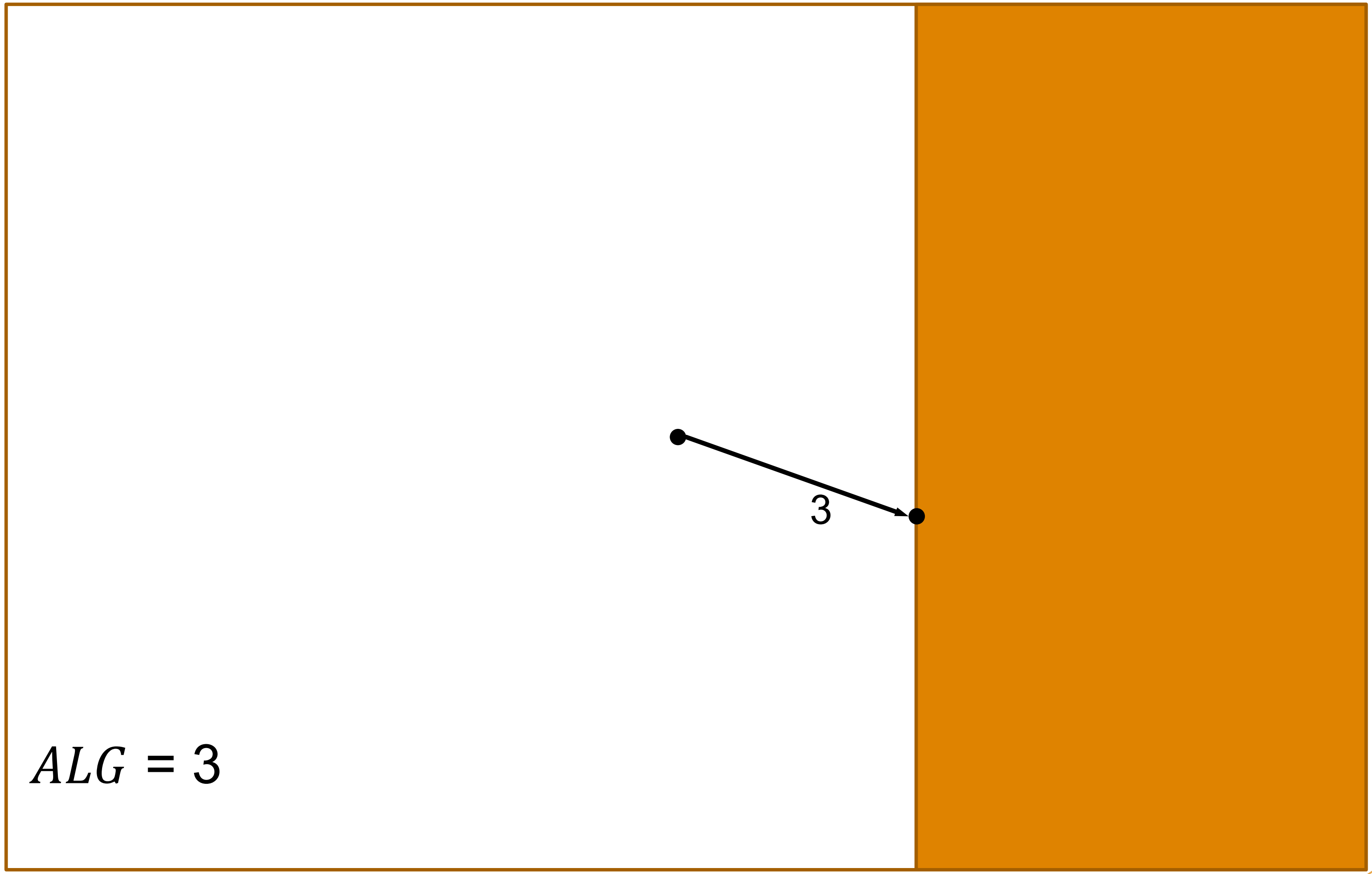
Convex Body Chasing

C.J. Argue

Joint with Anupam Gupta, Guru Guruganesh, Ziye Tang

some slides courtesy of Guru Guruganesh

Convex Body Chasing – The Problem



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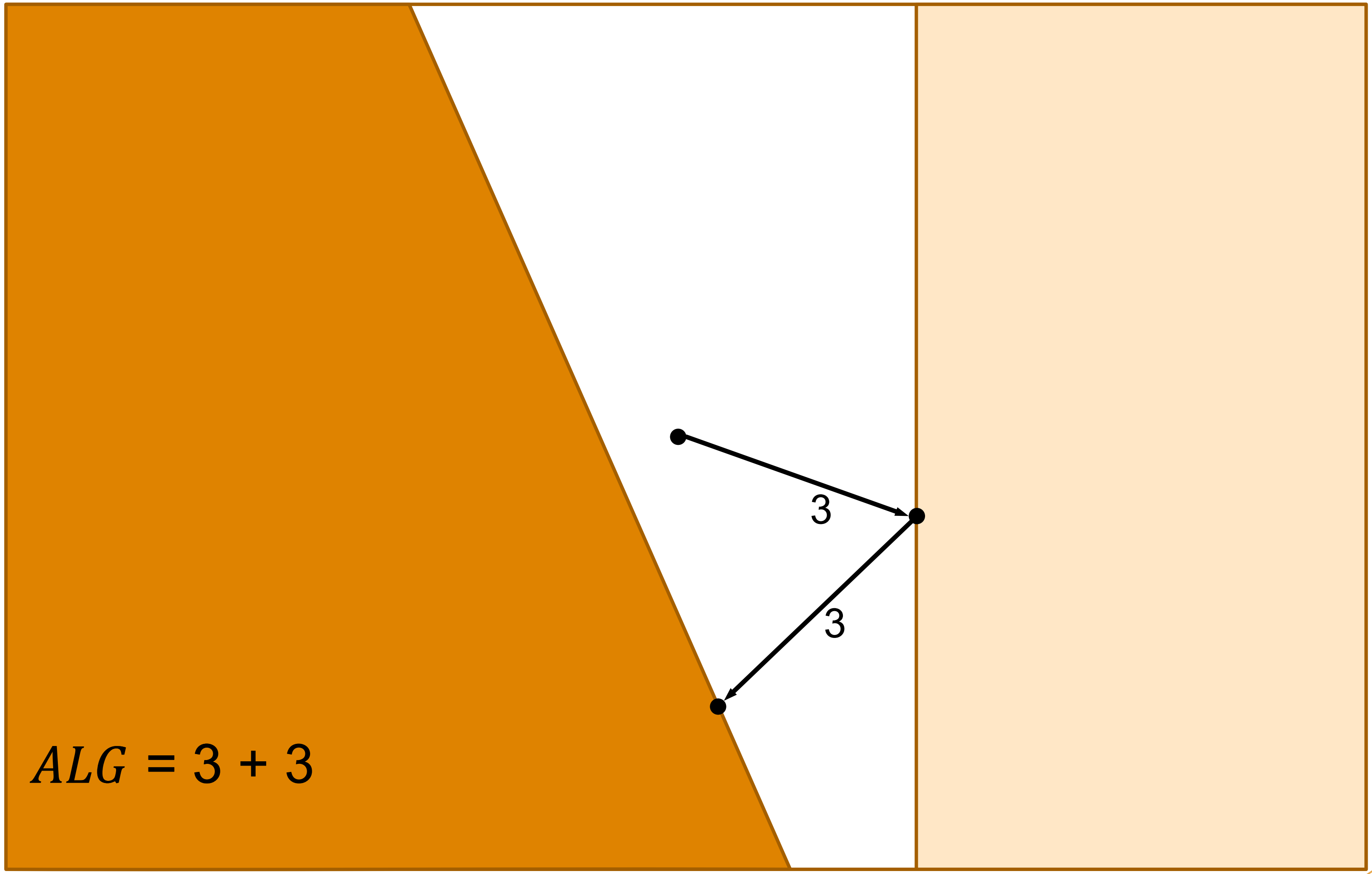

