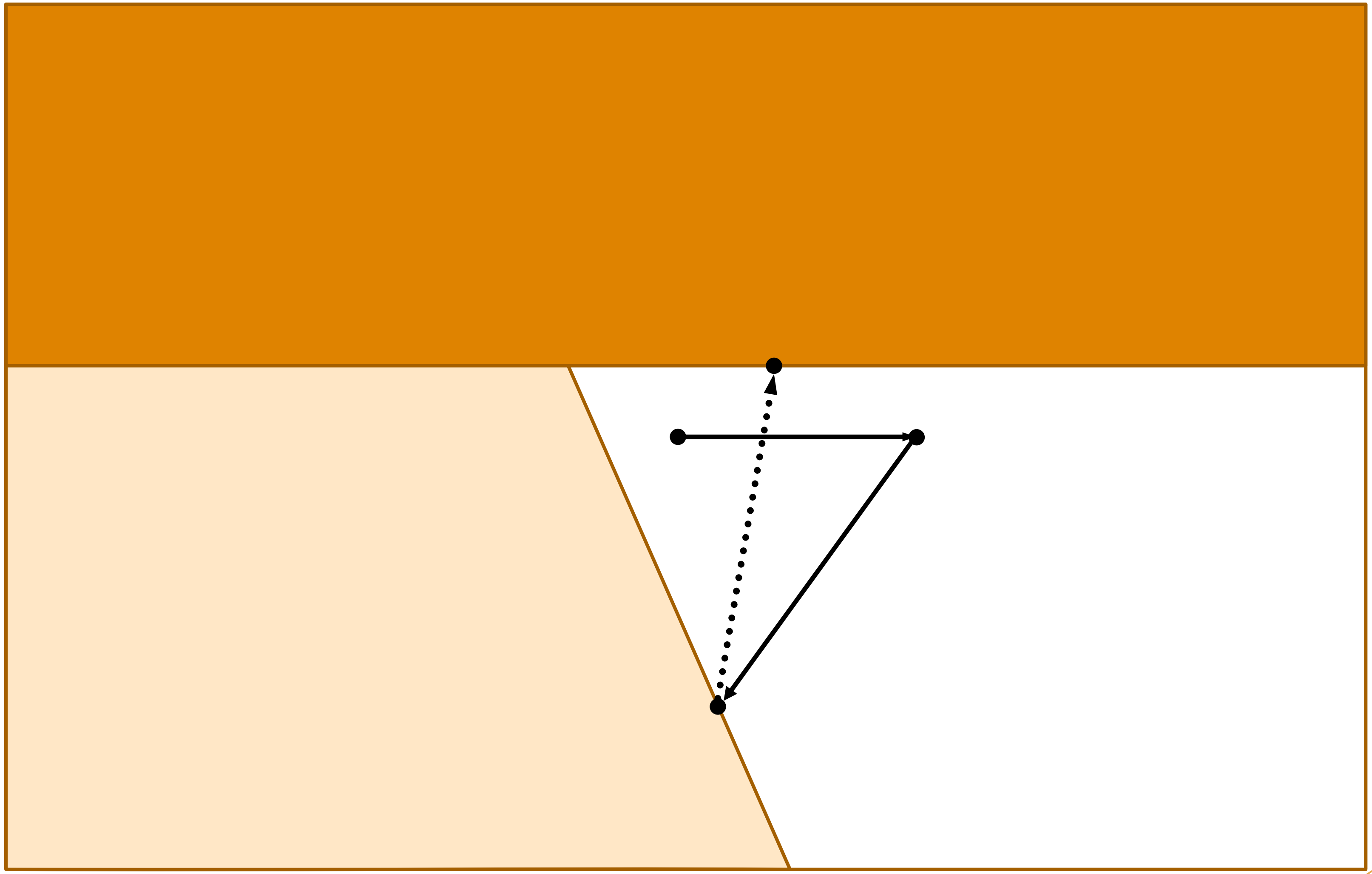

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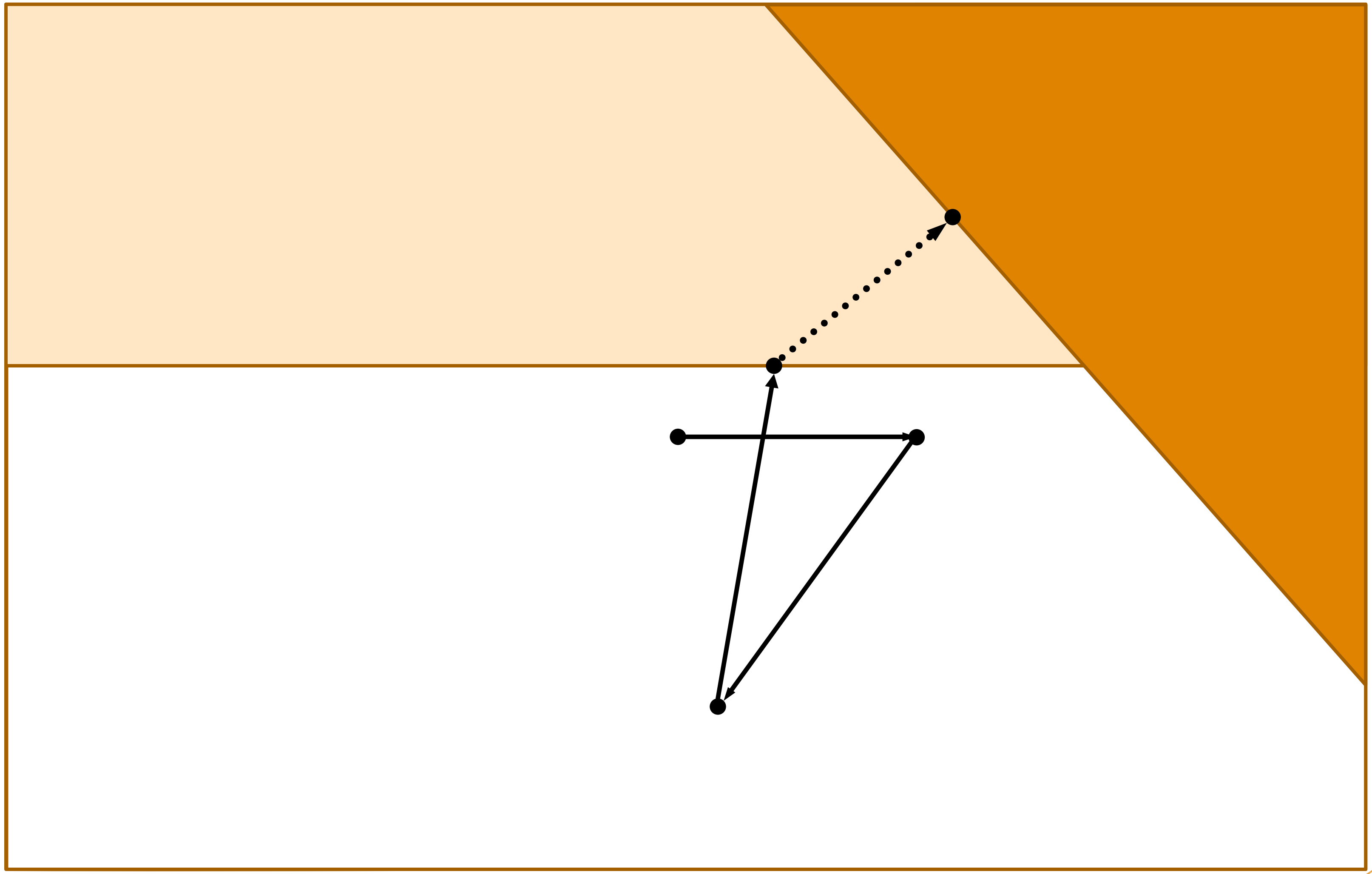

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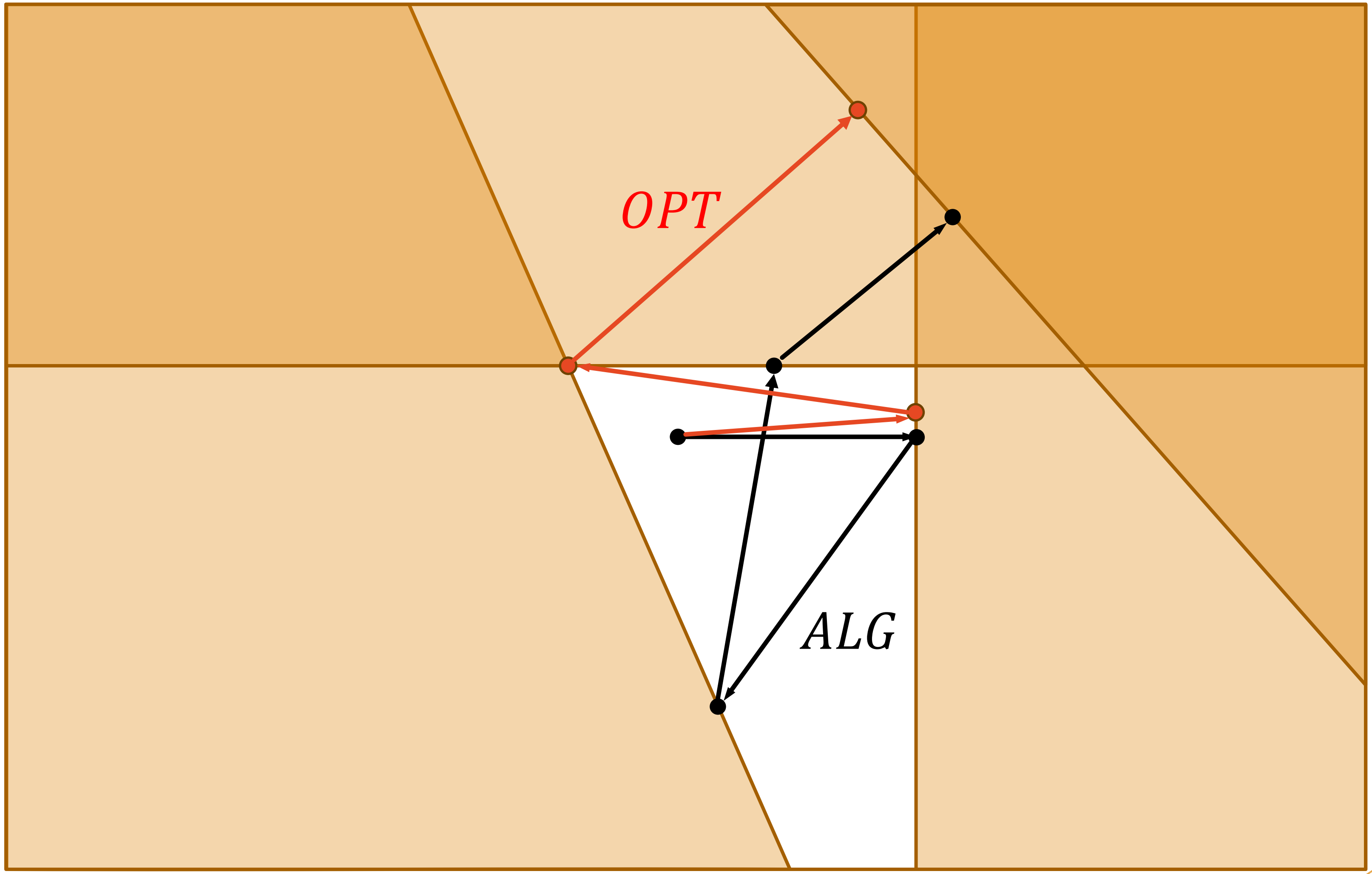


The Problem – Formal Definition

- Given convex sets $K^1, K^2, K^3, \ldots$ in $\mathbb{R}^d$
- Choose $x^i \in K^i$ *online* $(x^0 = 0)$
- Cost $ALG^t = \sum_{i=1}^{t} ||x^i - x^{i-1}||$
- Goal – minimize competitive ratio

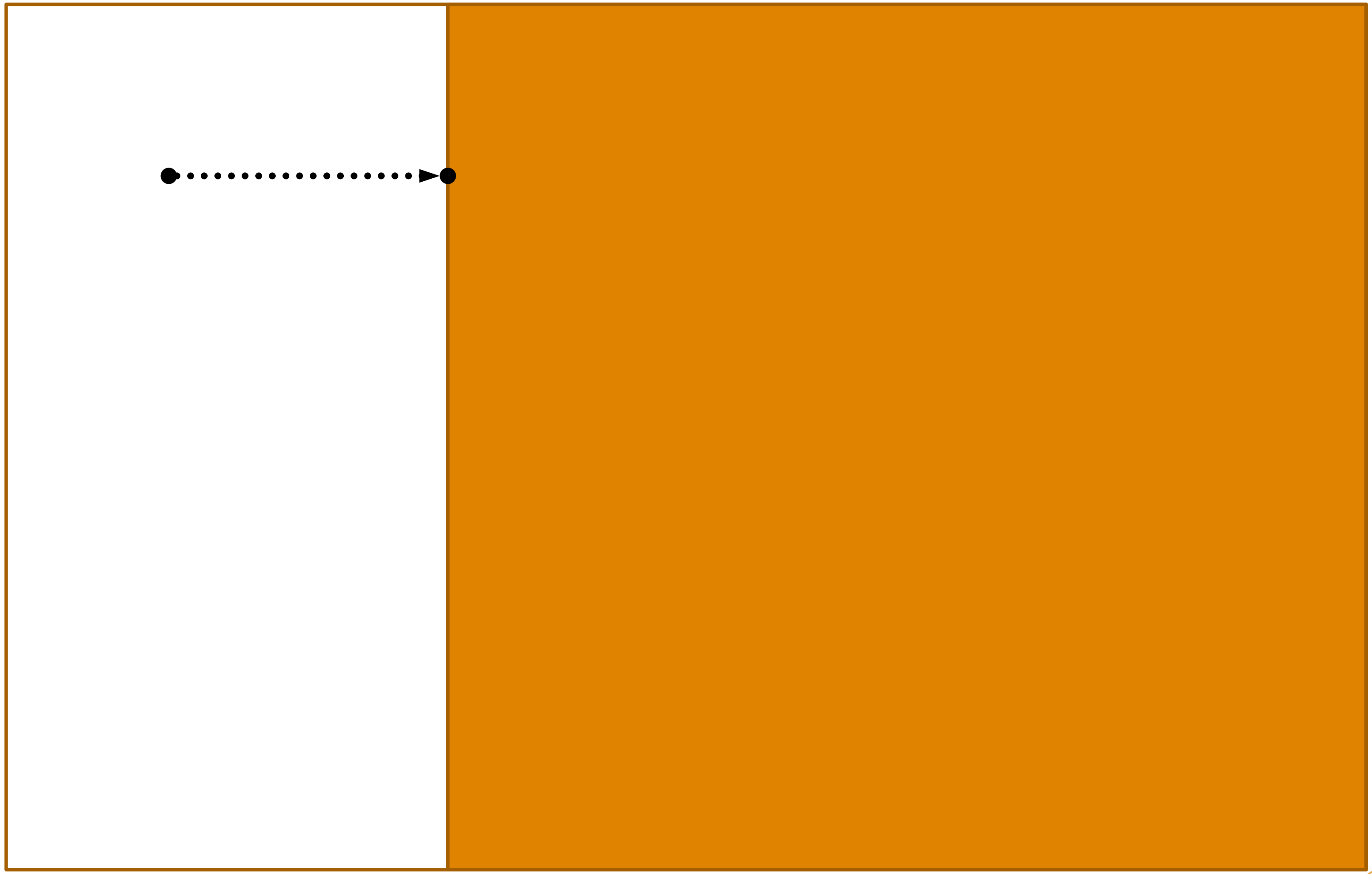
$$\mathrm{cr}(ALG) \coloneqq \max_{\sigma,t} \frac{ALG^t(\sigma)}{OPT^t(\sigma)}$$

  - σ arbitrary instance
  - $OPT^t(\sigma)$ optimal *offline* cost

Motivation

- Metrical task systems (MTS)
  - Given convex functions $f_1, f_2, f_3, \ldots$
  - Choose x^i *online* $(x^0 = 0)$
  - Cost $ALG^t = \sum_{i=1}^{t} ||x^i - x^{i-1}|| + f_i(x^i)$
  - Convex body chasing: role of geometry in MTS
- Related to k-server