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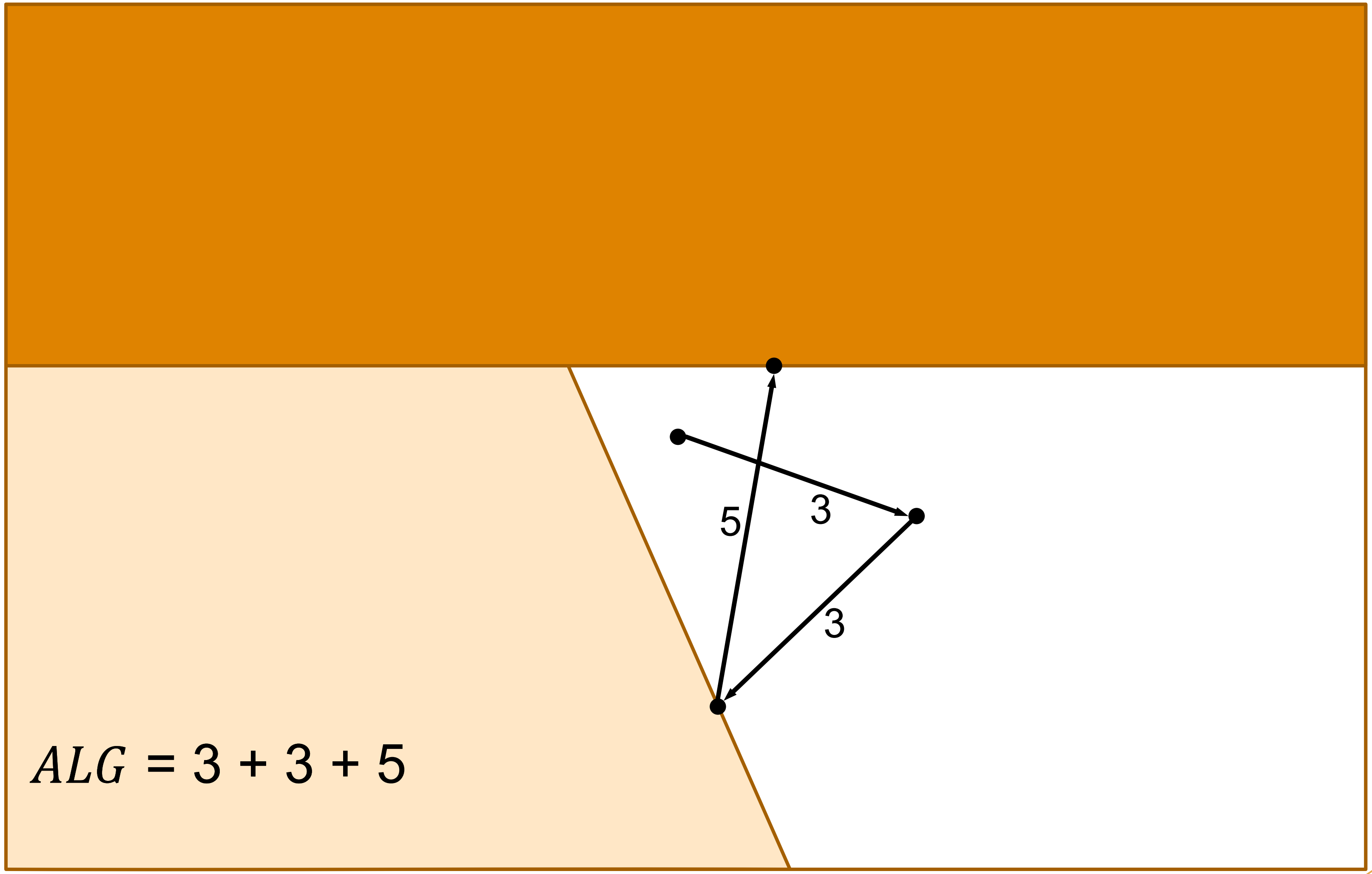
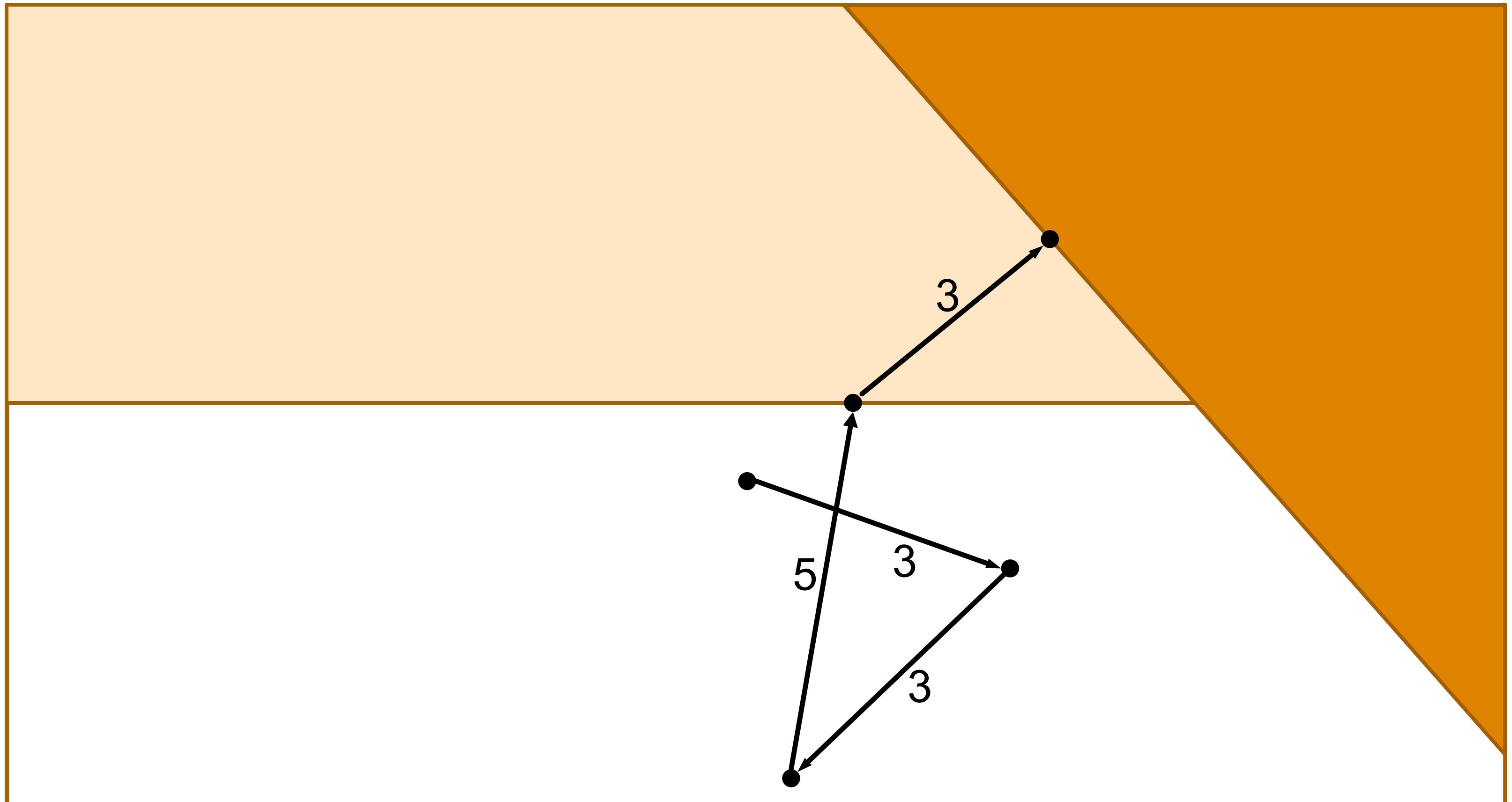
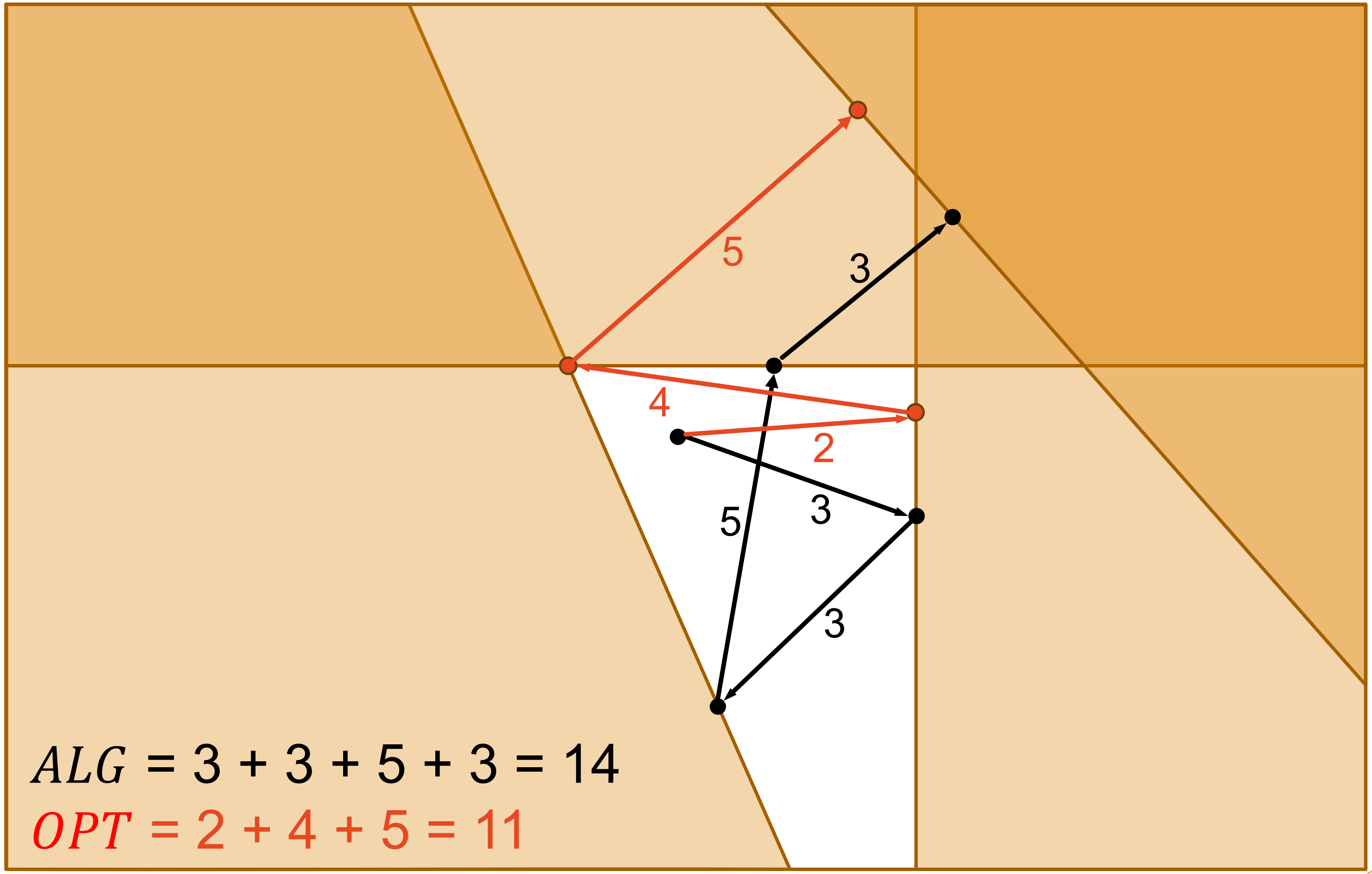


Convex Body Chasing – The Problem



$ALG = 3 + 3 + 5 + 3$

Convex Body Chasing – The Problem



ALG = 3 + 3 + 5 + 3 = 14

OPT = 2 + 4 + 5 = 11

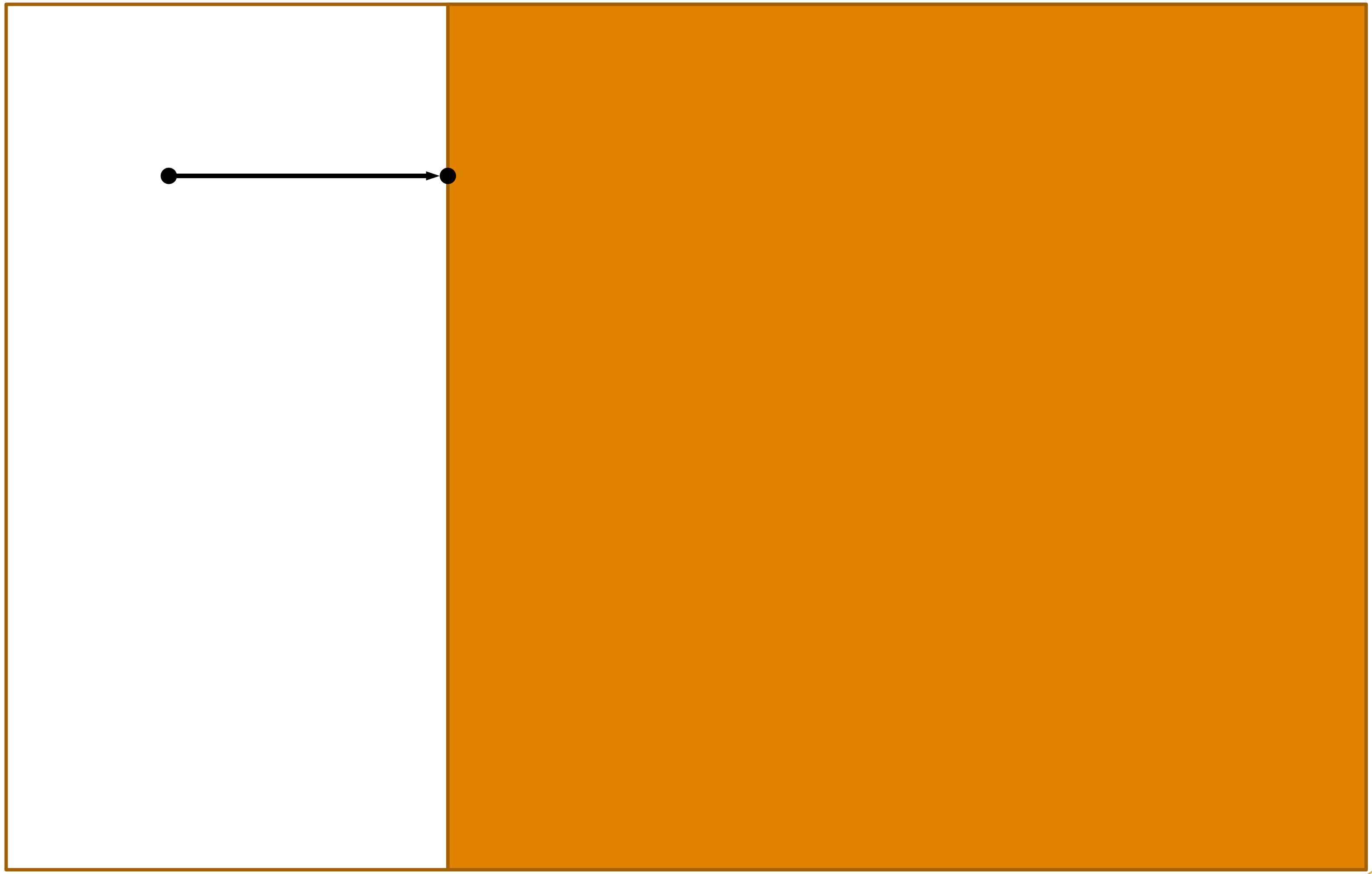
Formal Definition

- Input: convex sets $K_1, K_2, K_3, \dots, K_T \subseteq \mathbb{R}^d$
- Choose *online* $x_i \in K_i$
- Cost $ALG = \sum_{i=1}^{T} ||x_i - x_{i-1}||$
- Goal – minimize competitive ratio

$$\mathrm{cr}(ALG) \coloneqq \max_{\text{instance } \sigma} \frac{ALG(\sigma)}{OPT(\sigma)}$$

  - OPT optimal *offline* cost

Nested Version



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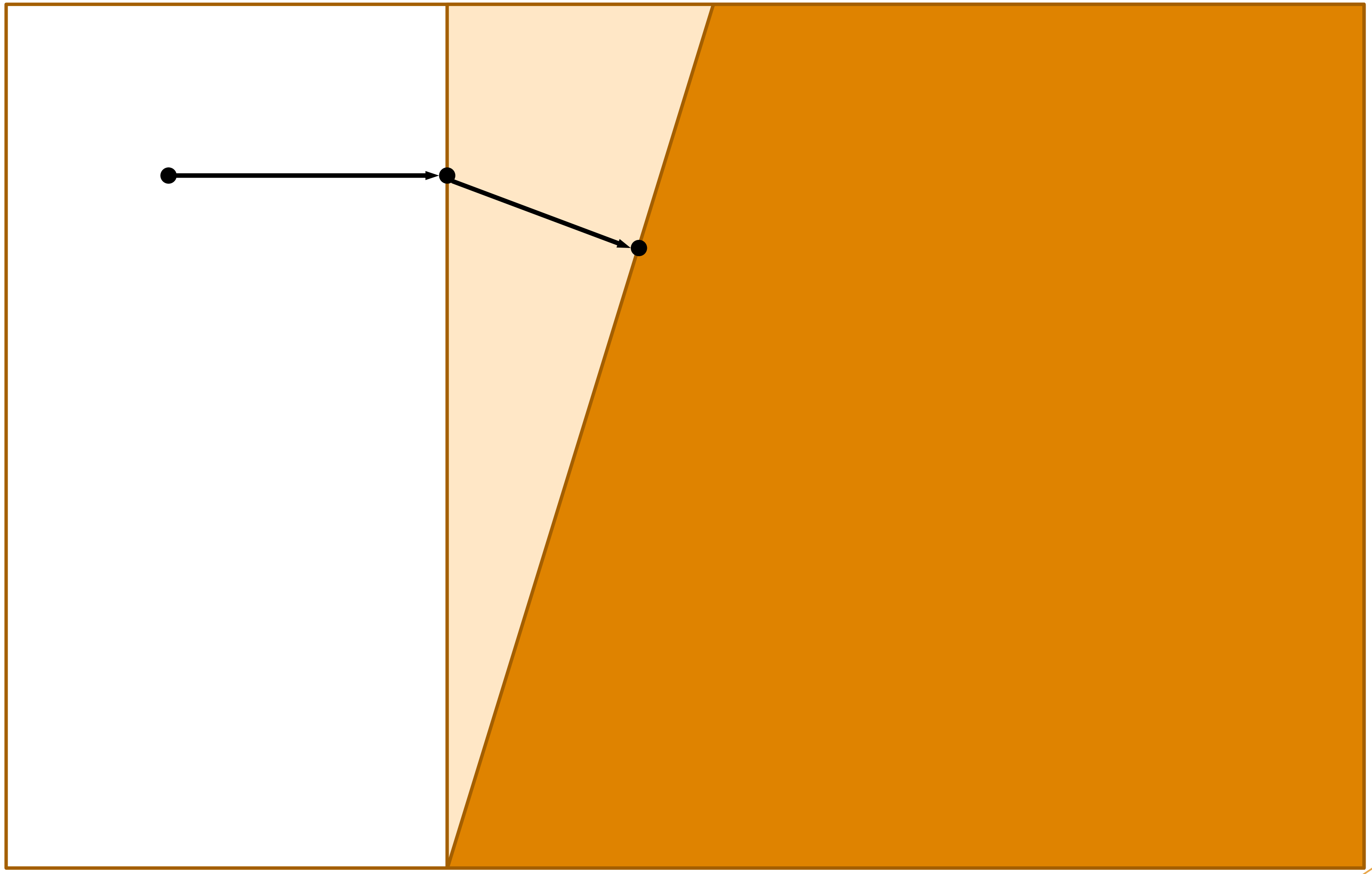

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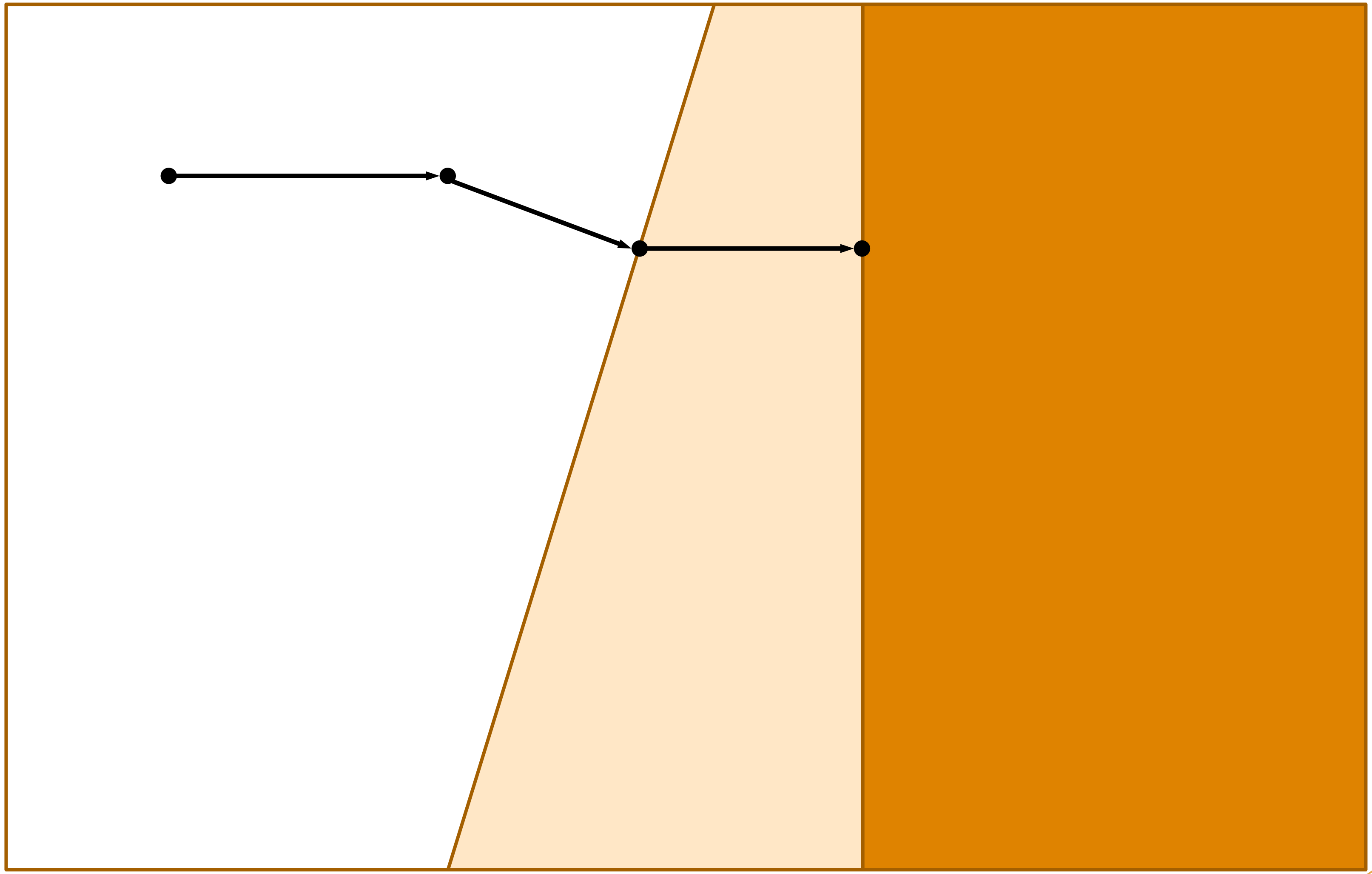