Nested Version



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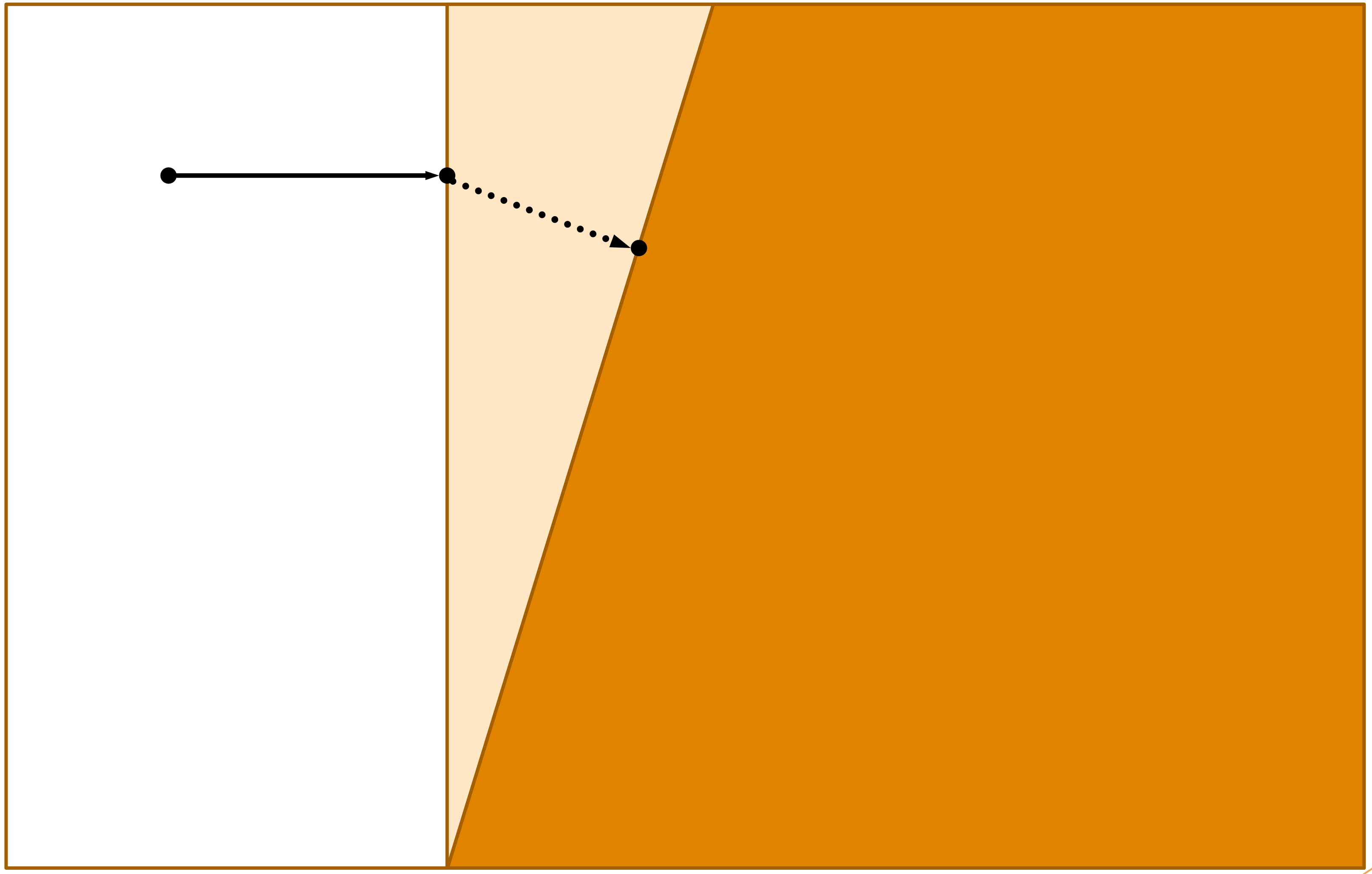

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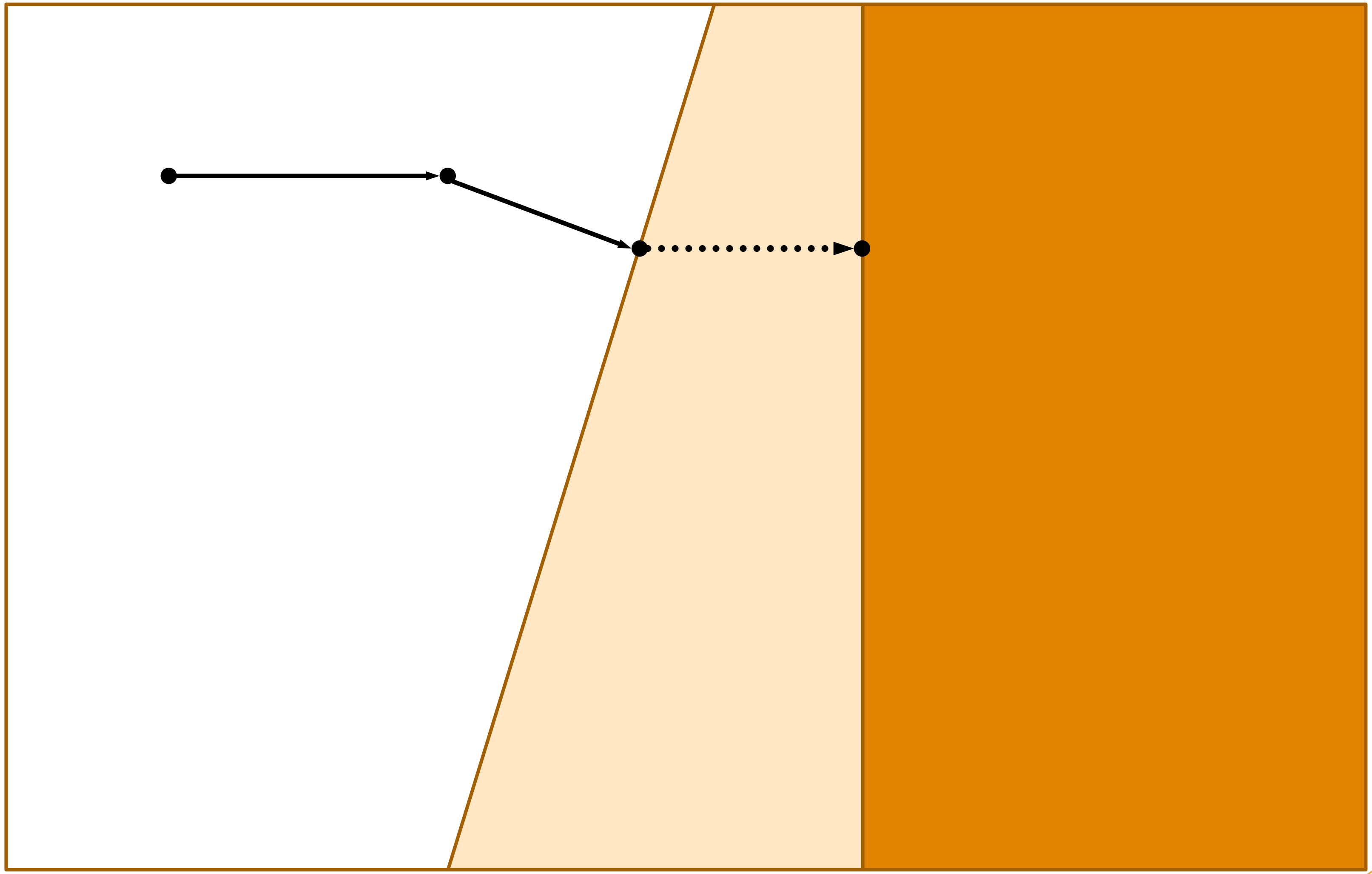

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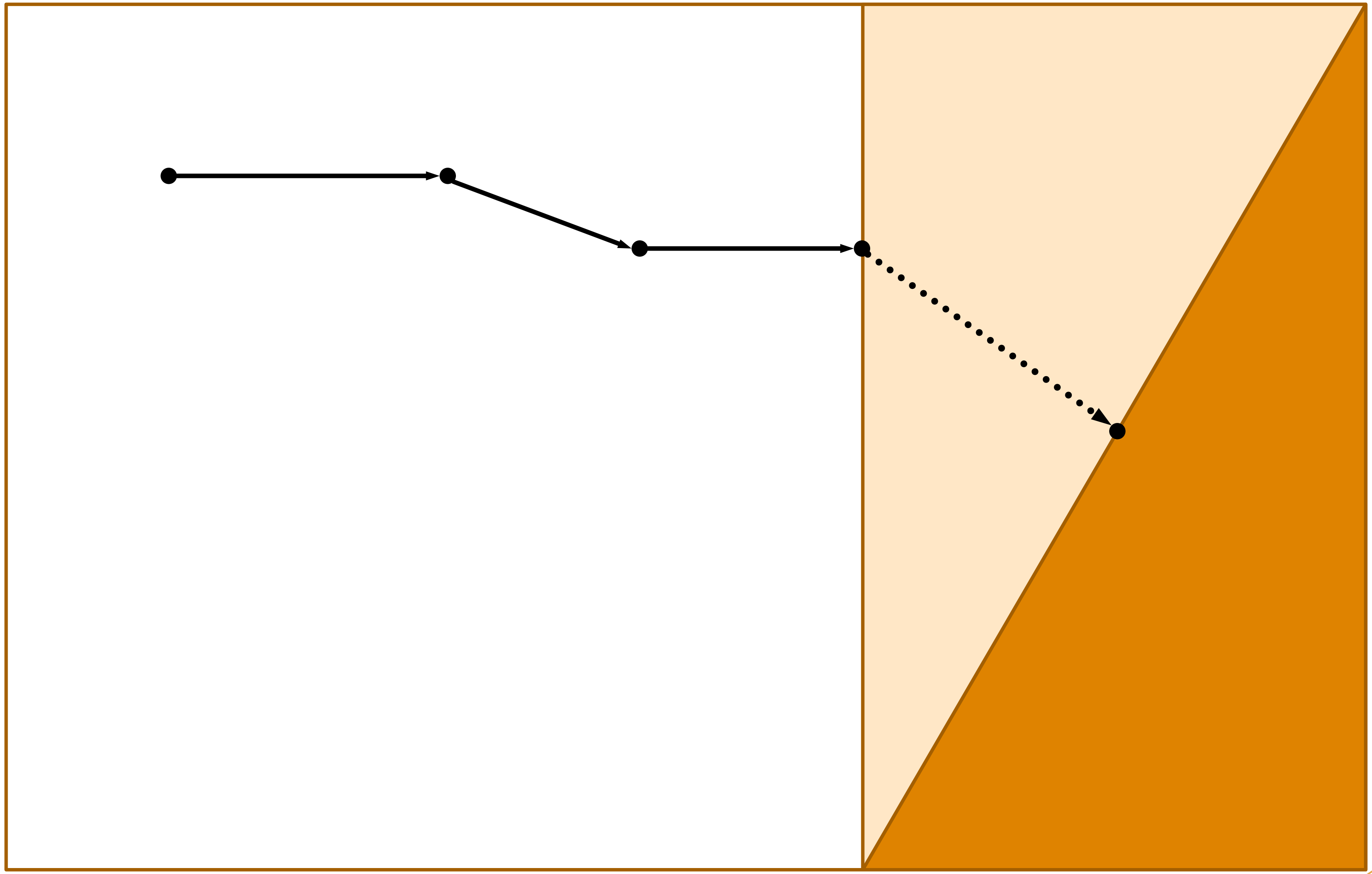

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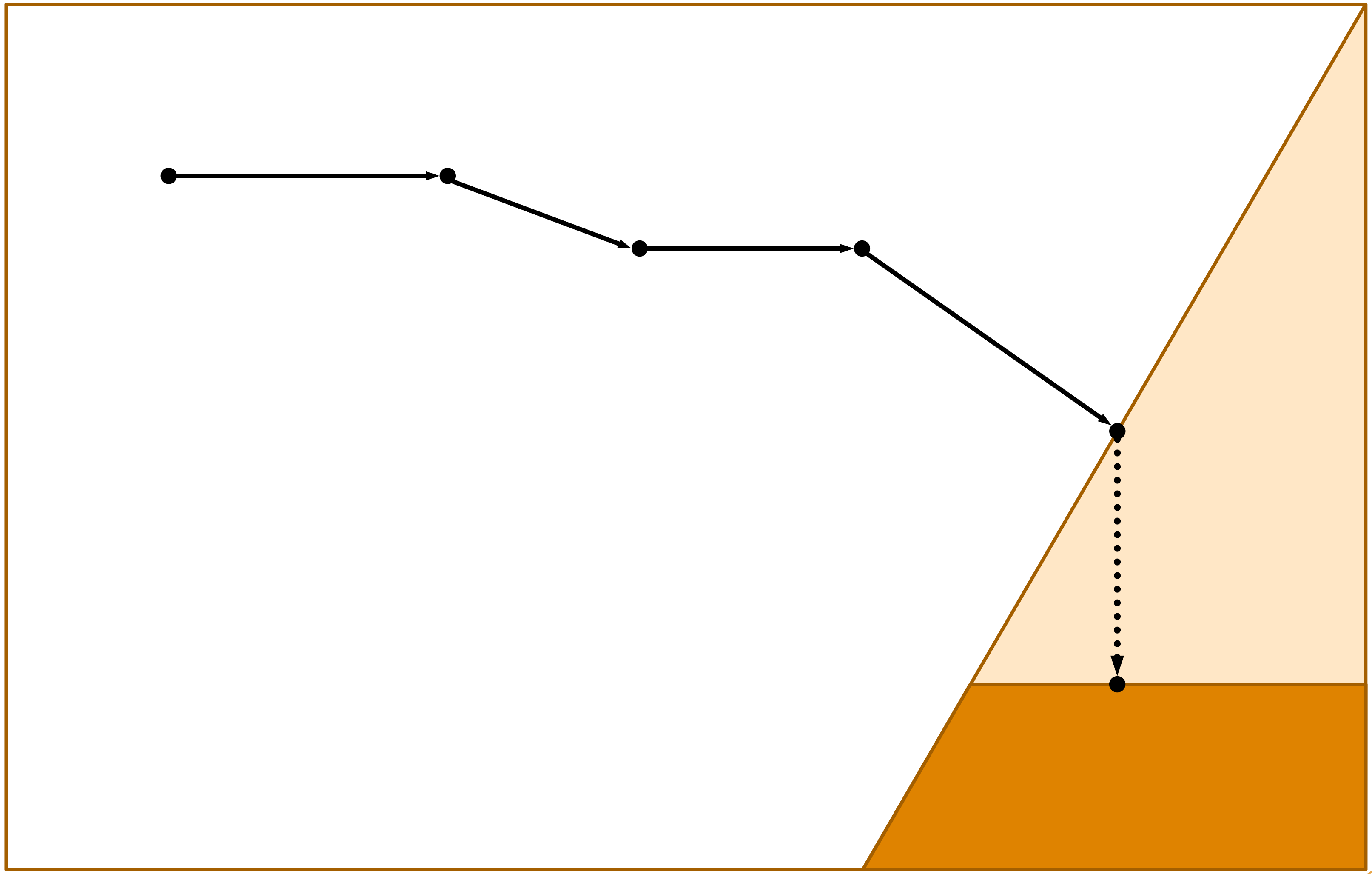

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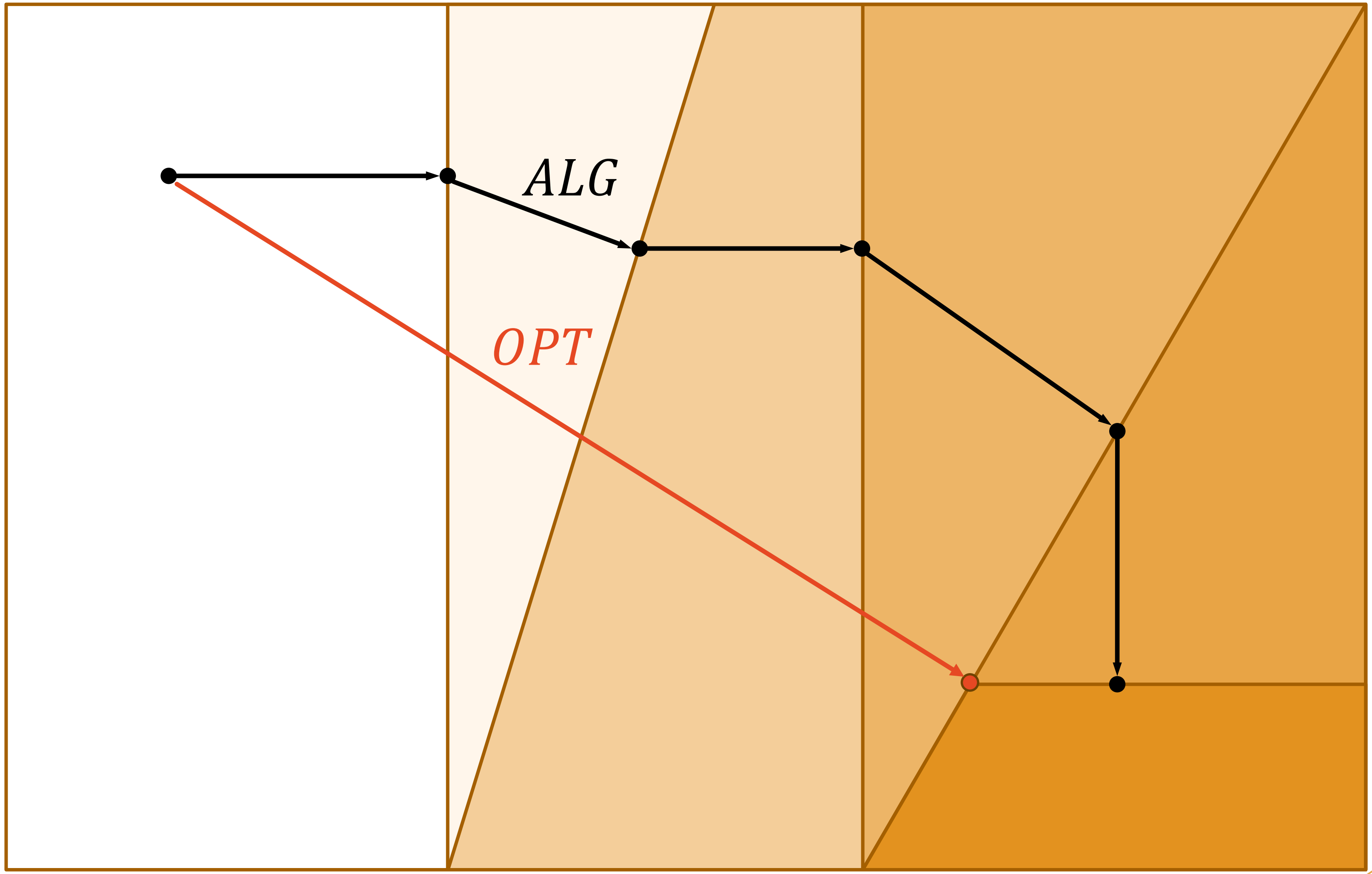


Results

- [FL 93] $\sqrt{d}$ lower bound,
  Competitive general chasing ($d = 2$ case)
- [BB+ 17] $d^{O(d)}$-competitive nested chasing
- **[AB+ 18] $O(d \log d)$-competitive nested chasing**
- [BL+ 18] $O\left(\sqrt{d \log d}\right)$-competitive nested chasing,
  $exp(d)$-competitive general chasing

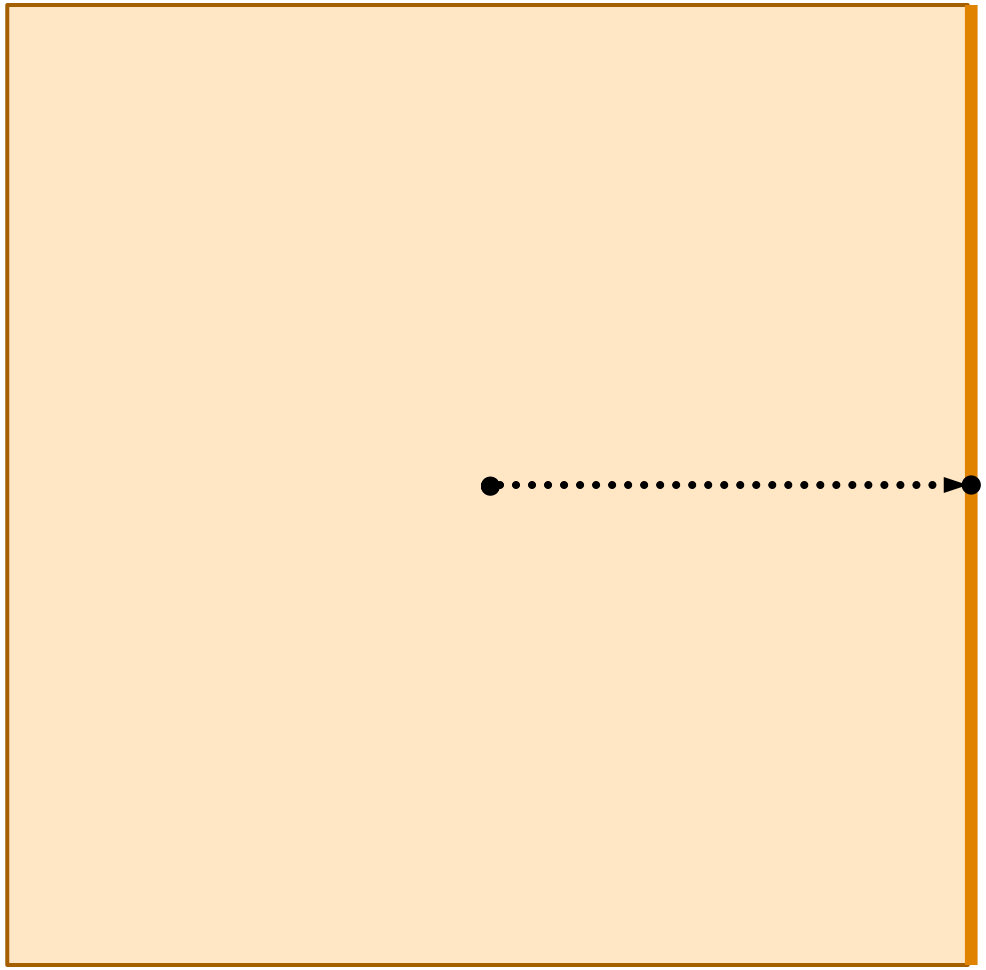
Talk outline

1. Warm-up ideas from general chasing
2. *Centroid* and *Recursive Greedy* – two motivating ideas
3. *Recursive Centroid* – $O(d \log d)$-competitive, analysis

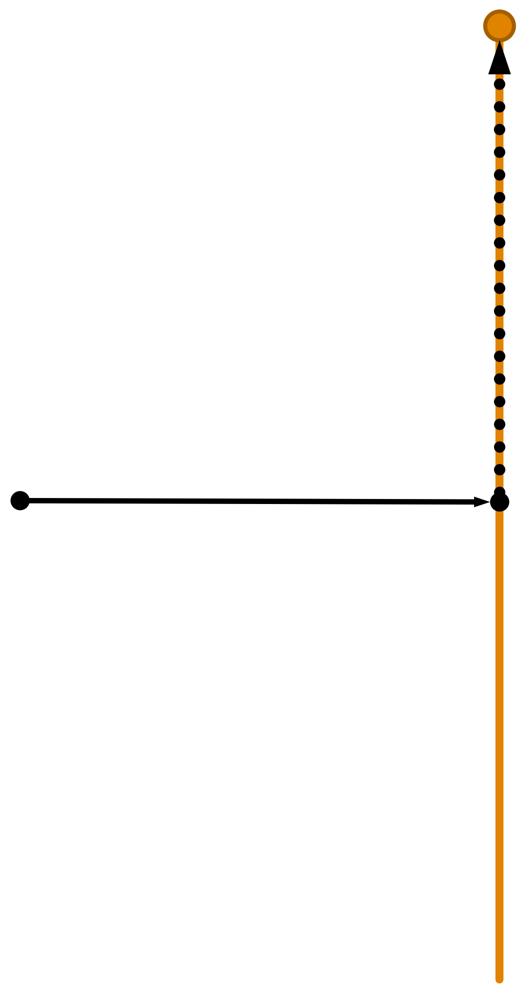
Part 1 – Warm-up ideas

A lower bound, a bad algorithm, and two reductions

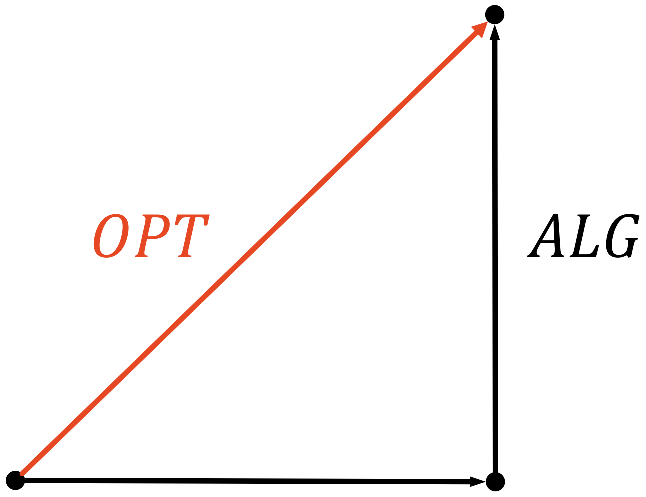
Lower Bound



Lower Bound



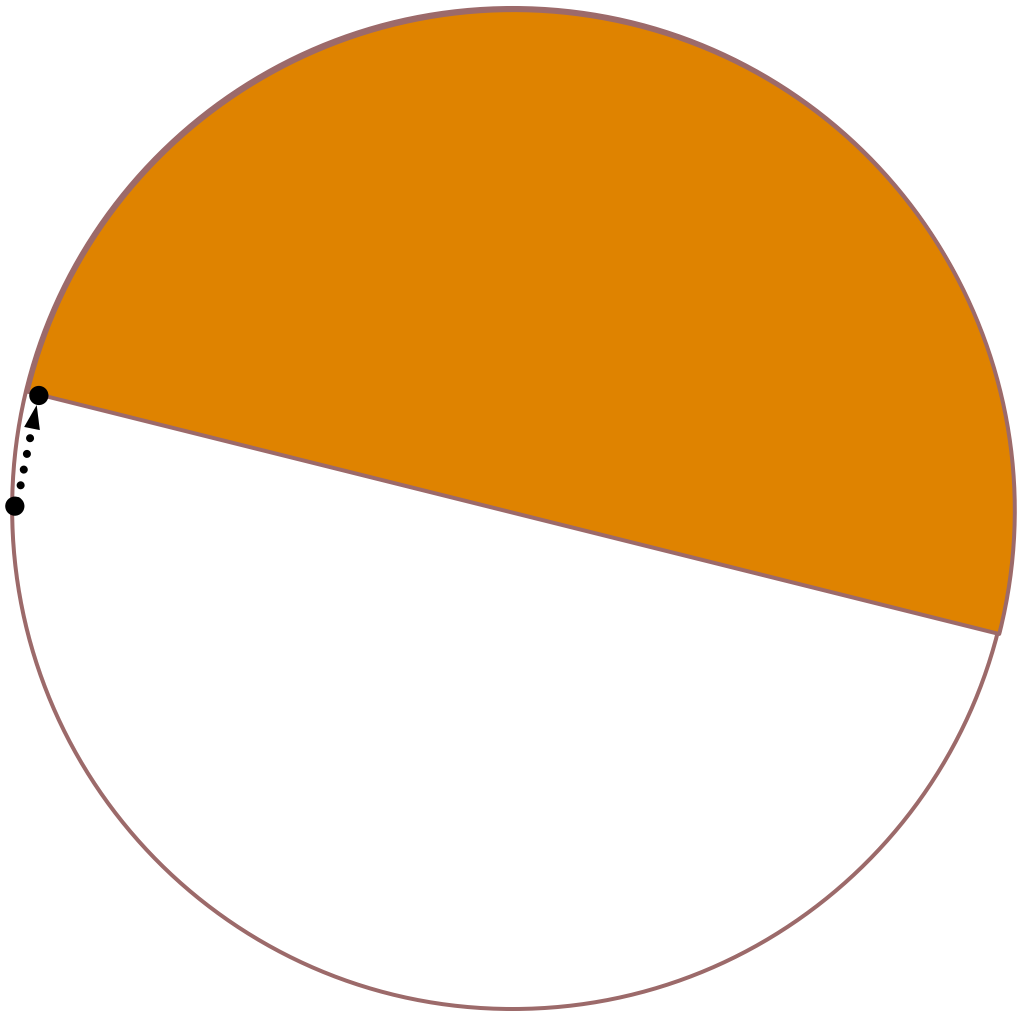
Lower Bound



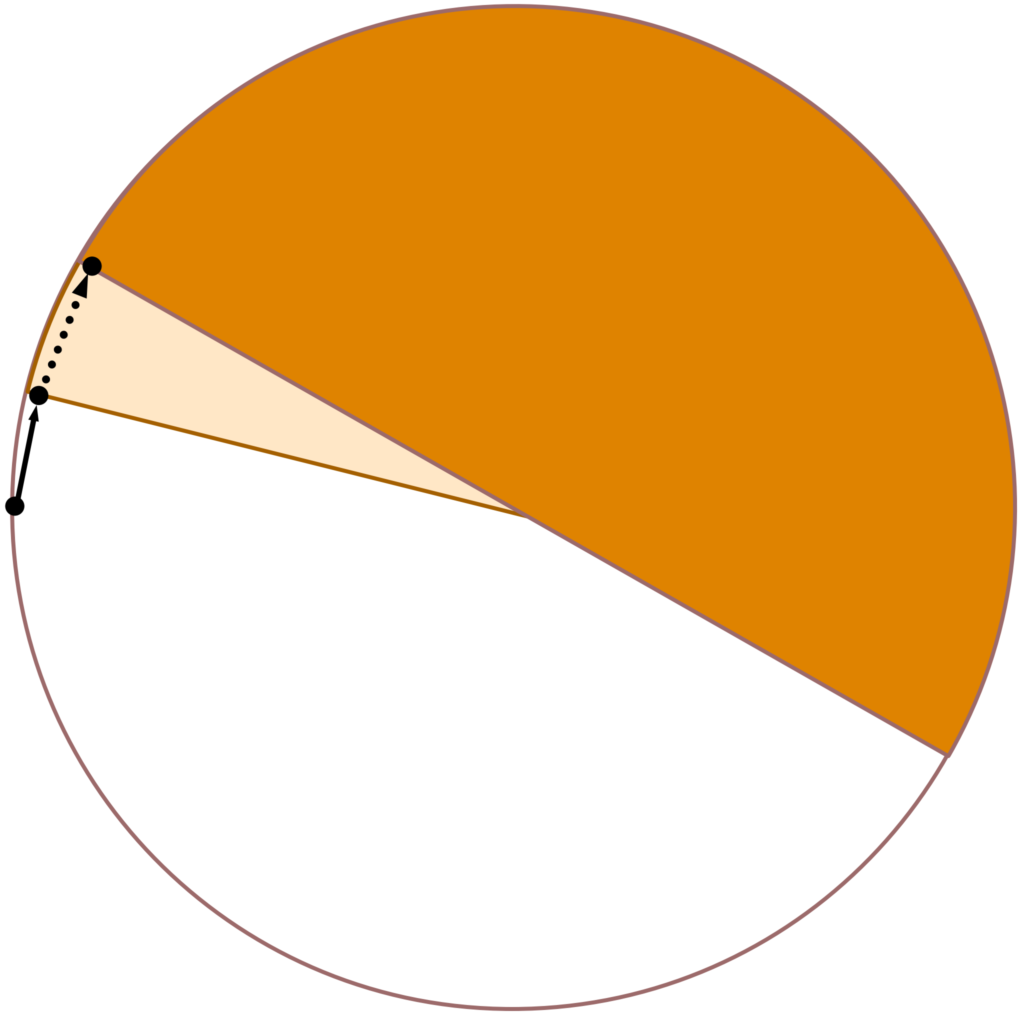
$$ALG \geq \sqrt{2} \cdot OPT$$

$$\boxed{ALG \geq \sqrt{d} \cdot OPT}$$

Bad Idea – *Greedy*



Bad Idea – *Greedy*



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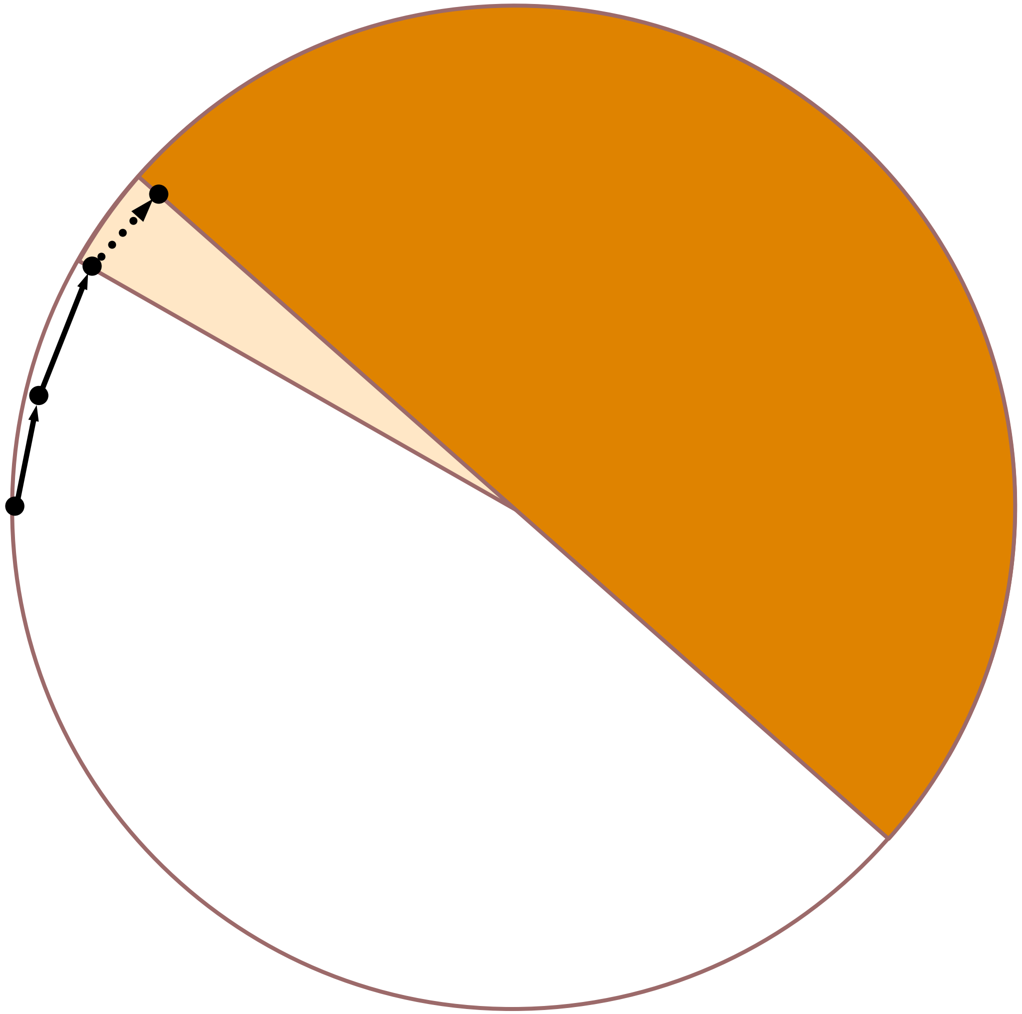