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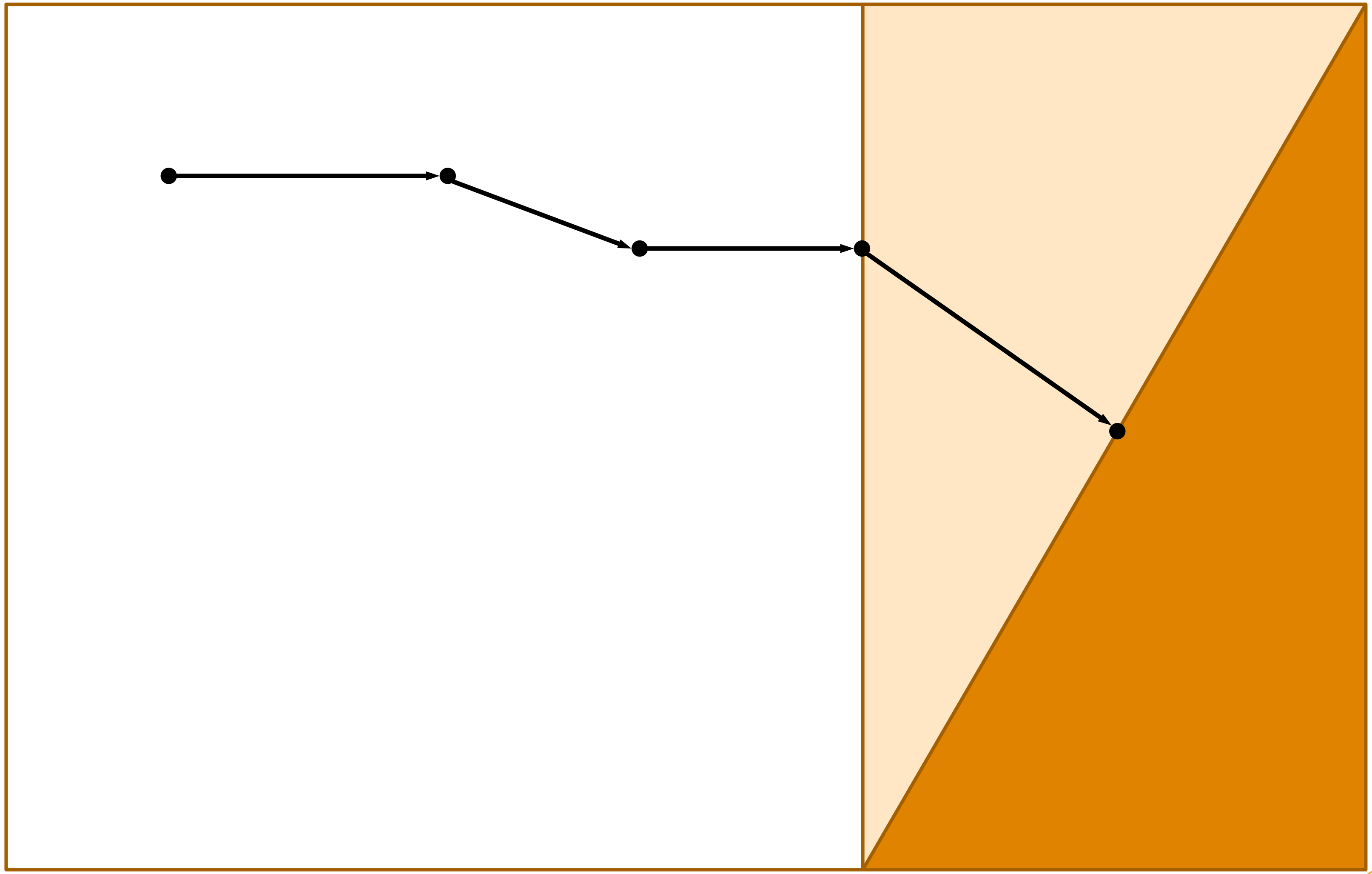

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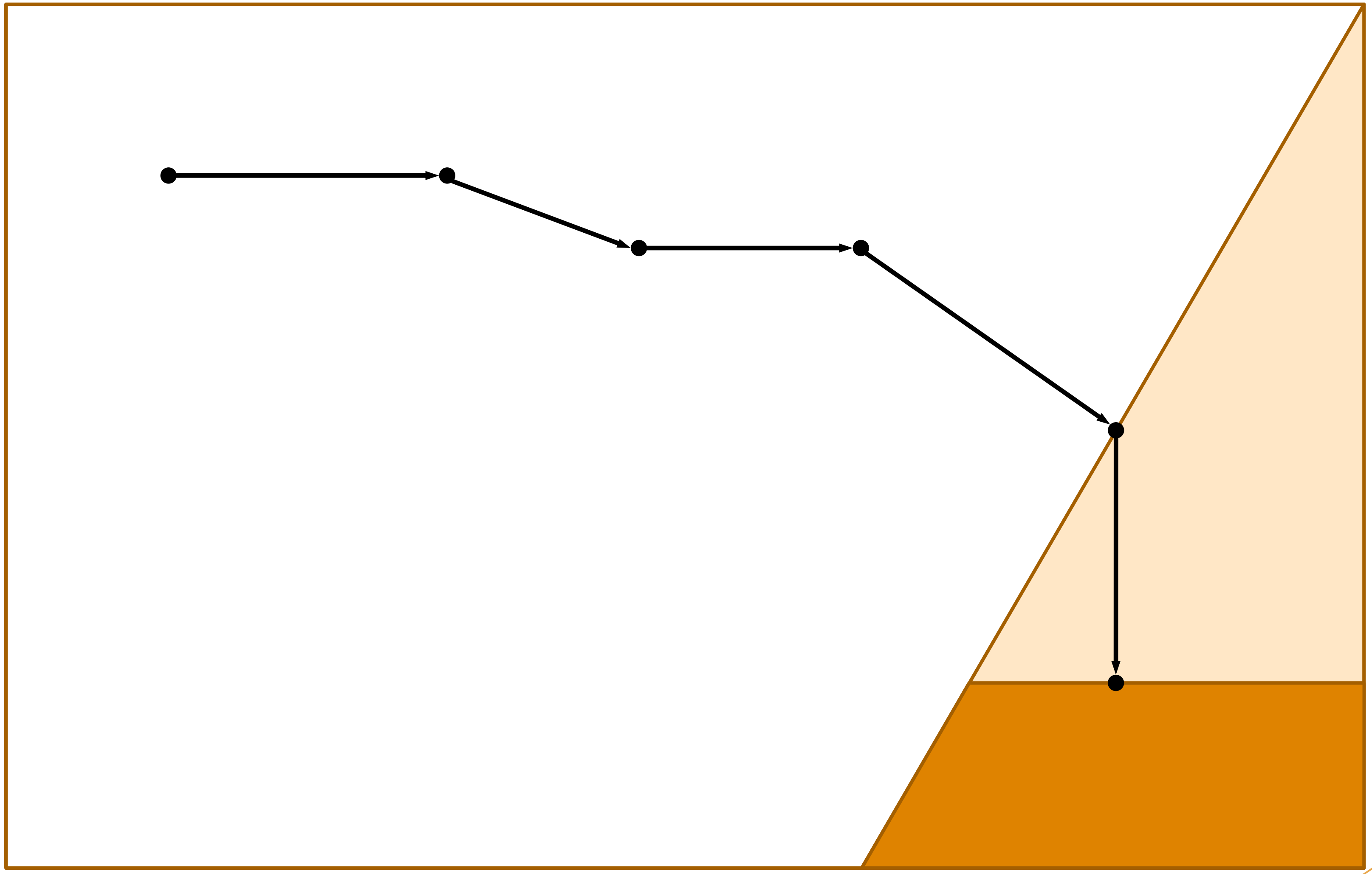