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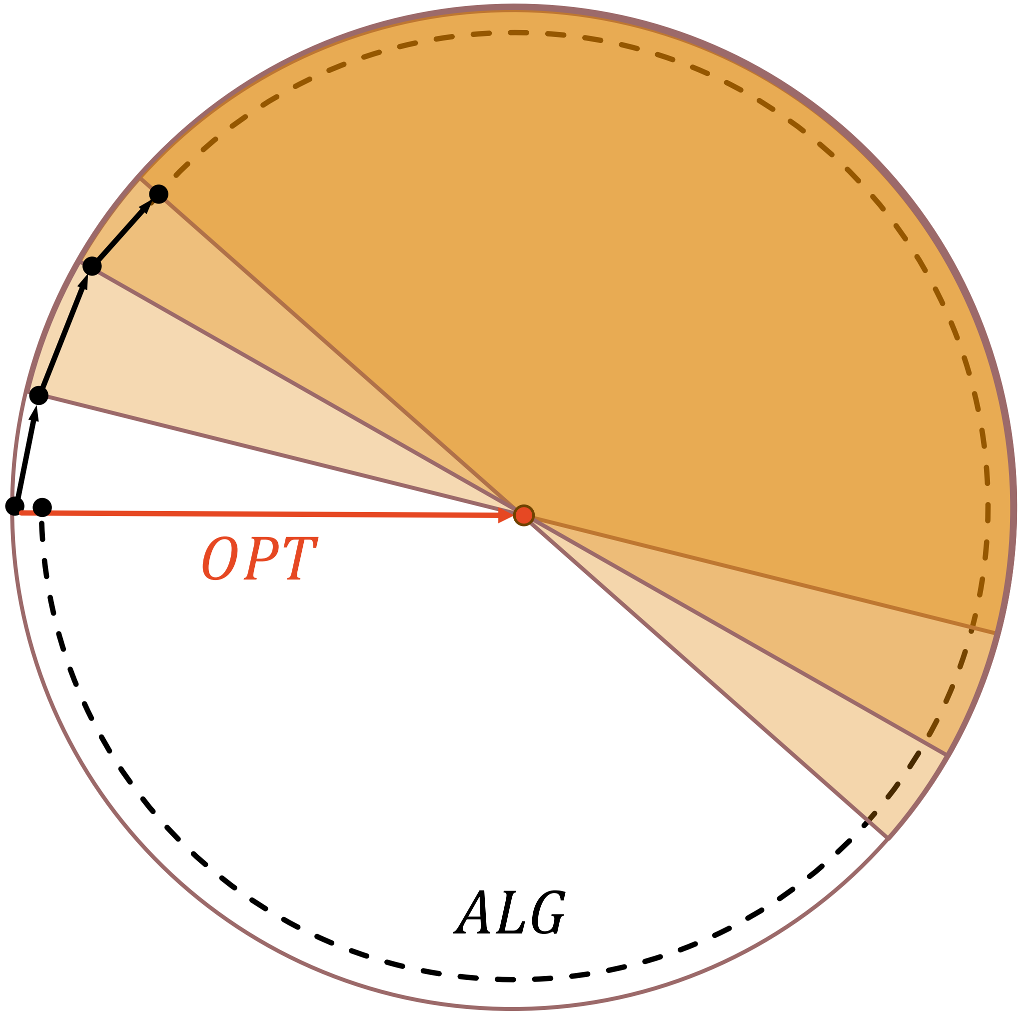


- ALG unbounded
- $OPT = O(1)$
- Not competitive ☹
- Bounded: $d^{O(d)}$-competitive

Reductions

- *Bounded:* $diam(K^1) = O(1), \;\; OPT = \Omega(1)$
  - $f(d) \cdot diam(K^1)$ total cost $\Rightarrow f(d)$-competitive
  - Guess-and-double
- *Tighten:* end when $diam(K^t) \leq \frac{1}{2}\, diam(K^1)$
  - Apply repeatedly
  - Cost decreases geometrically

Recap of Part 1

- $\sqrt{d}$ lower bound
- Greedy is not good
- Suffices to halve diameter with bounded cost

Part 2 – Two initial ideas

Centroid, recursive greedy, and why neither is good enough

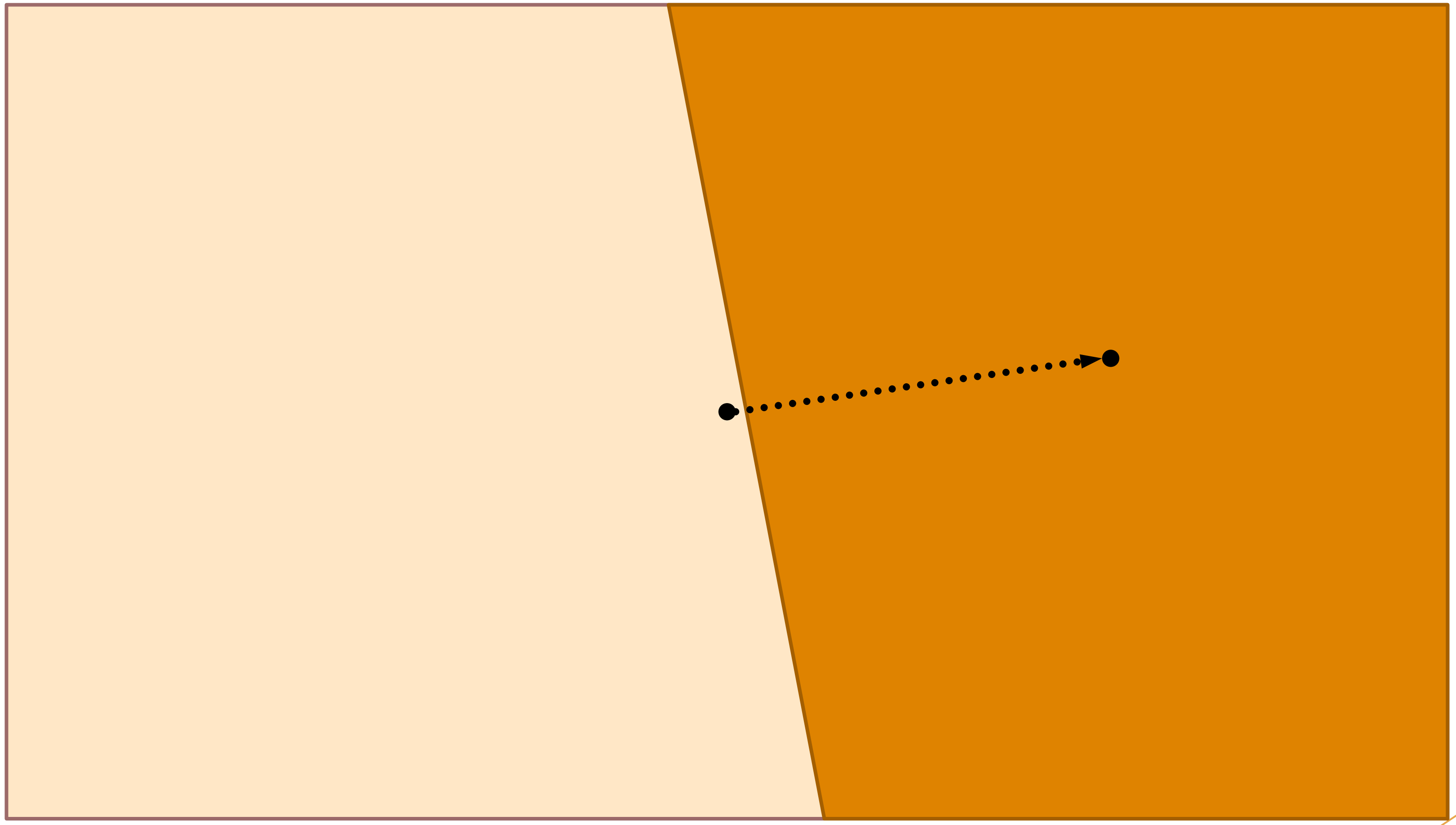
Idea 1 – *Centroid*

- Move to "center" of K^t
  - (K^t bounded)
- Centroid of $A \subseteq \mathbb{R}^n$ is $\mu(A) \coloneqq \int_A x\, dx$

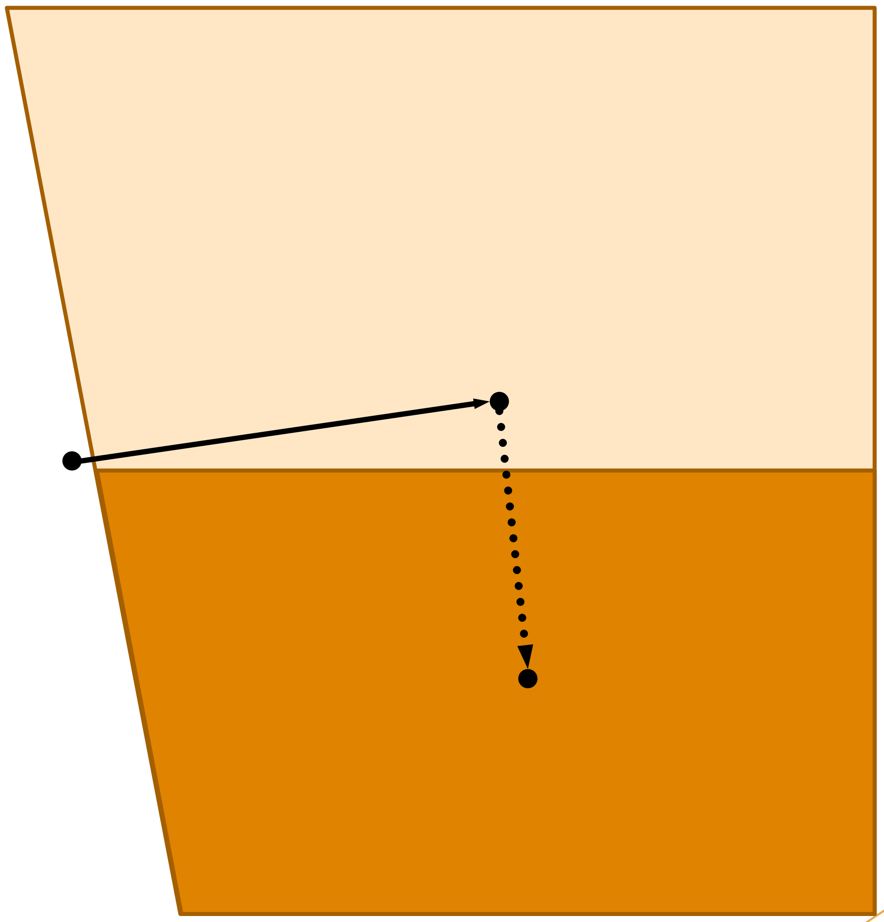
Centroid Algorithm: $\boxed{x^t = \mu(K^t)}$

- Motivation: cut large portion of K^t each step

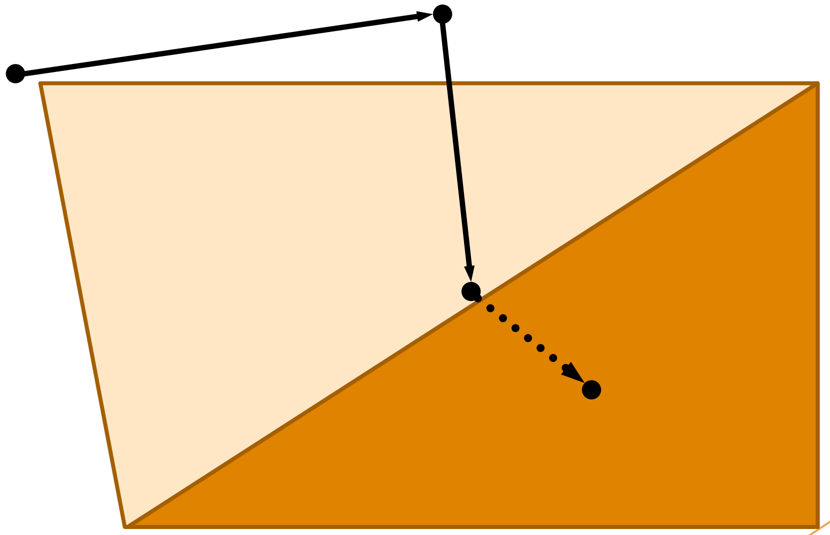
Idea 1 – *Centroid*



Idea 1 – *Centroid*



Idea 1 – *Centroid*



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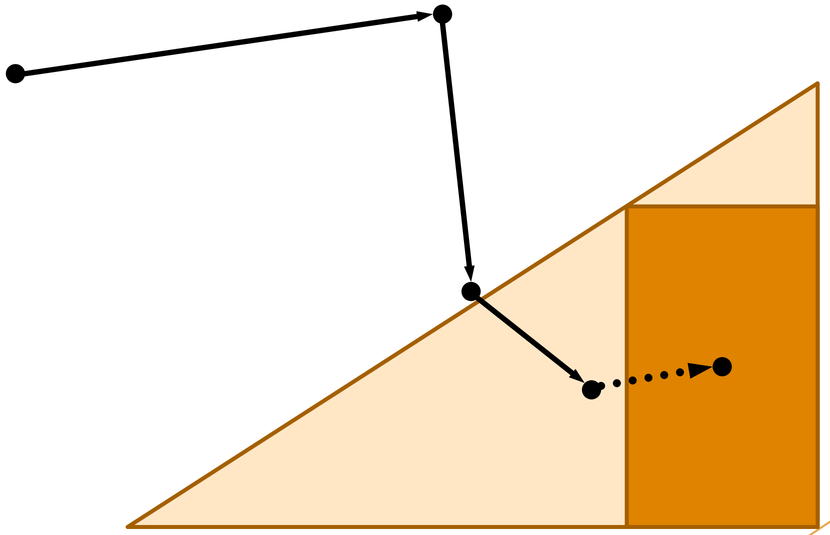
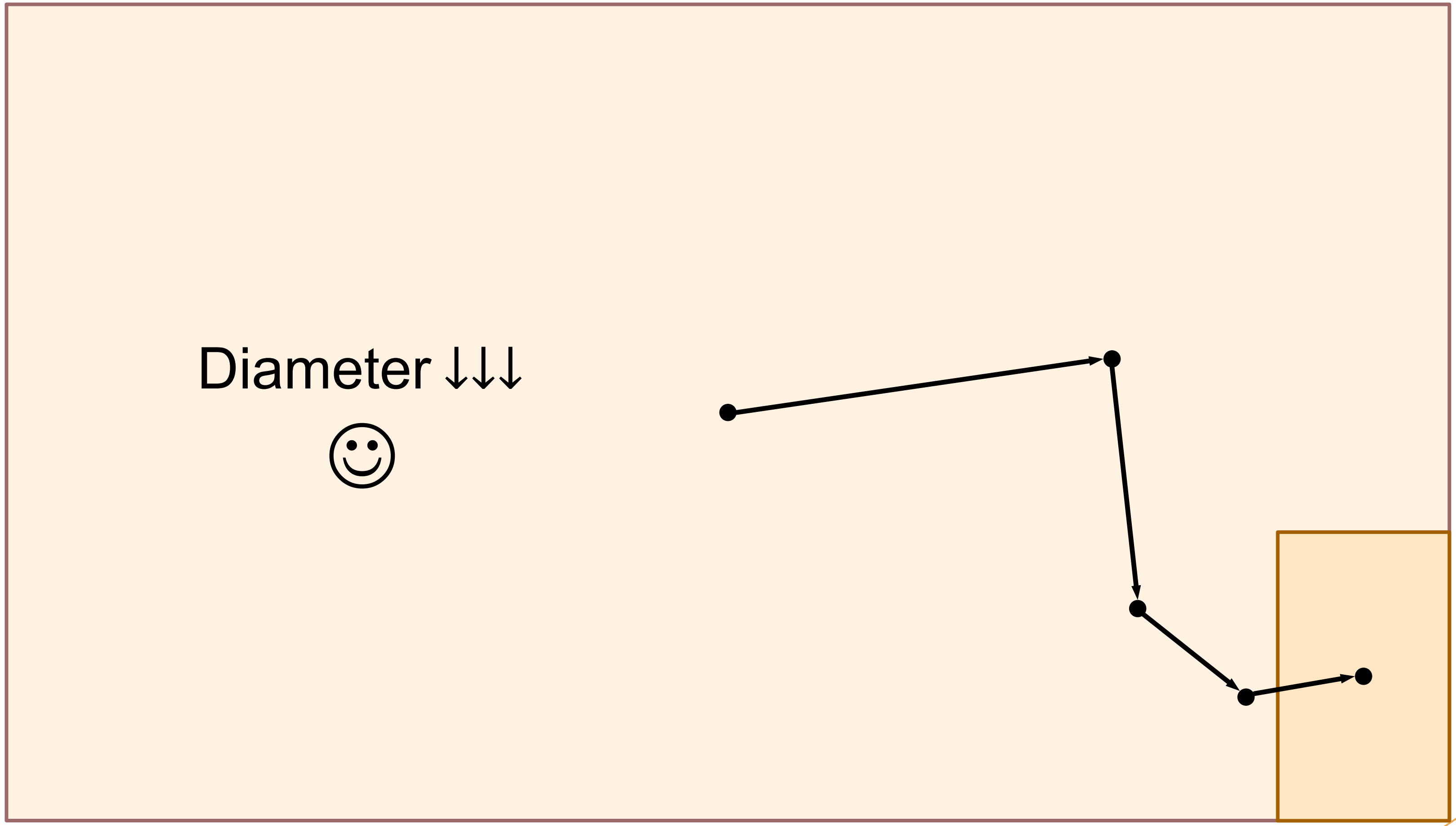