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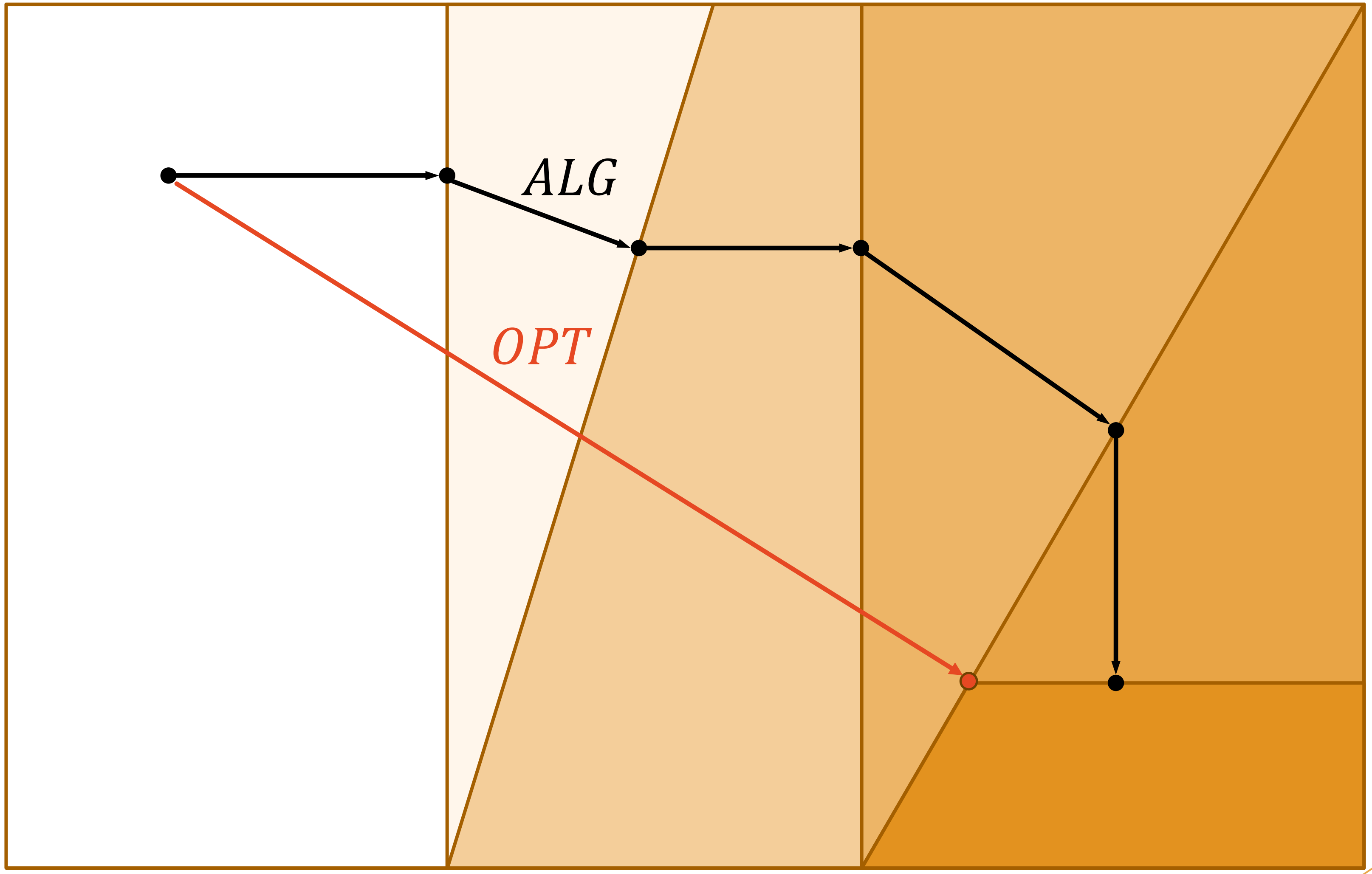

Motivation – Function Chasing

- Instance σ: convex sets $K_1, K_2, K_3, \dots, K_T \subseteq \mathbb{R}^d$
- Choose *online* $x_i \in K_i$
- Cost $ALG(\sigma) = \sum_{i=1}^{T} ||x_i - x_{i-1}||$

Motivation – Function Chasing

functions $f_1, f_2, f_3, \ldots, f_t : \mathbb{R}^d \to \mathbb{R}$

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- Choose *online* ~~$x_i \in K_i$~~ $x_i \in \mathbb{R}^d$
- Cost $ALG(\sigma) = \sum_{i=1}^{T} ||x_i - x_{i-1}|| + f_i(x_i)$

Motivation – Function Chasing

functions $f_1, f_2, f_3, \dots, f_t : \mathbb{R}^d \rightarrow \mathbb{R}$

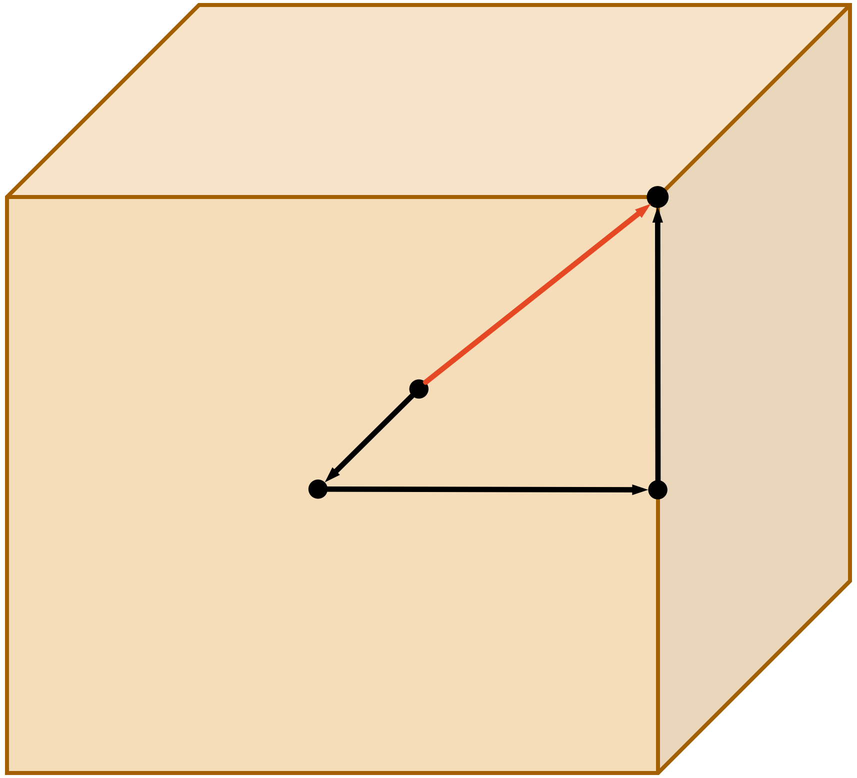
- Instance σ: convex ~~sets $K_1, K_2, K_3, \dots, K_T \subseteq \mathbb{R}^d$~~
- Choose *online* ~~$x_i \in K_t$~~ $x_i \in \mathbb{R}^d$
- Cost $ALG(\sigma) = \sum_{i=1}^{T} ||x_i - x_{i-1}|| + f_i(x_i)$

- Captures many well-studied problems
- Function chasing $\cong$ body chasing

  [Bubeck, Lee, Li, Sellke 18]

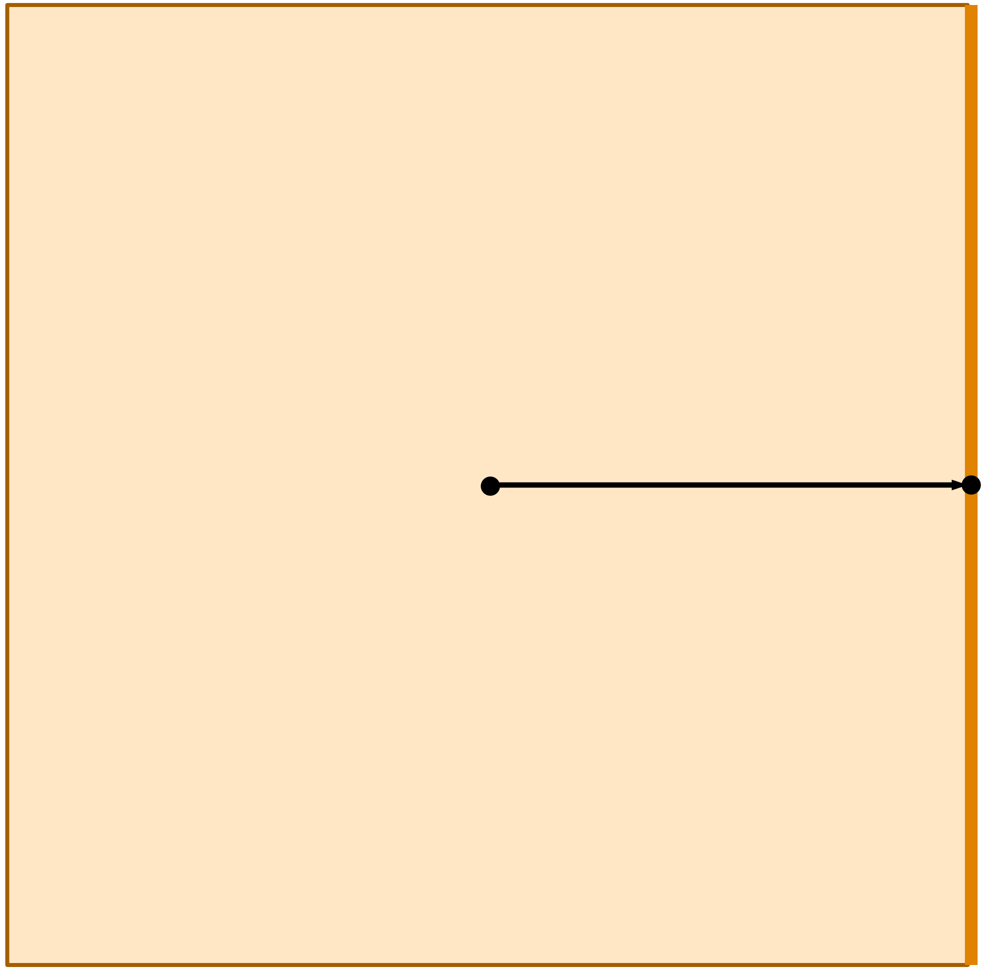
Lower Bound of $\sqrt{d}$

[Friedman, Linial 93]



Lower Bound of $\sqrt{d}$

[Friedman, Linial 93]



Lower Bound of $\sqrt{d}$

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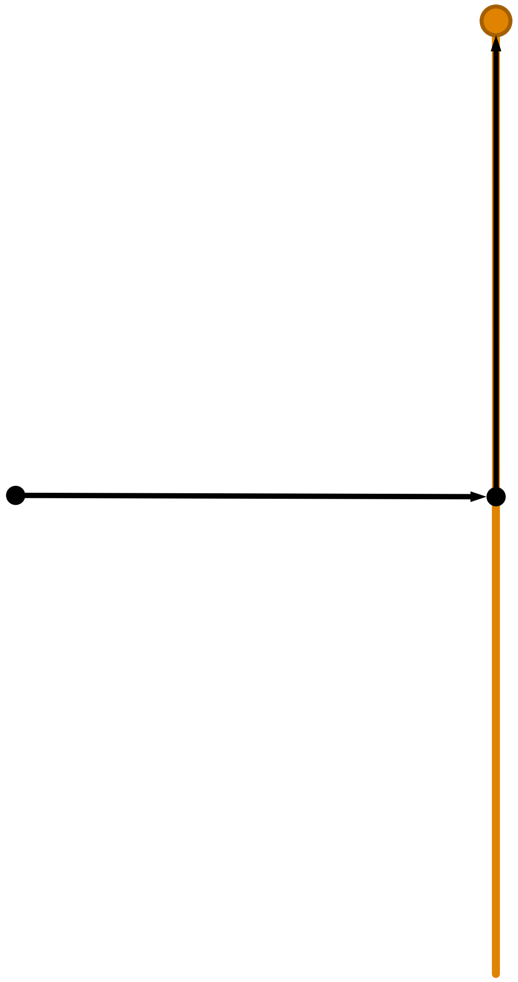

Lower Bound of $\sqrt{d}$

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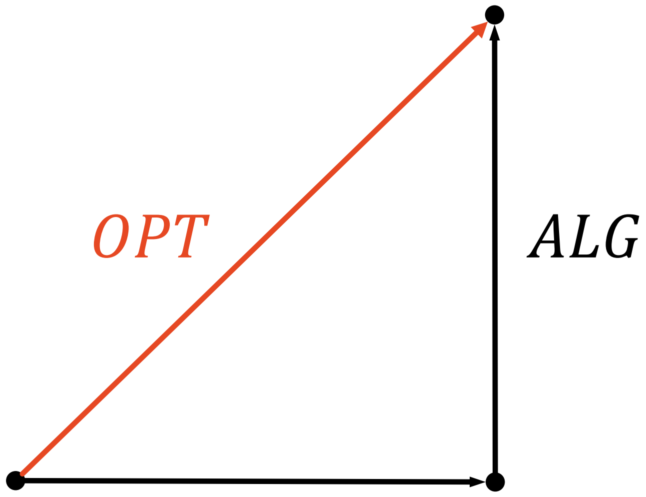
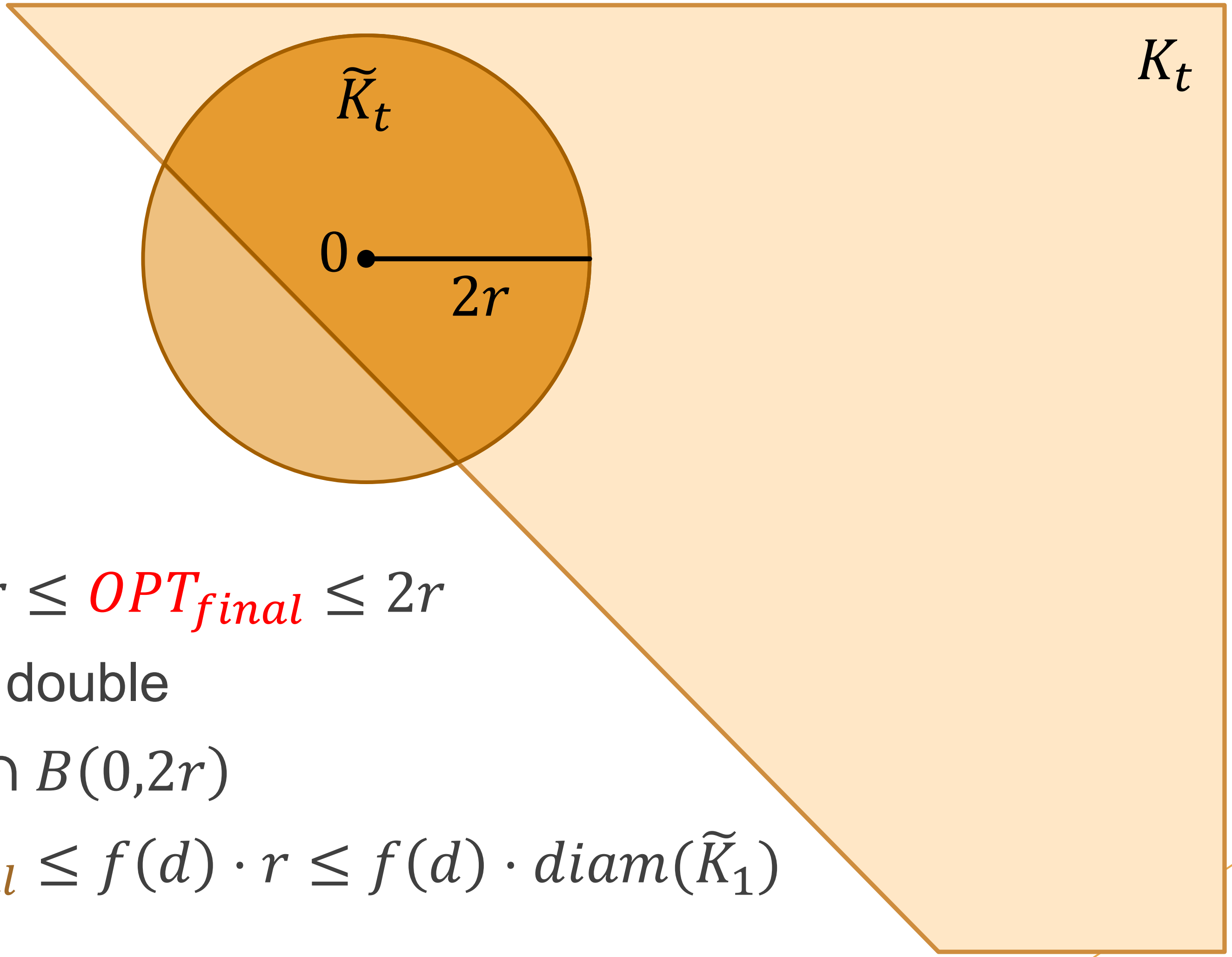