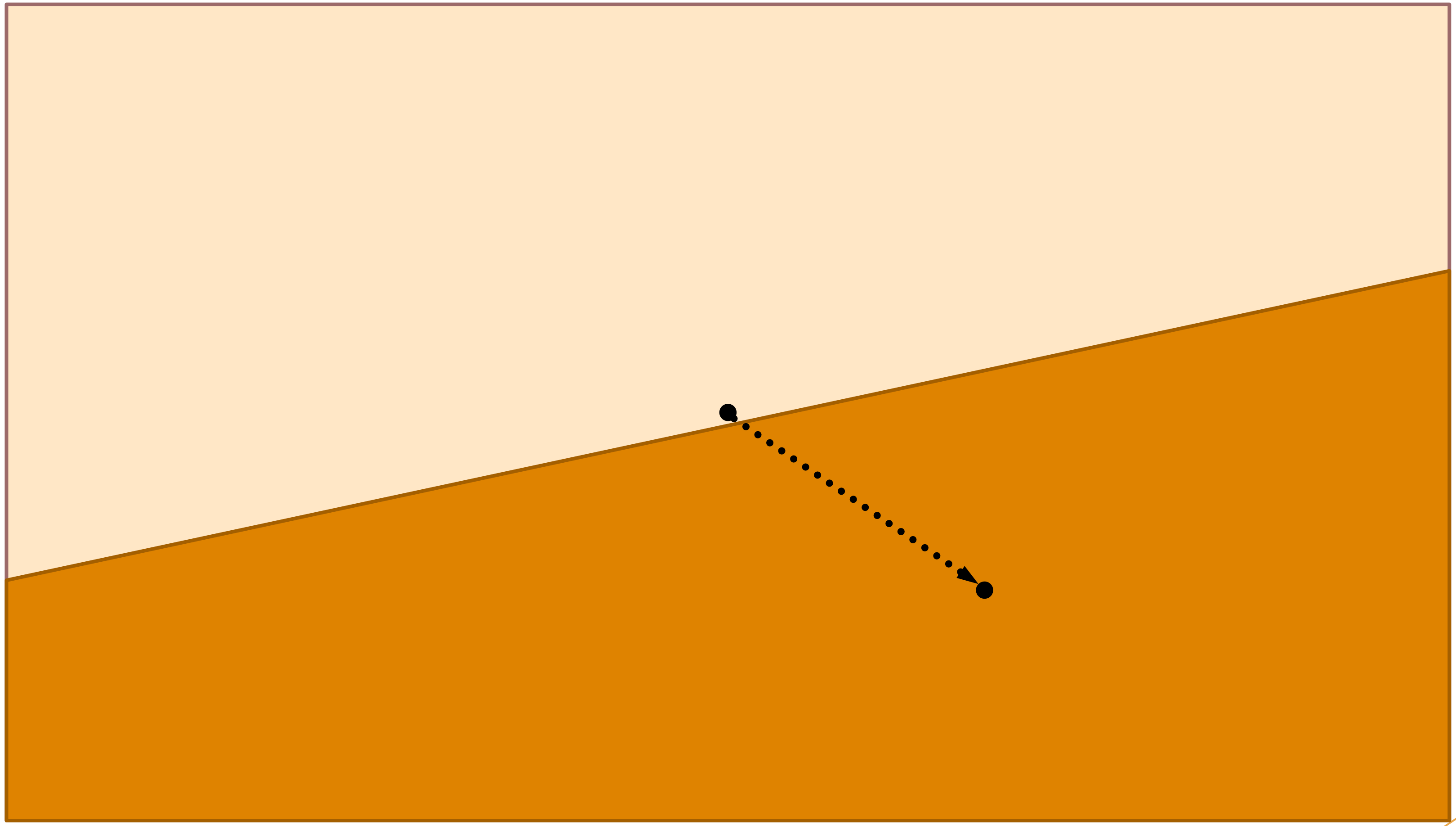
Idea 1 – *Centroid*



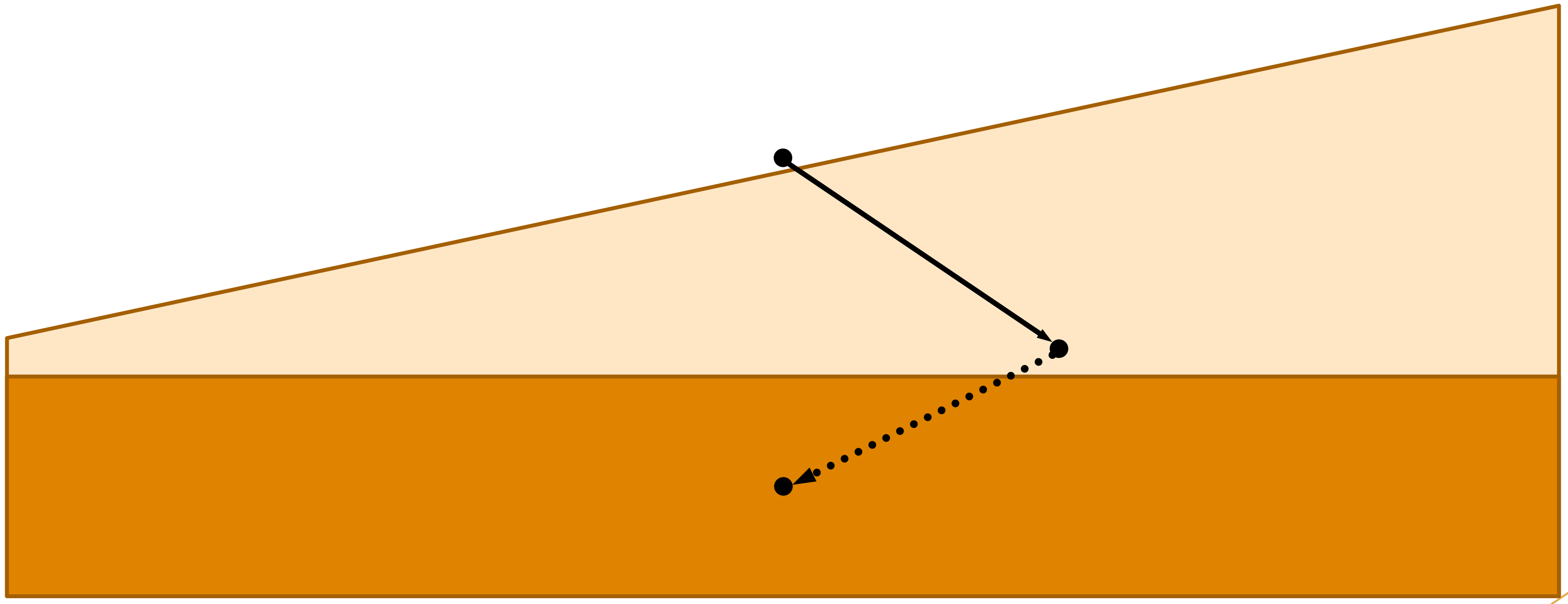
Advantage of *Centroid*

- Grünbaum ['60] $\Rightarrow Vol(K^t) \leq (1-c) \cdot Vol(K^{t-1})$
  $\leq (1-c)^t \cdot Vol(\mathrm{K}^0)$
- Volume drops $O(2^d)$ in $O(d)$ steps
- Step cost at most $diam(K^t) = O(1)$
- $O(d)$ total cost?

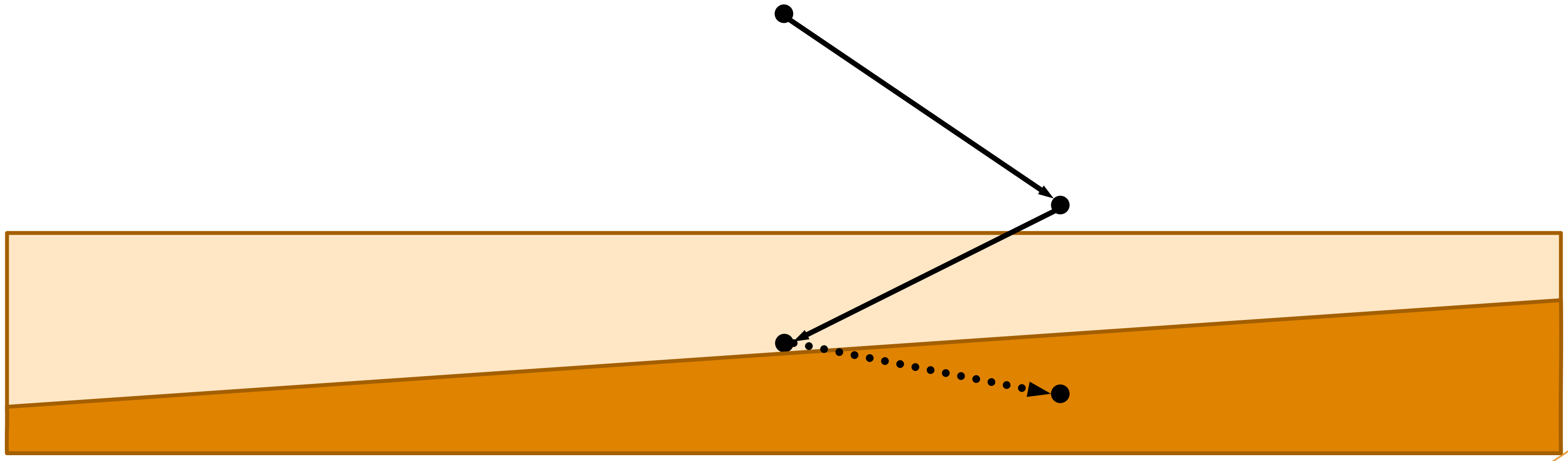
Problem with *Centroid*



Problem with *Centroid*

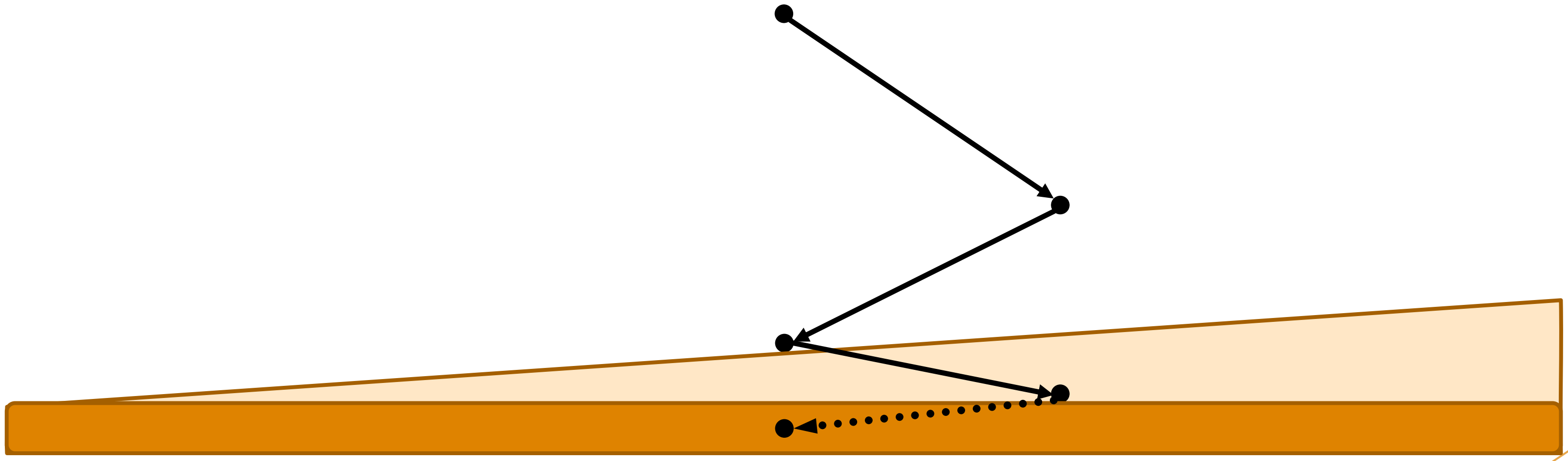


Problem with *Centroid*



Problem with *Centroid*

Diameter constant
Not competitive
☹



Summary – *Centroid*

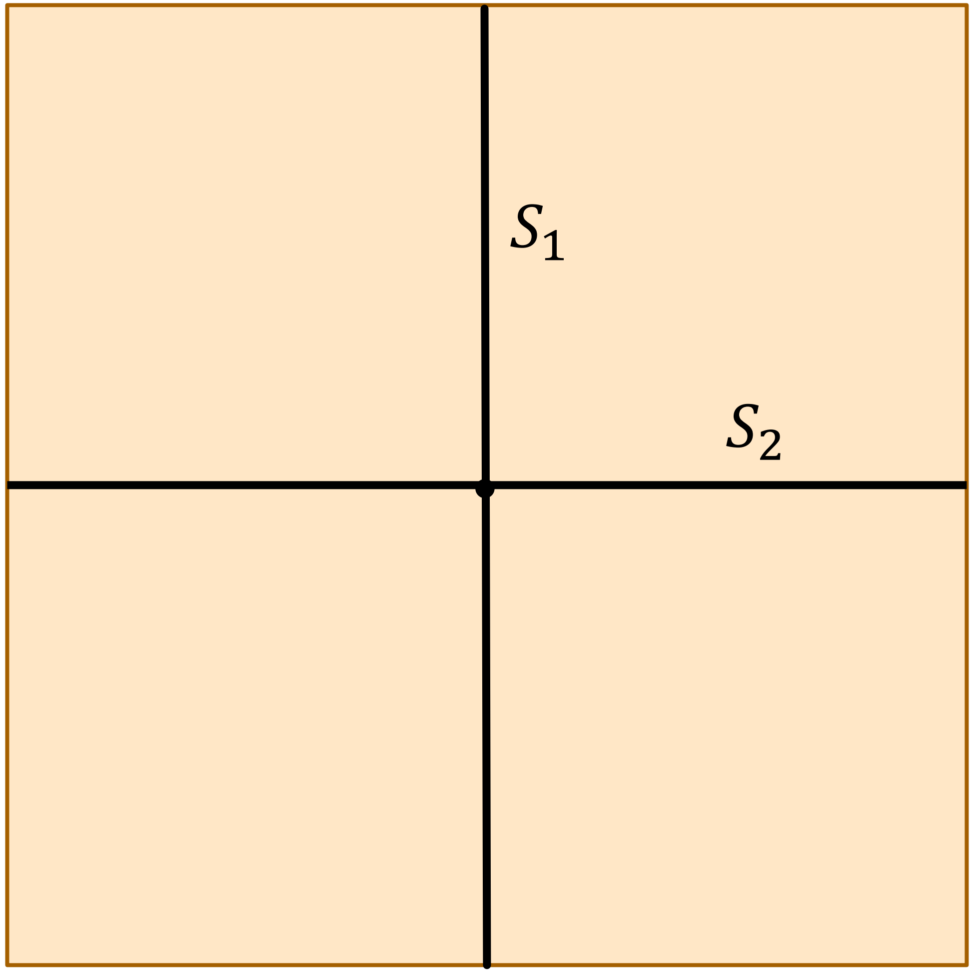
- $Vol(K^t)$ drops quickly
- $Diam(K^t)$ stays constant