$$ALG \geq \sqrt{2} \cdot OPT$$

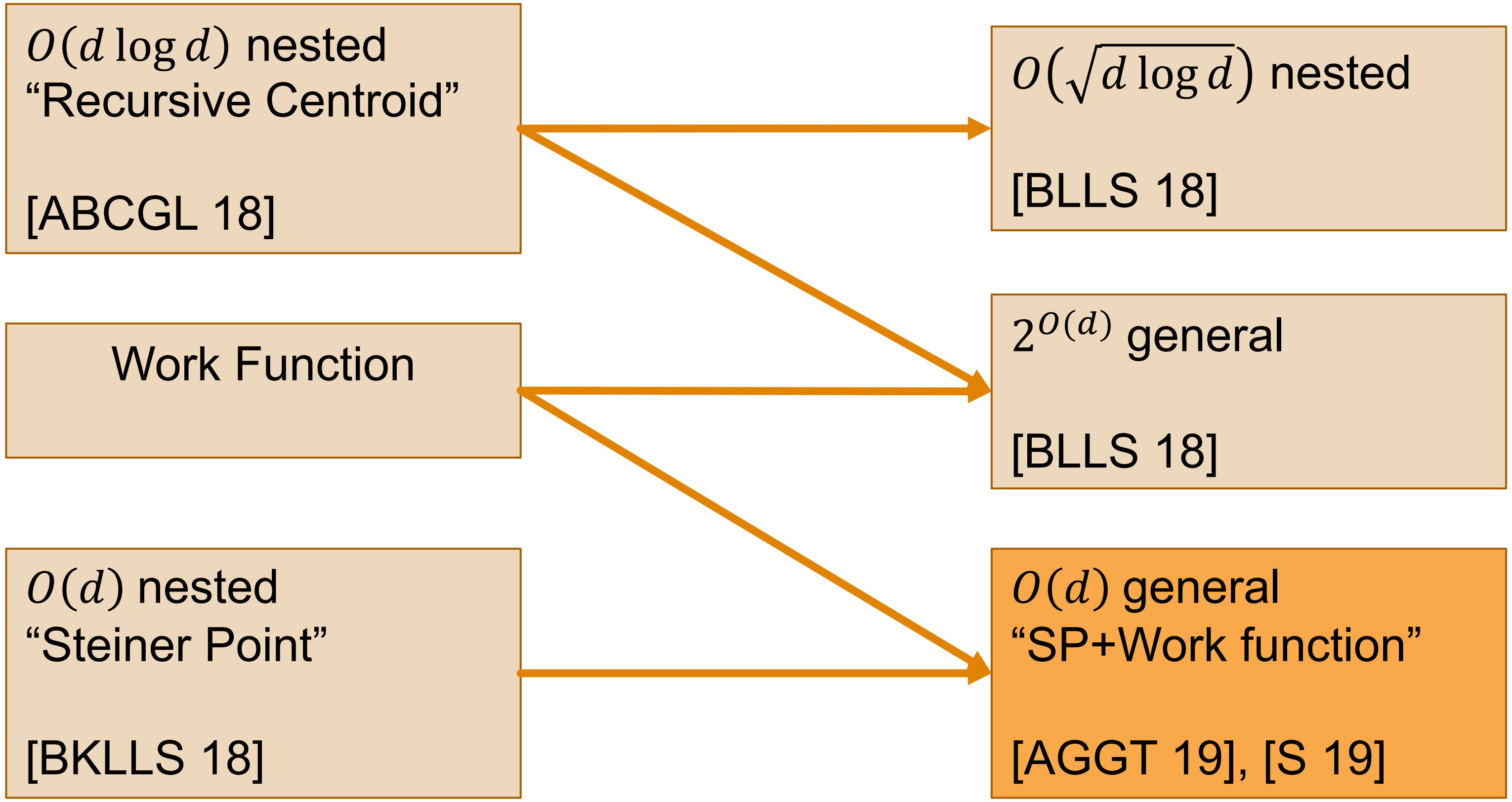
$$\boxed{ALG \geq \sqrt{d} \cdot OPT}$$

Reduction – Bounded Sets, Bound Cost

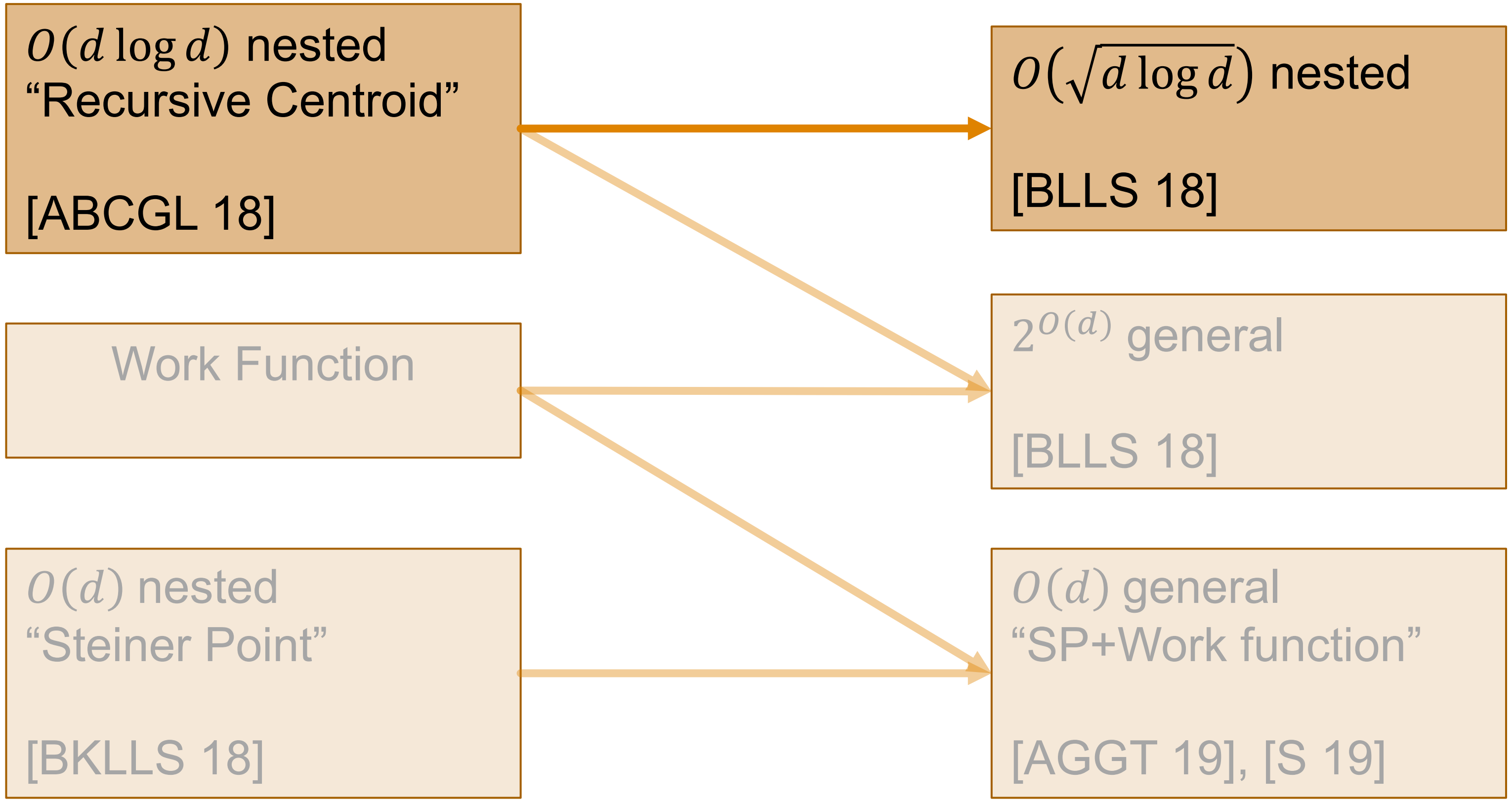


- Given r
- Guaranteed $r \leq OPT_{final} \leq 2r$
  - Guess and double
- Use $\widetilde{K}_t = K_t \cap B(0,2r)$
- Want $ALG_{final} \leq f(d) \cdot r \leq f(d) \cdot diam(\widetilde{K}_1)$

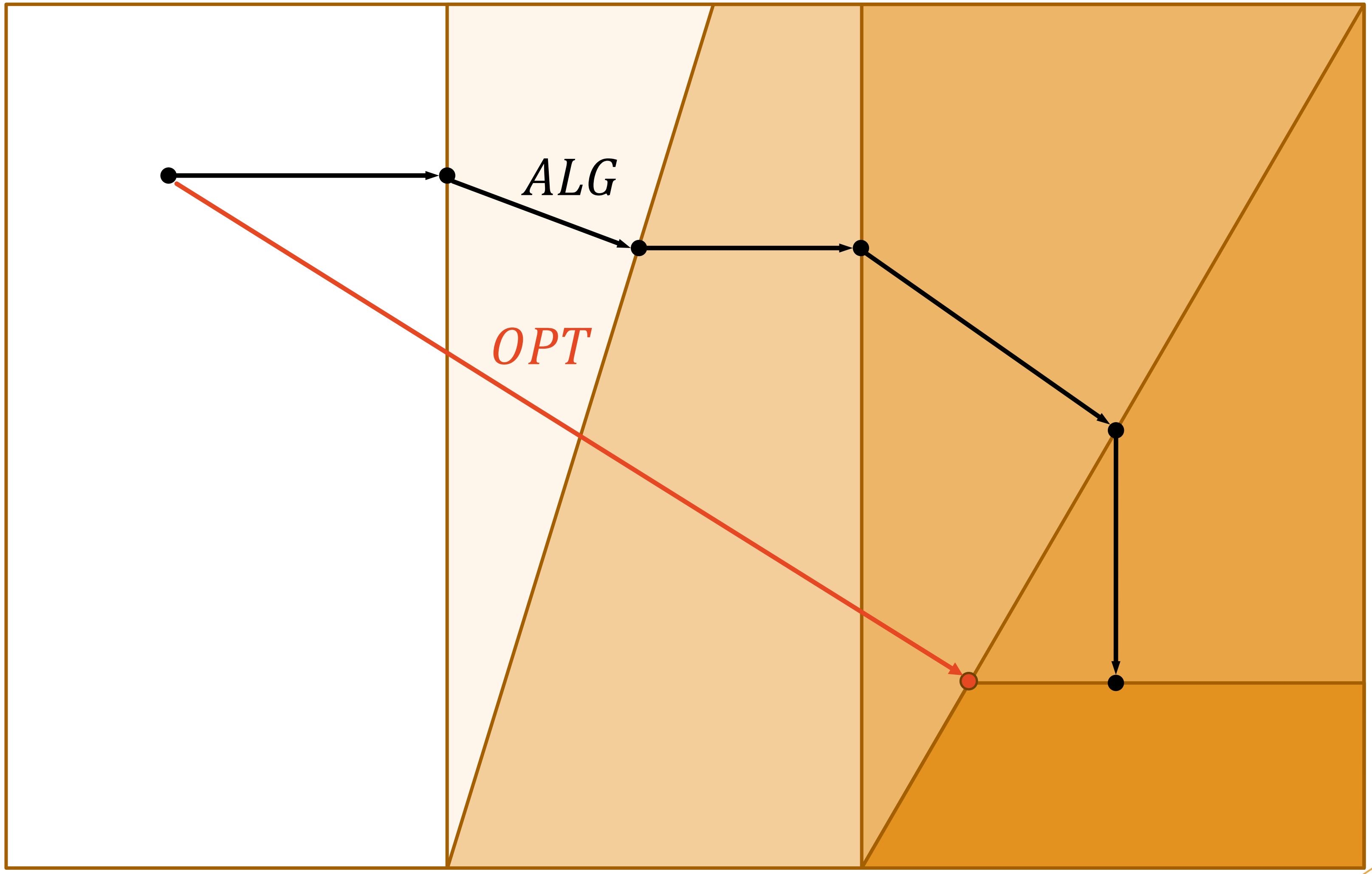
Progress


$O(d \log d)$ nested
"Recursive Centroid"
[ABCGL 18]
Work Function
$O(d)$ nested
"Steiner Point"
[BKLLS 18]
$O(\sqrt{d \log d})$ nested
[BLLS 18]
$2^{O(d)}$ general
[BLLS 18]
$O(d)$ general
"SP+Work function"
[AGGT 19], [S 19]