Idea 2 – *Recursive Greedy*

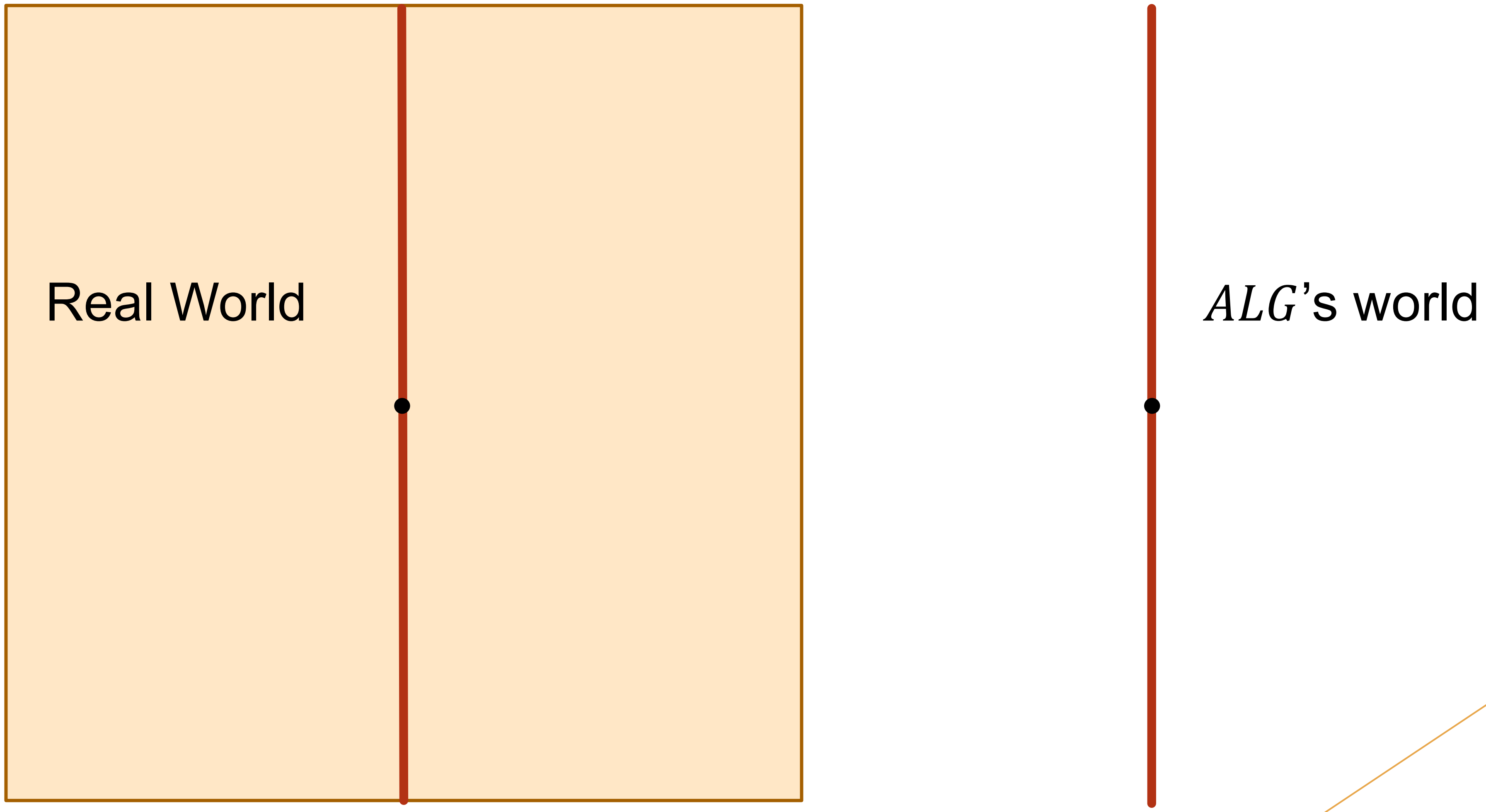
- “Refuse to move back and forth”
- In $\mathbb{R}^1$, run *Greedy*
- In $\mathbb{R}^d$
  - Fix orthogonal hyperplanes $S_1, \dots, S_d$
  - For $i = 1, \dots, d$
    - Run RG^{d-1} on sets $S_i \cap K^t$

RG^{d-1} – *Recursive Greedy* in $(d-1)$ dimensions

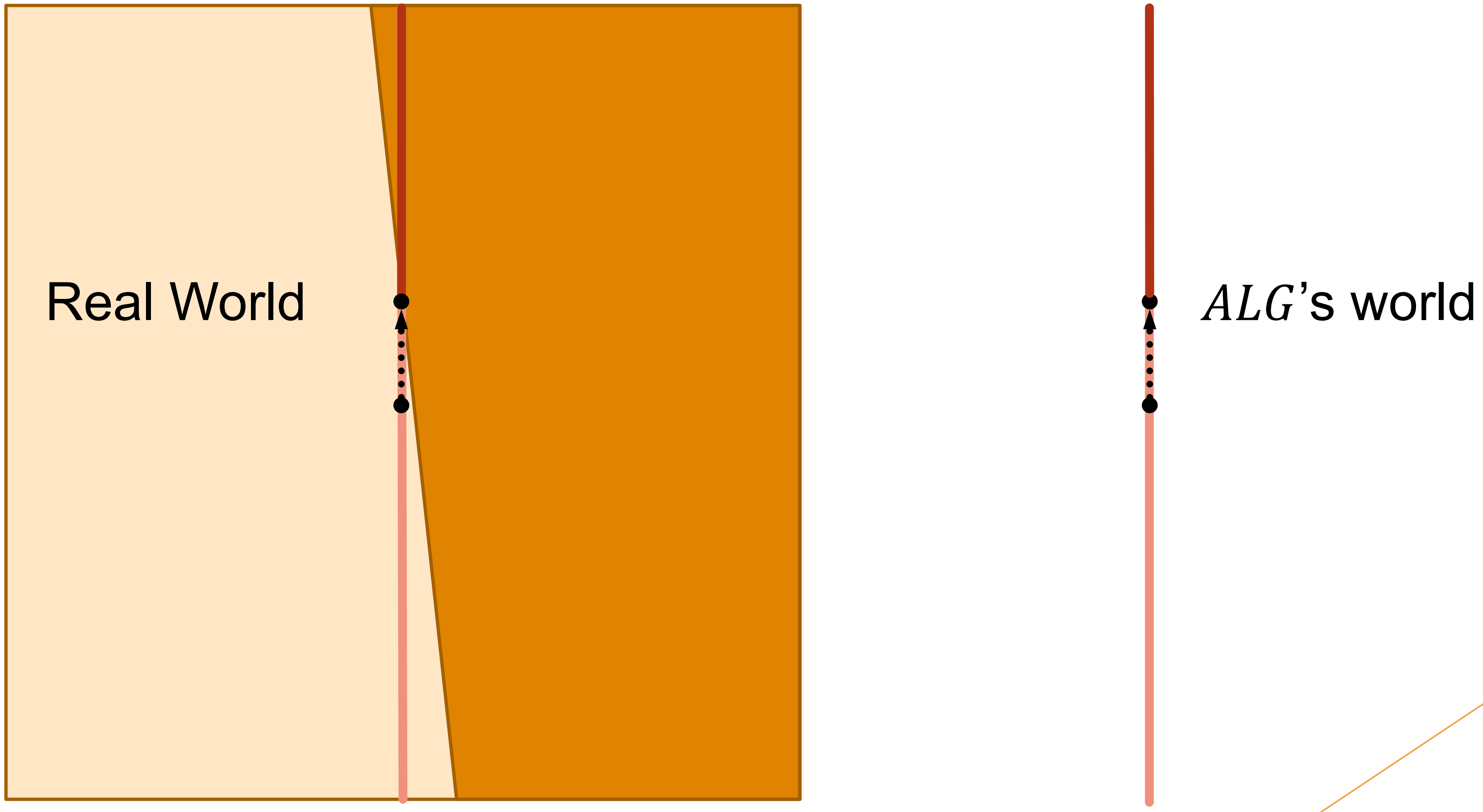
Idea 2 – *Recursive Greedy*



Idea 2 – *Recursive Greedy*



Idea 2 – *Recursive Greedy*



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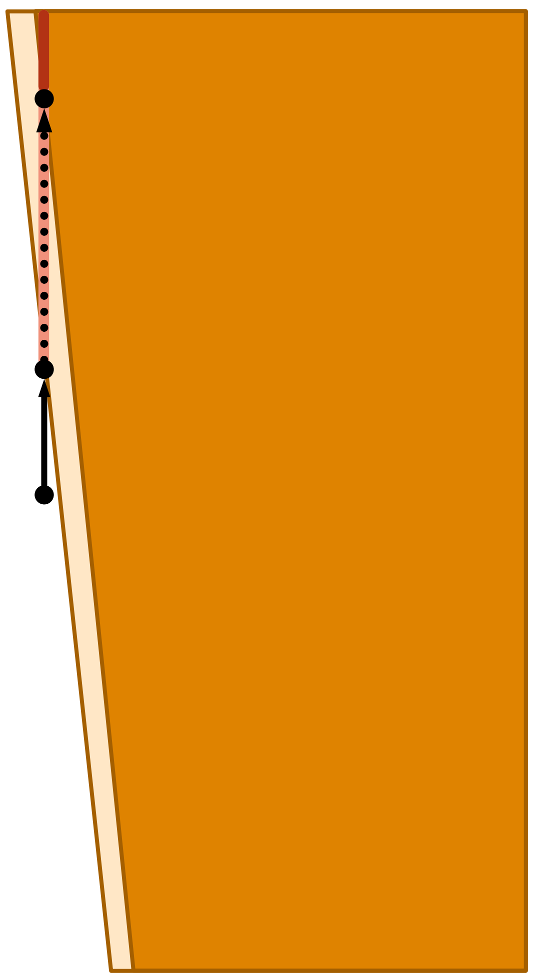
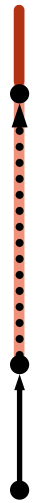
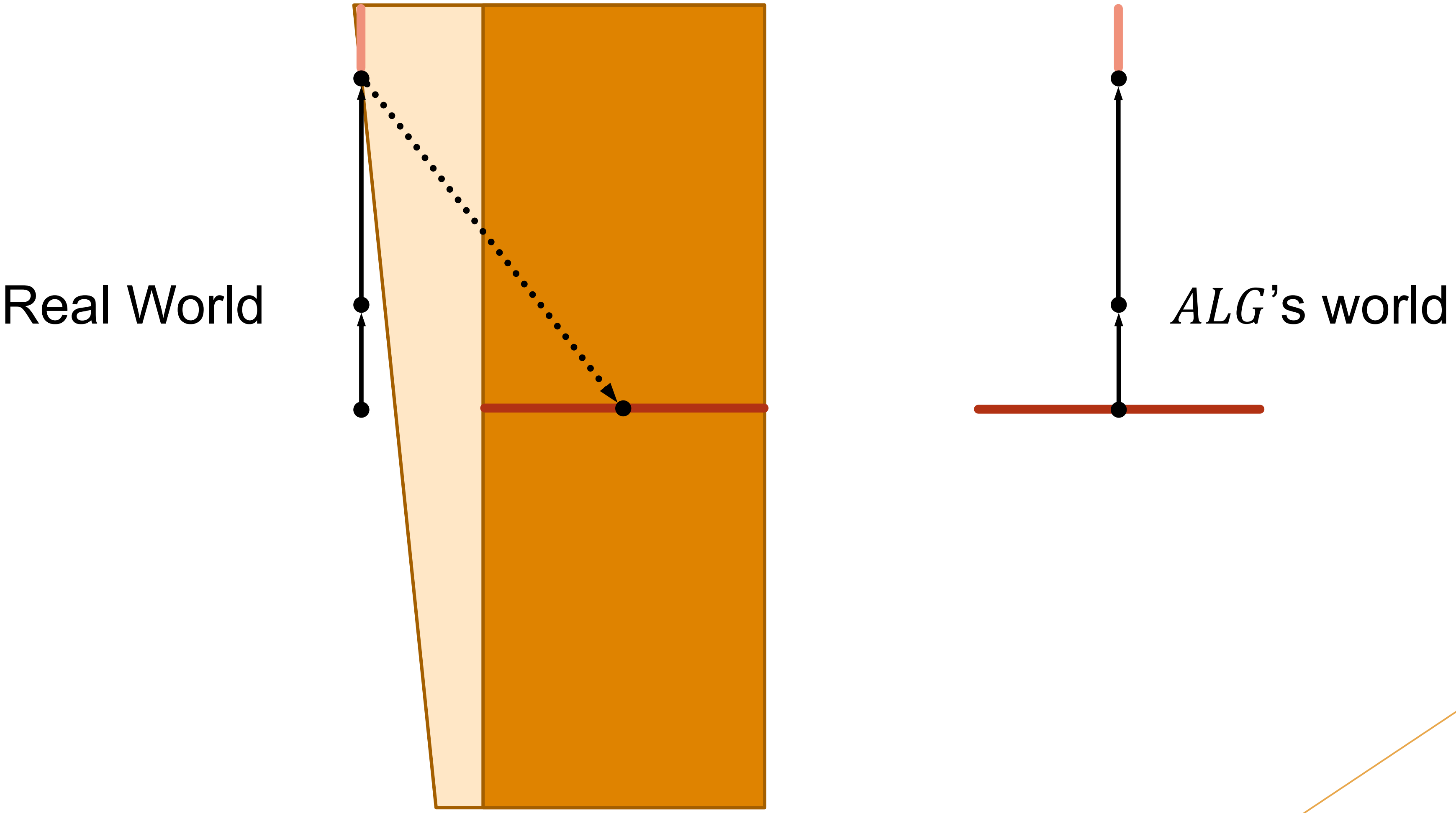




Idea 2 – *Recursive Greedy*



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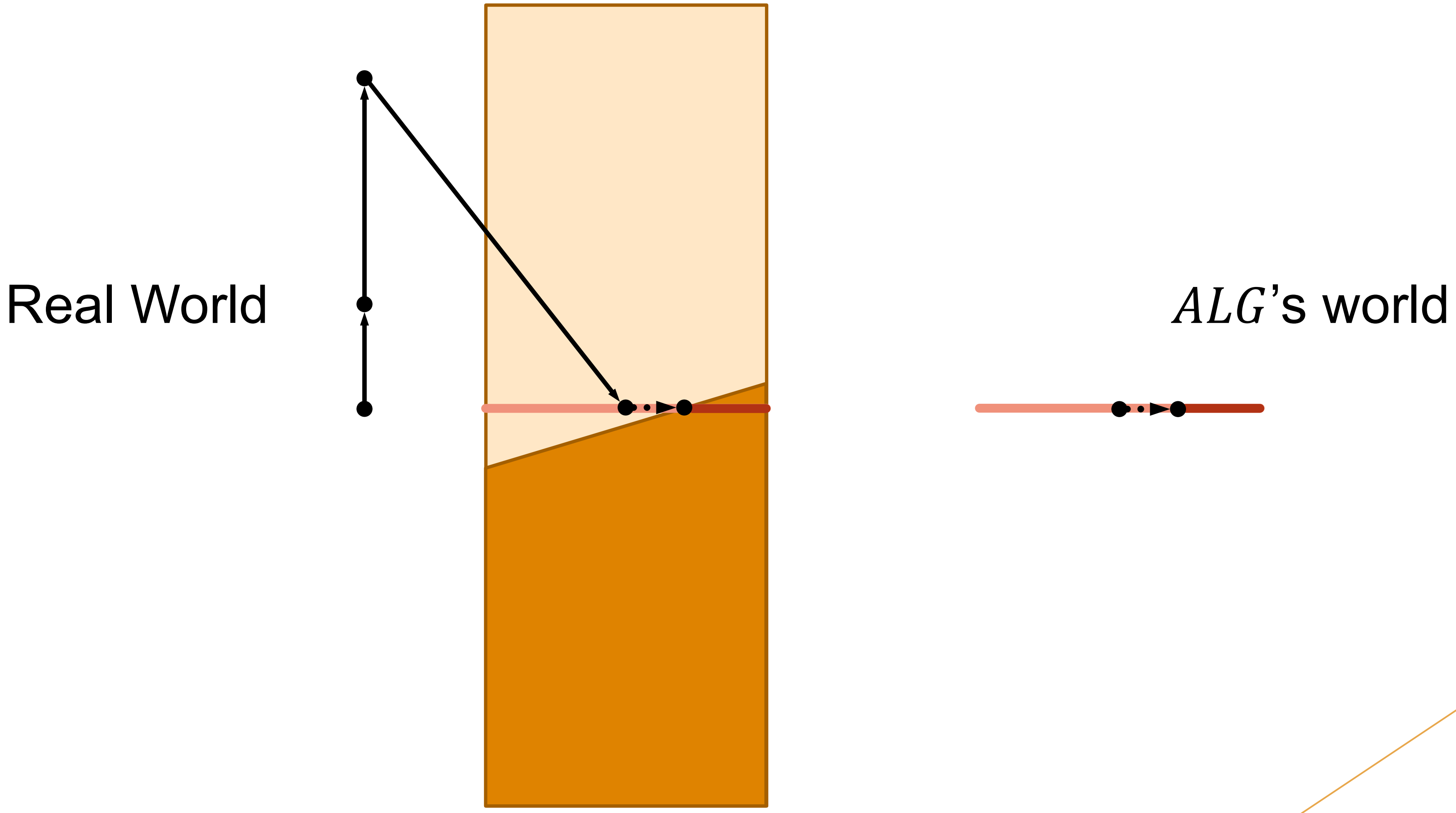
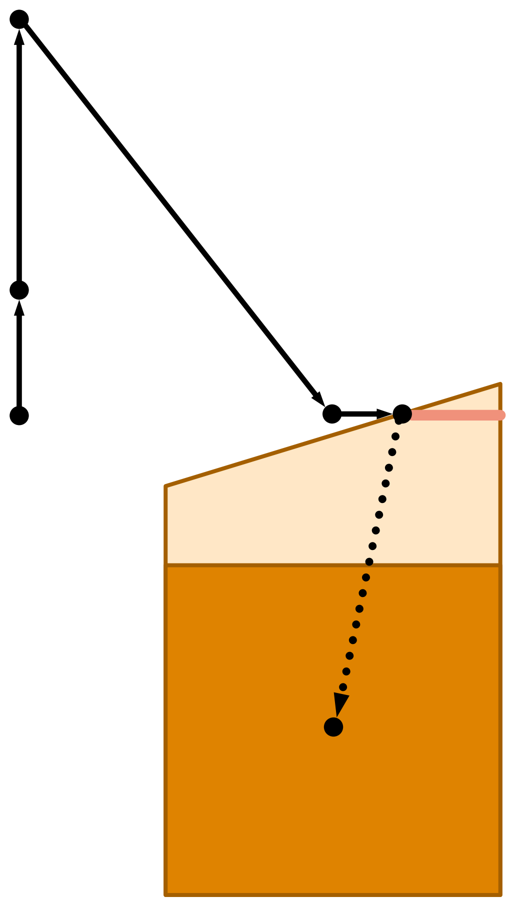
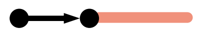


Idea 2 – *Recursive Greedy*

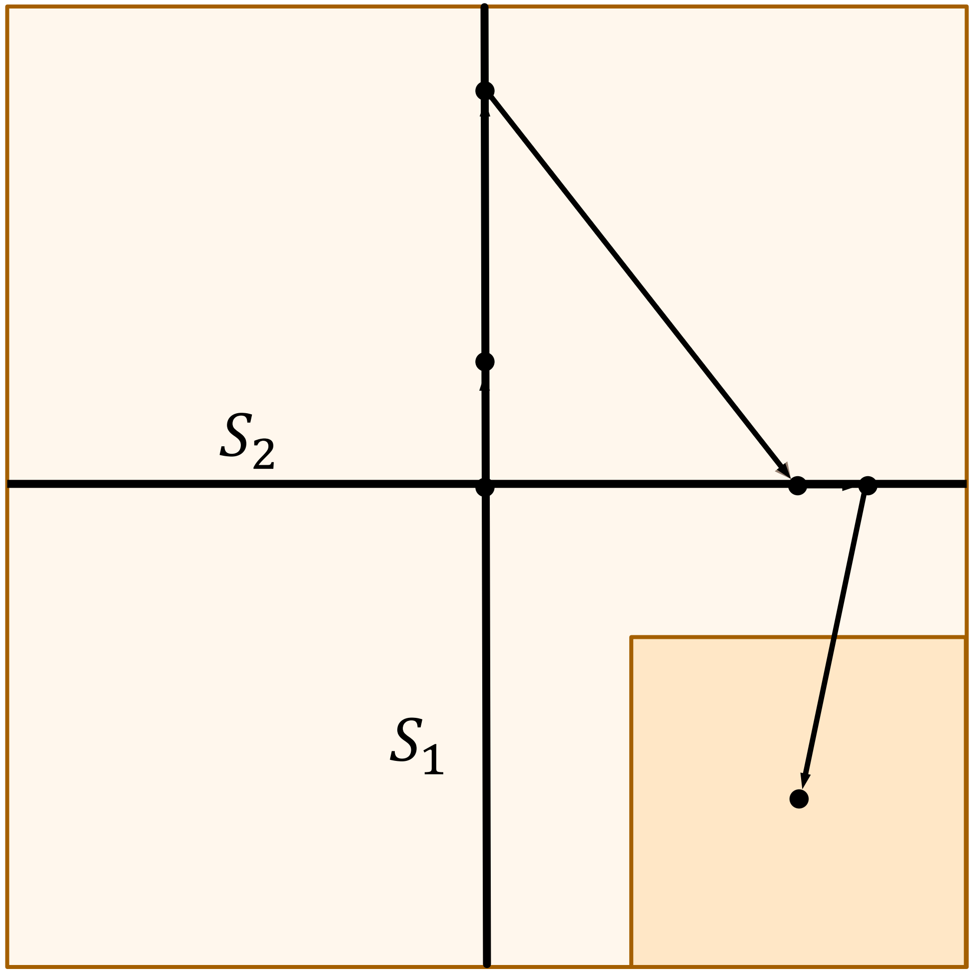
Real World



ALG's world



Idea 2 – *Recursive Greedy*



Diameter ↓↓↓

☺

Competitive algorithm
[BB+ ‘17]

Problem with *Recursive Greedy*

- $d^{O(d)}$-competitive
  - Worse than *Greedy!*
- Expensive recursive calls
- Diameter ↓ only $O\left(\sqrt{1-1/d}\right)$ after d recursive calls

Recap of Part 2

- *Centroid*
  - Volume drops quickly
  - Diameter stays constant
- *Recursive Greedy*
  - Controls individual dimensions
  - Expensive recursive calls
  - Diameter shrinks slowly

Part 3 – A better idea

Recursive Centroid: fusion of *Centroid* and *Recursive Greedy*

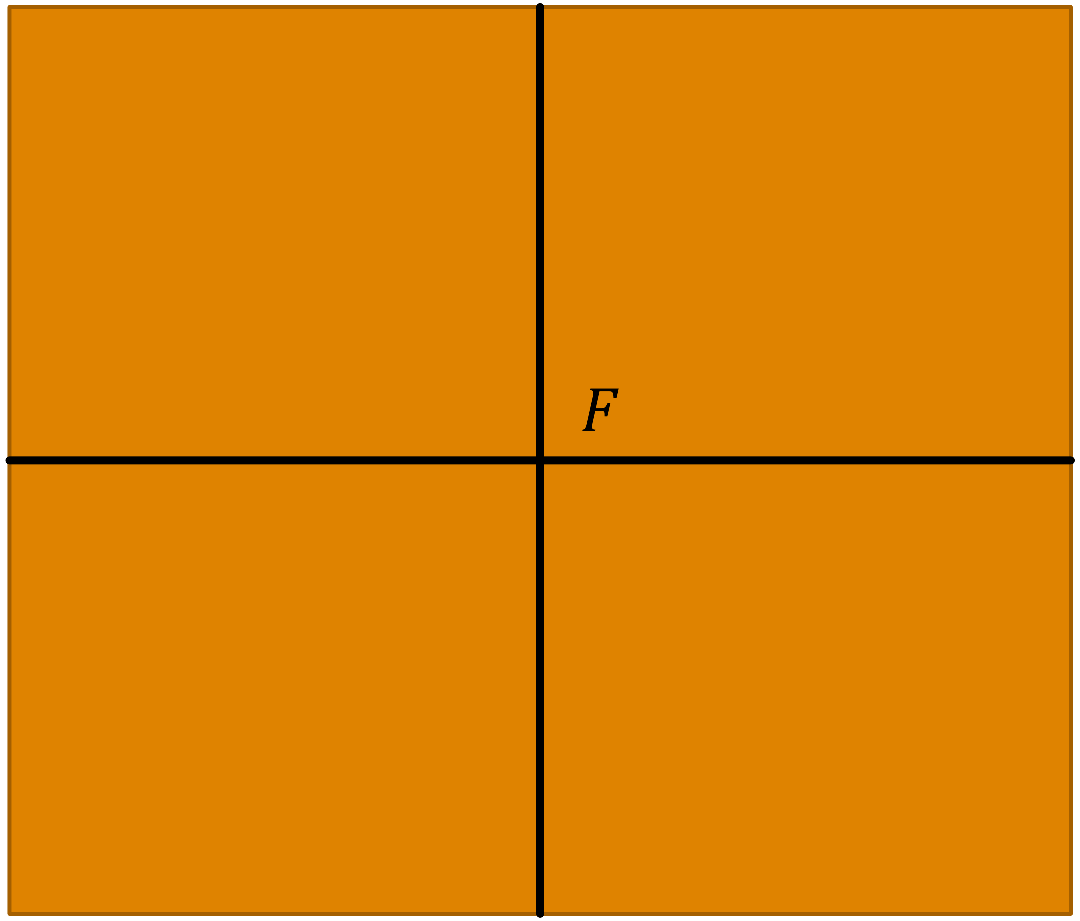
New Ideas

- Play centroid in recursion
- Recursion on skinny subspace
  - Cheap
  - Hyperplane separation $\Rightarrow$ cut parallel to skinny subspace
    - Progress on fat subspace

Skinny subspace

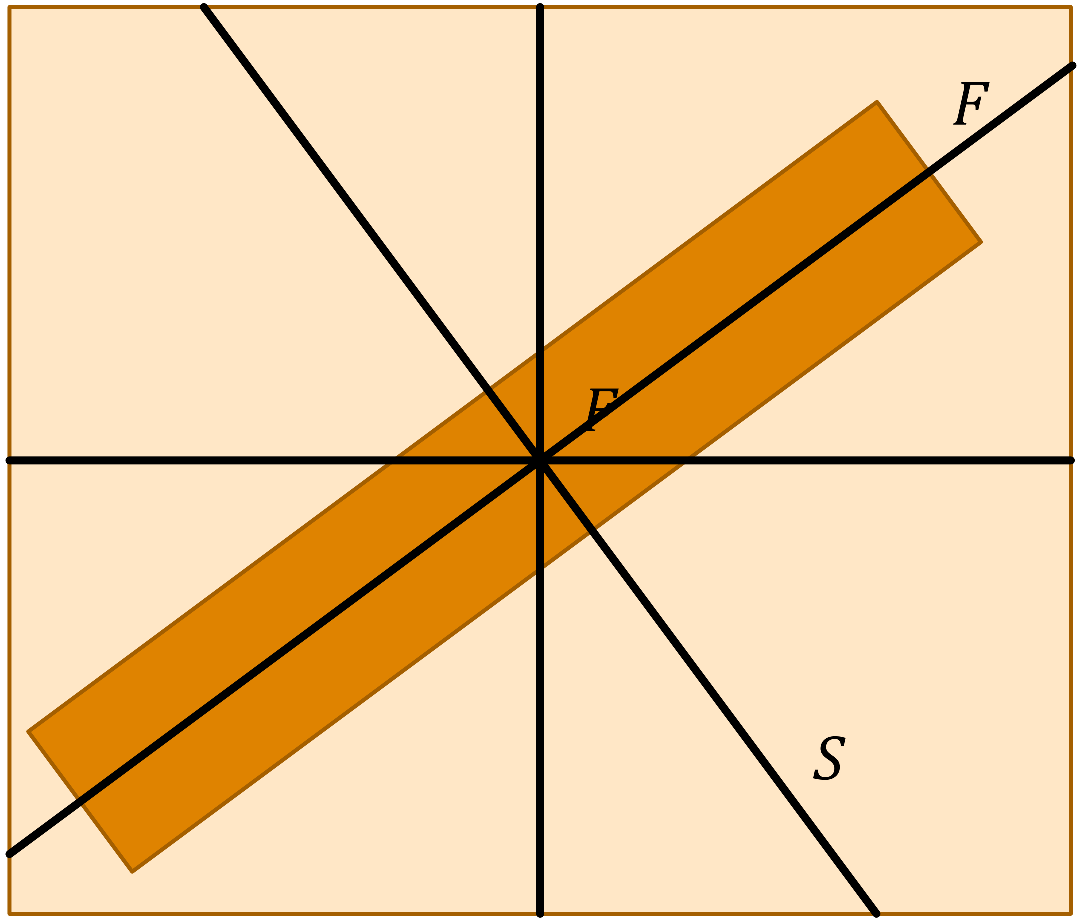
- Directional width $w(K, v) \coloneqq \max_{x,y \in K} \langle x - y, v \rangle$
- Skinny direction – v such that $w(K^t, v) \lesssim 1/d^2$
- $S \coloneqq$ span of k skinny directions
- $F \coloneqq S^{\perp}$ (fat subspace)

Skinny and Fat subspace



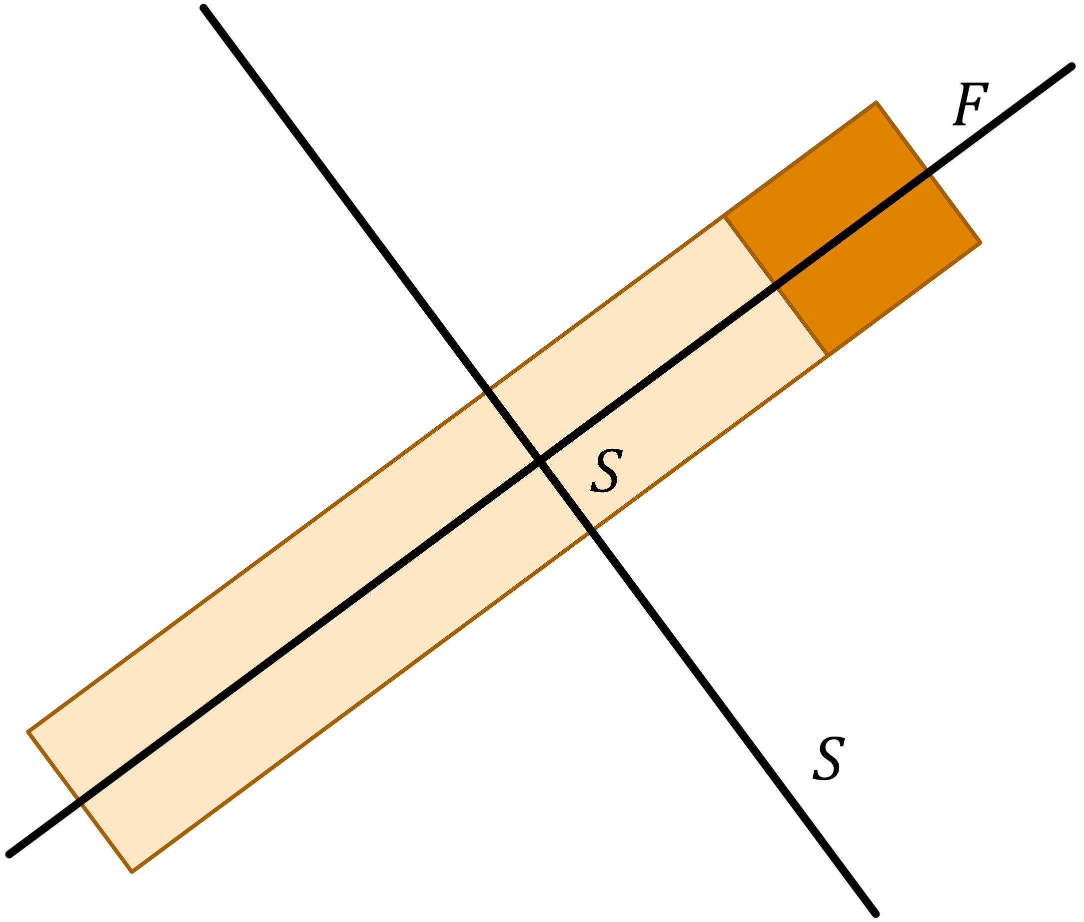
$S = \{0\}$

Skinny and Fat subspace



$S = \{0\}$

Skinny and Fat subspace



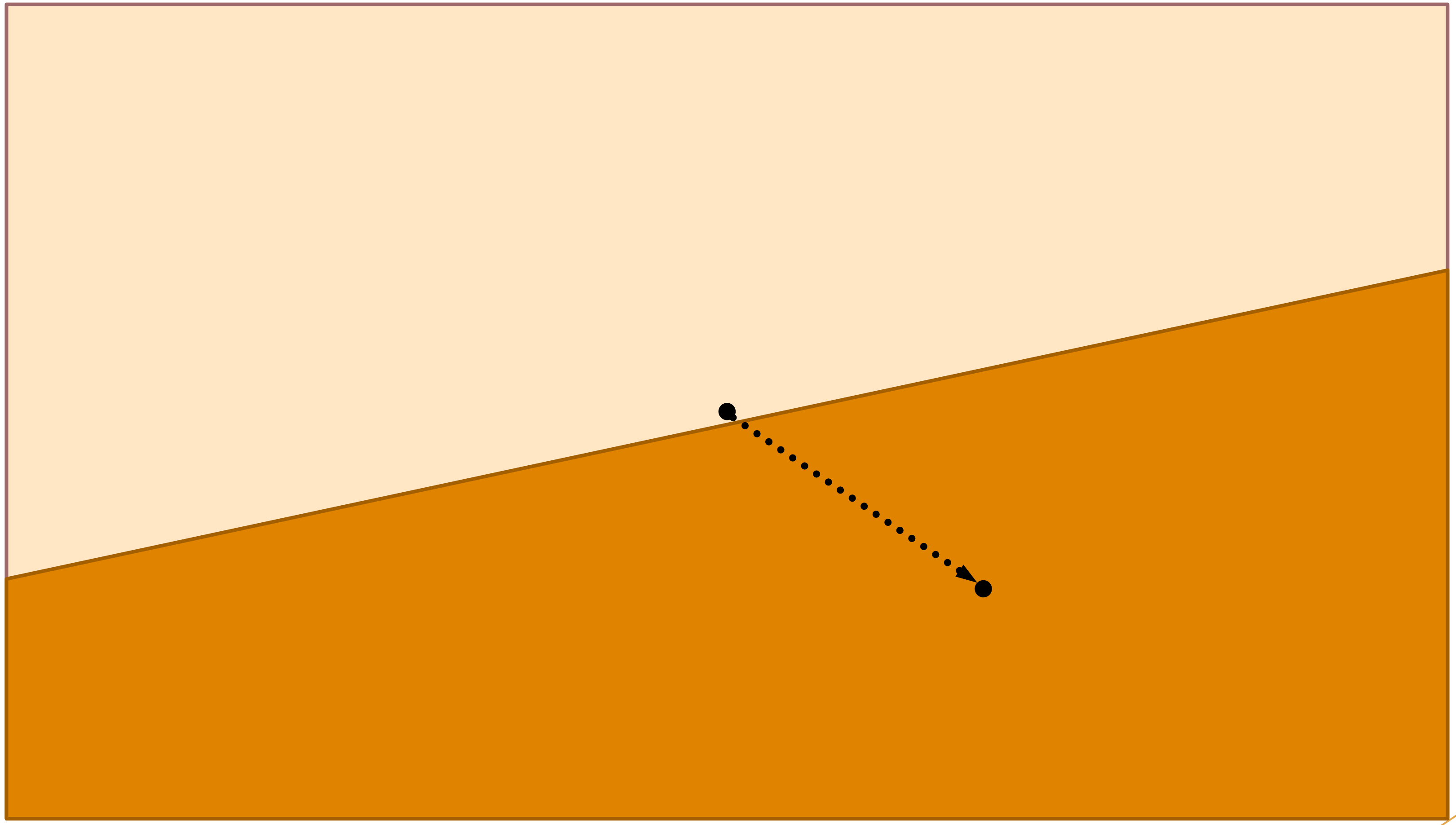
$F = \{0\}$

Recursive Centroid

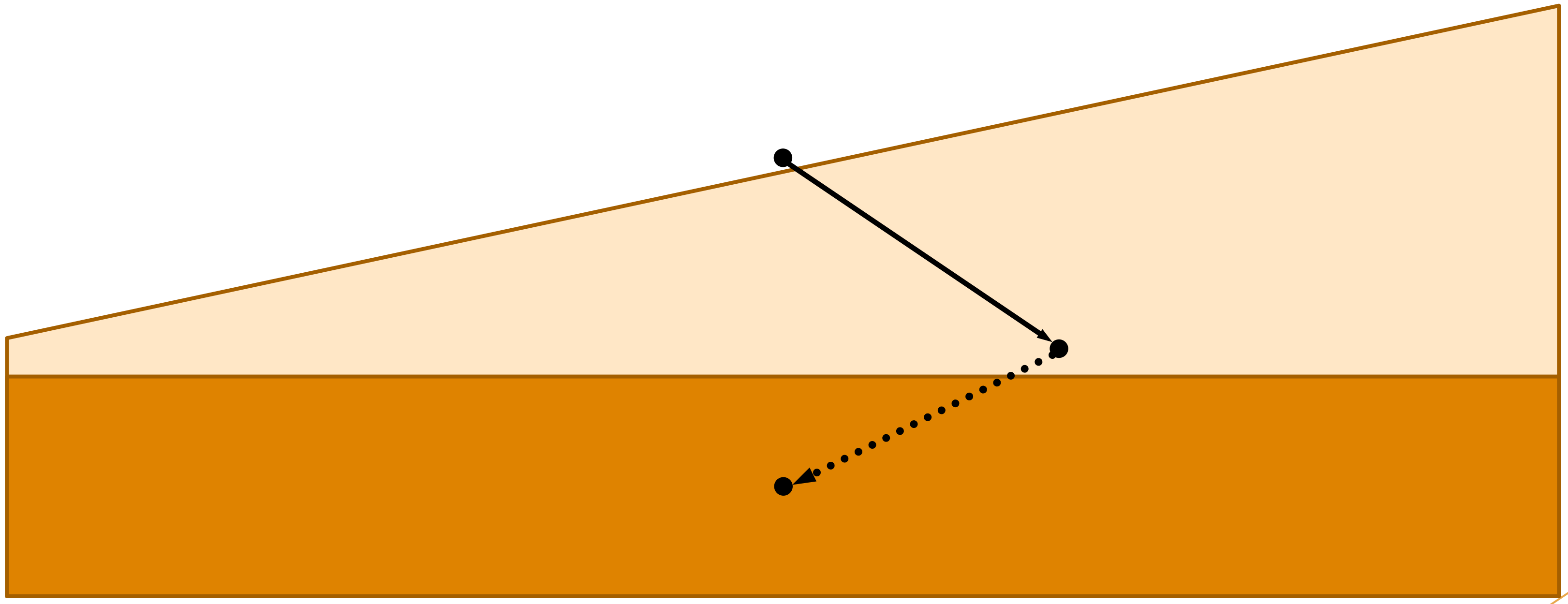
- While $diam(K^t) \geq 1/2 \cdot diam(K^1)$
  - If $S_t \neq \{0\}$
    - $\bar{t} \leftarrow t$
    - Run $RC^{\dim(S_{\bar{t}})}$ on $K^t \cap (x_{\bar{t}} + S_{\bar{t}})$ until empty
  - $x_t \leftarrow \mu(K^t)$
  - While ∃ skinny direction $v \in S_t^{\perp}$
    - $S_t \leftarrow span(S_t, v)$

$RC^{\dim(S_{\bar{t}})}$ – *Recursive Centroid* in $\dim(S_{\bar{t}})$ dimensions

Recursive Centroid



Recursive Centroid



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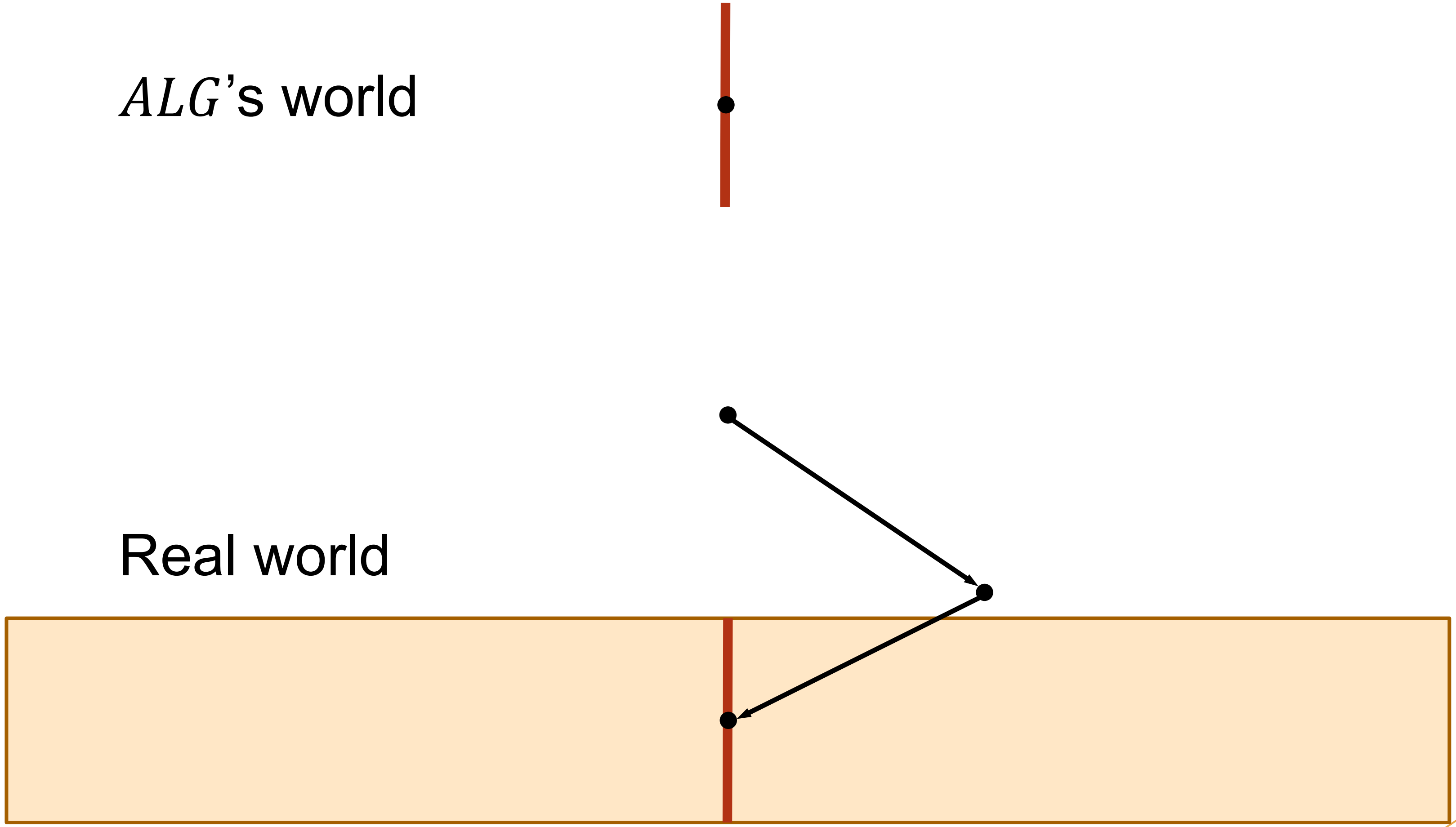
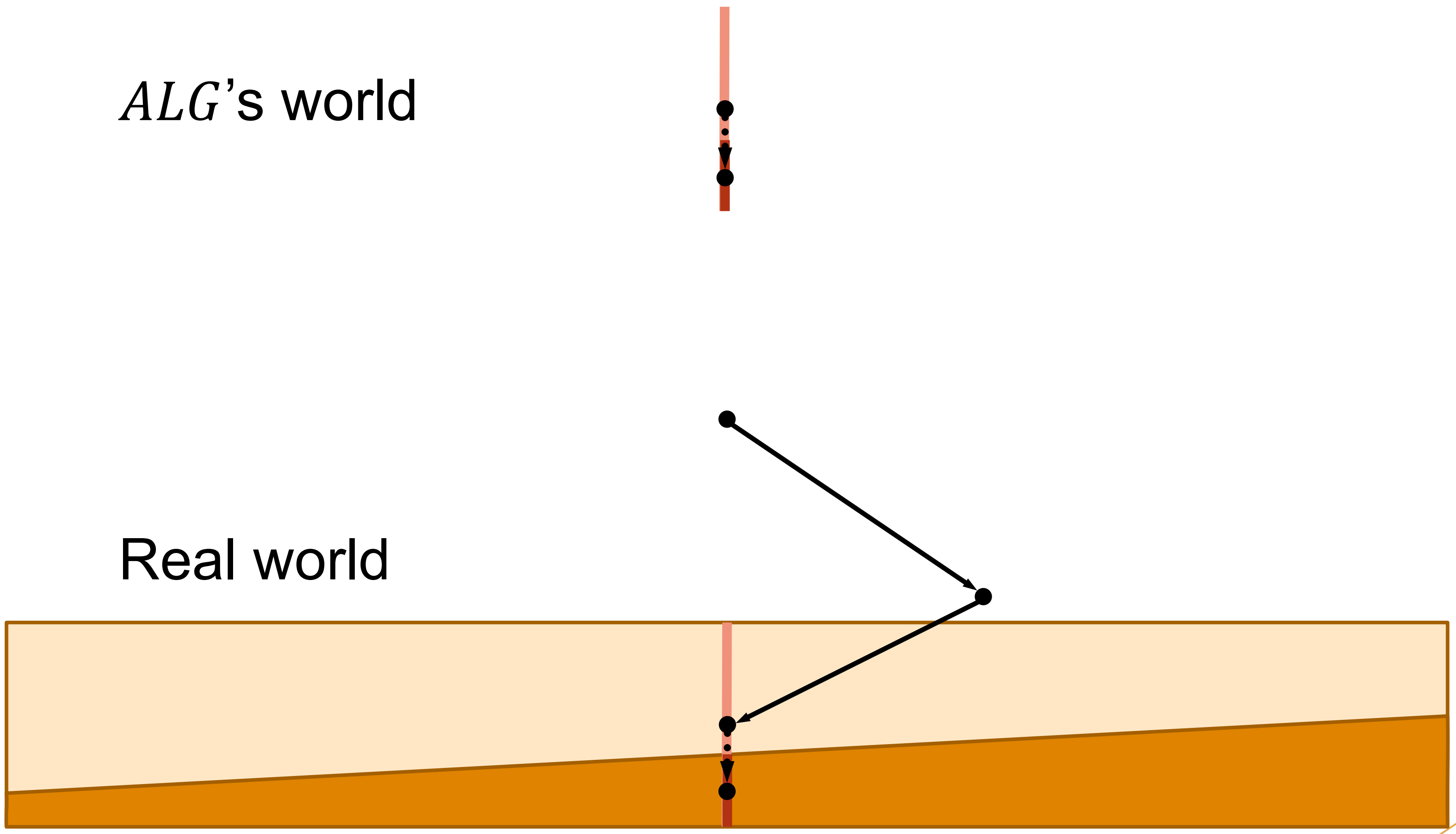


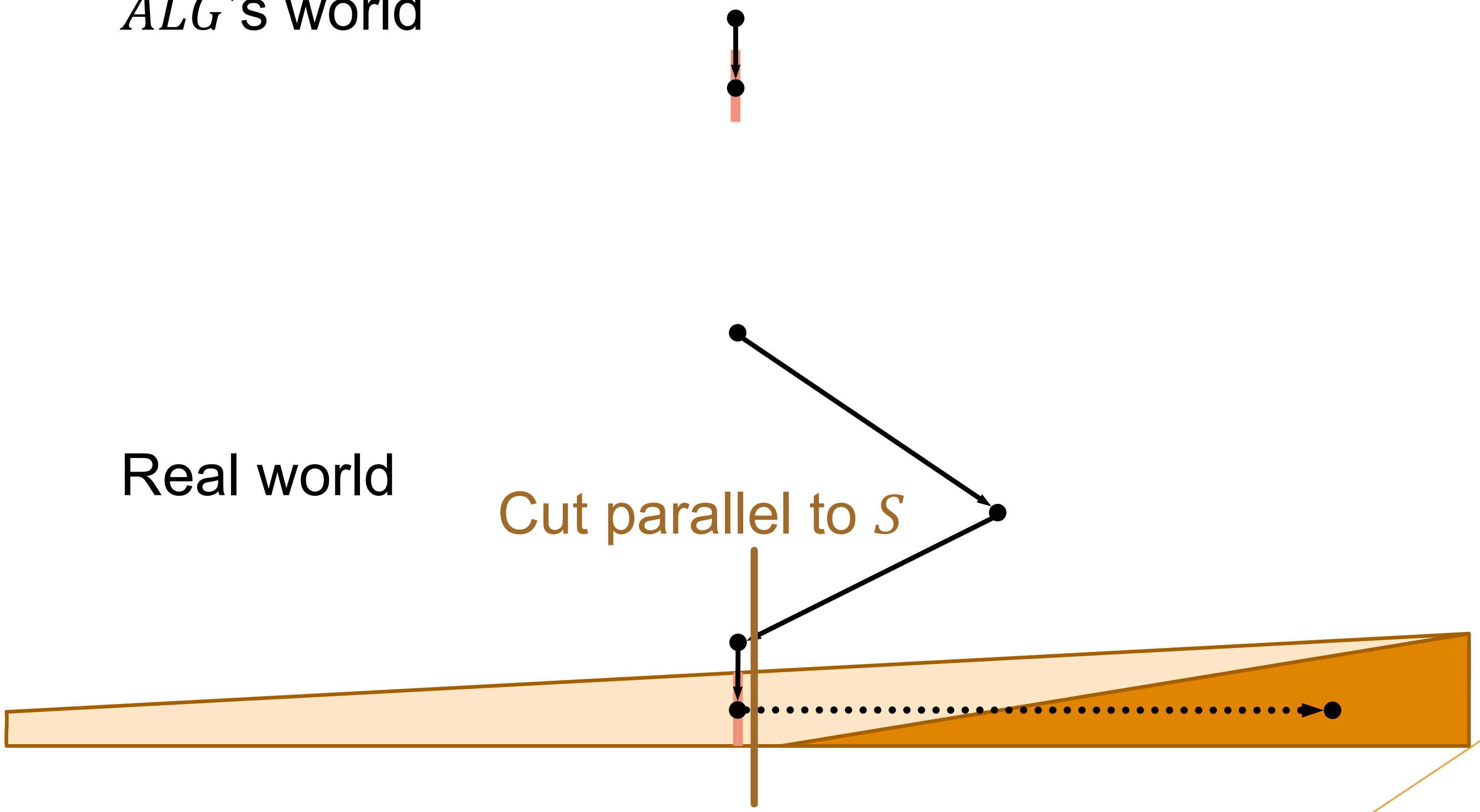
Recursive Centroid



Recursive Centroid

ALG's world

Real world



Main theorem

Recursive Centroid is $O(d \log d)$-competitive [ABCGL ‘18]

- Recall $\sqrt{d}$ lower bound

Proof outline

- Potential $\Phi^t := Vol\left(Proj_F(K^t)\right)$
- 'Step' = Recursive call + move to centroid of K^t

- Cost of 1 step = $O(1)$
- $O(d \log d)$ steps
- $O(d \log d)$ total cost

Proof part I – A single step

$$\Phi^t = Vol\left(Proj_F(K^t)\right)$$

- Cost $O(1)$
  - Recursion: $O(d \log d) \cdot 1/d^2 = o(1)$
  - Move to centroid: $O(1)$
- Φ^t drops $(1-c)$
  - K^t cut by halfspace *parallel to* S

Proof part II – $O(d \log d)$ steps

$$\Phi^t = Vol\left(Proj_F(K^t)\right)$$

- Φ^t drops $\geq (1-c)^m$
  - m steps
- Φ^t increases $\leq d^{O(d)}$
  - F changes
- $\Phi^{T-1} \geq d^{-O(d)}$
  - $Proj_F(K^{T-1})$ contains ball of radius $1/poly(d) = d^{-O(1)}$

Proof part II – $O(d \log d)$ steps

$$\Phi^t = Vol\left(Proj_F(K^t)\right)$$

- Φ^t drops $\geq (1-c)^m$
- Φ^t increases $\leq d^{O(d)}$
- $\Phi^{T-1} \geq d^{-O(d)}$

$$d^{O(d)}(1-c)^{m-1} \geq \Phi^{T-1}/\Phi^0 \geq d^{-O(d)}$$

$$m \leq O(d \log d)$$

Recap of Part 3

- Recursion on skinny subspaces
  - Cheap, good cuts
- Play centroid
  - Volume drop
- K^t bounded, recursion cheap $\Rightarrow$ step cost $O(1)$
- $Vol\left(Proj_F(K^t)\right)$ drops, bounded $\Rightarrow O(d \log d)$ steps

Open questions

- $poly(d)$-competitive general chasing
- $exp(d)$ lower bound for general chasing
- Efficient algorithms

Thank you!

Questions?

In memory of Michael Cohen

References

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