Part 1 – Centroid

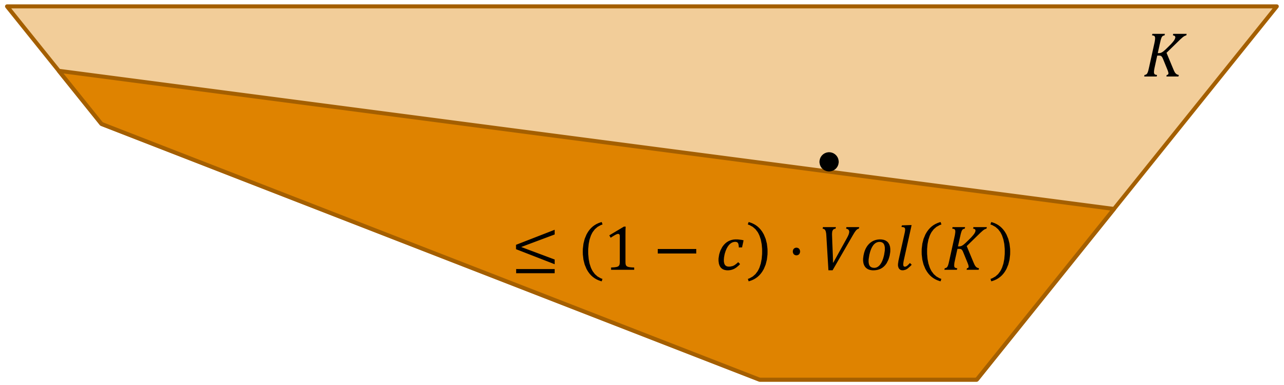


This Section – Nested

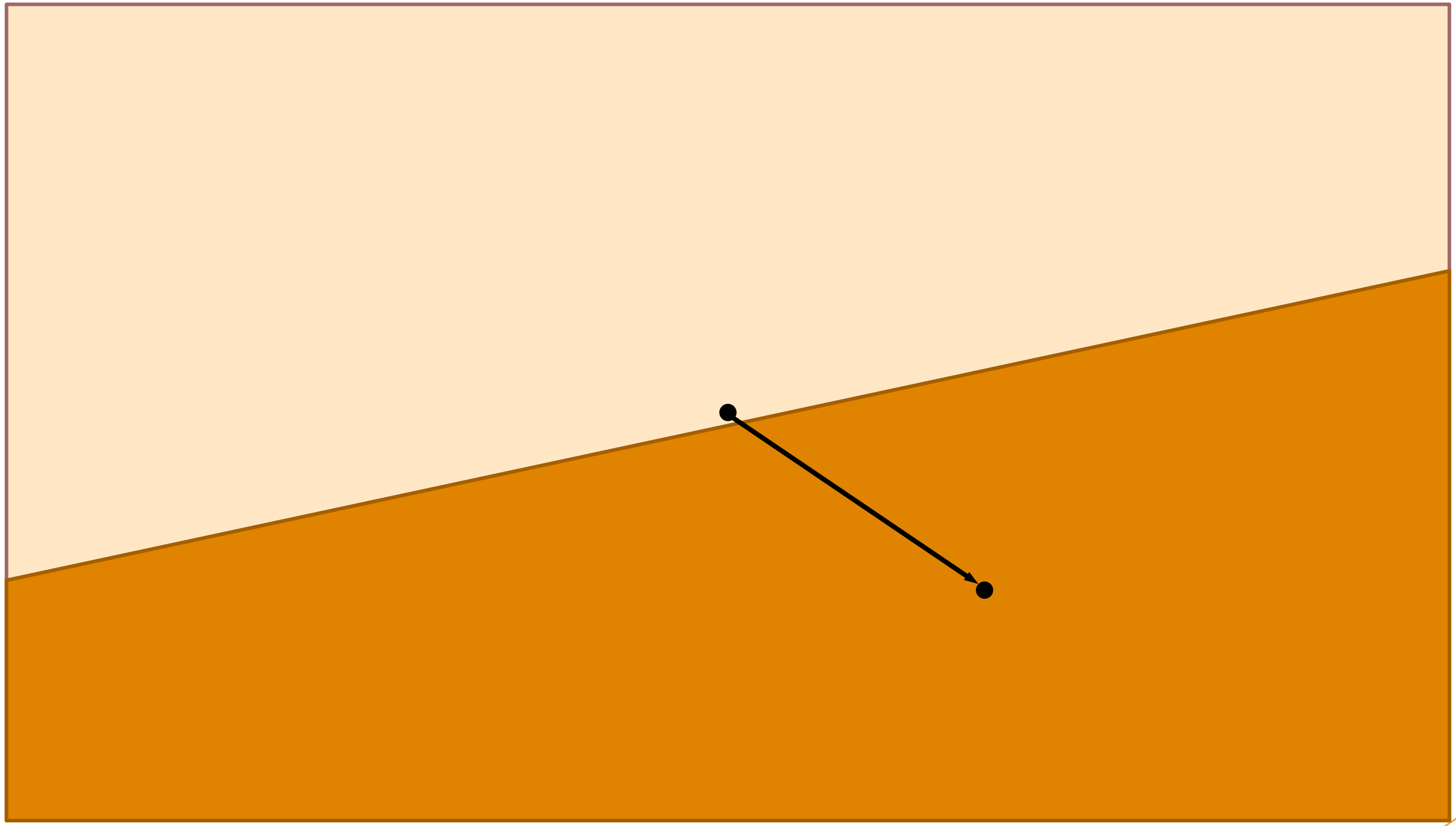


Idea – *Centroid / Center of Gravity*

- Algorithm: $x_t = cg(K_t)$
  - K_t bounded
- Grunbaum's Theorem
  - Cut off centroid ⇒ volume decreases by constant



Problem with *Centroid*



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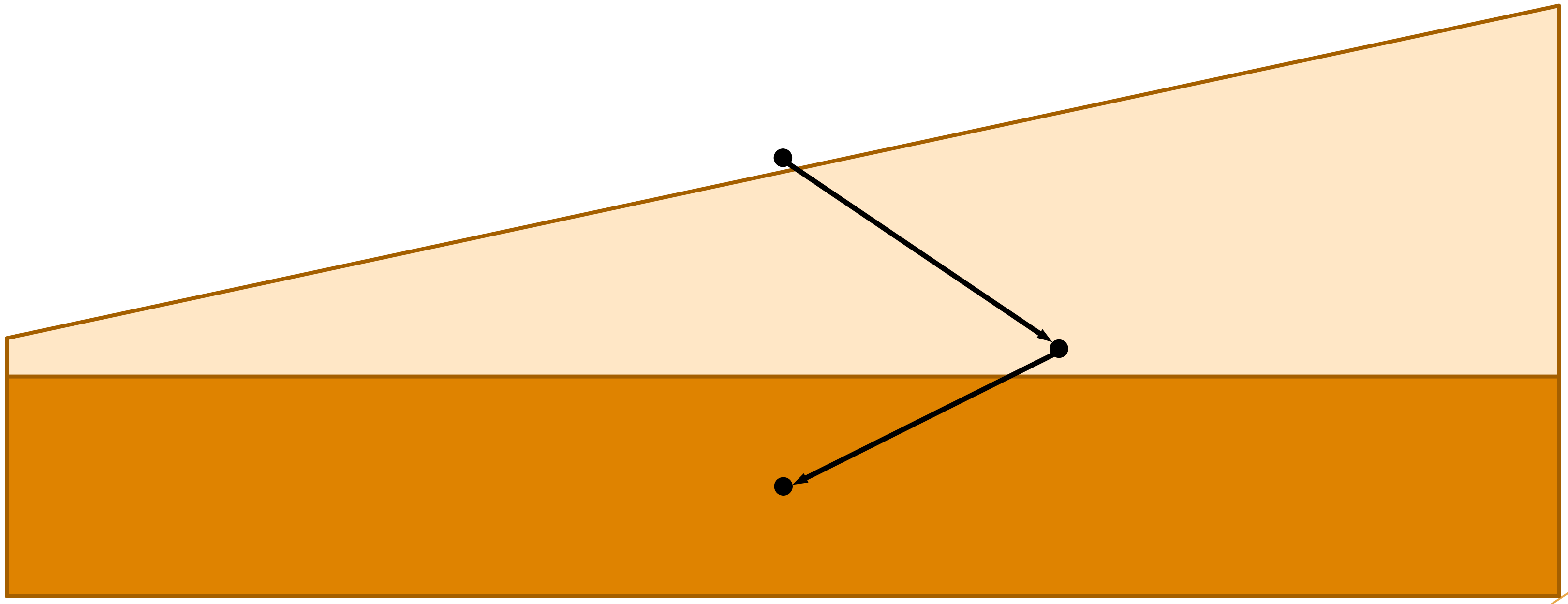
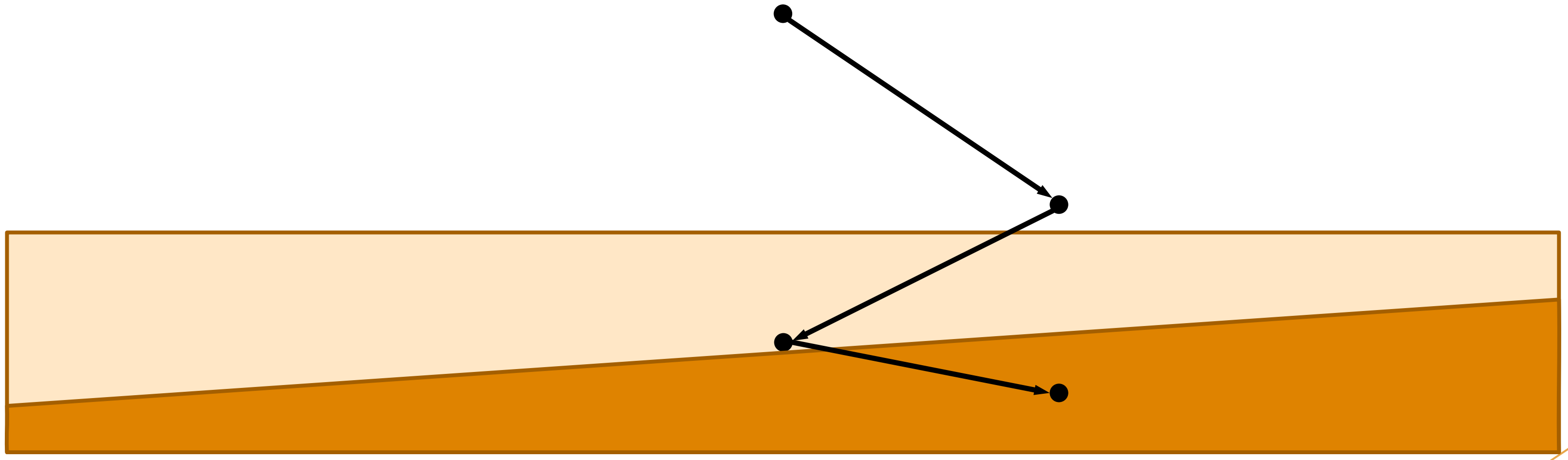
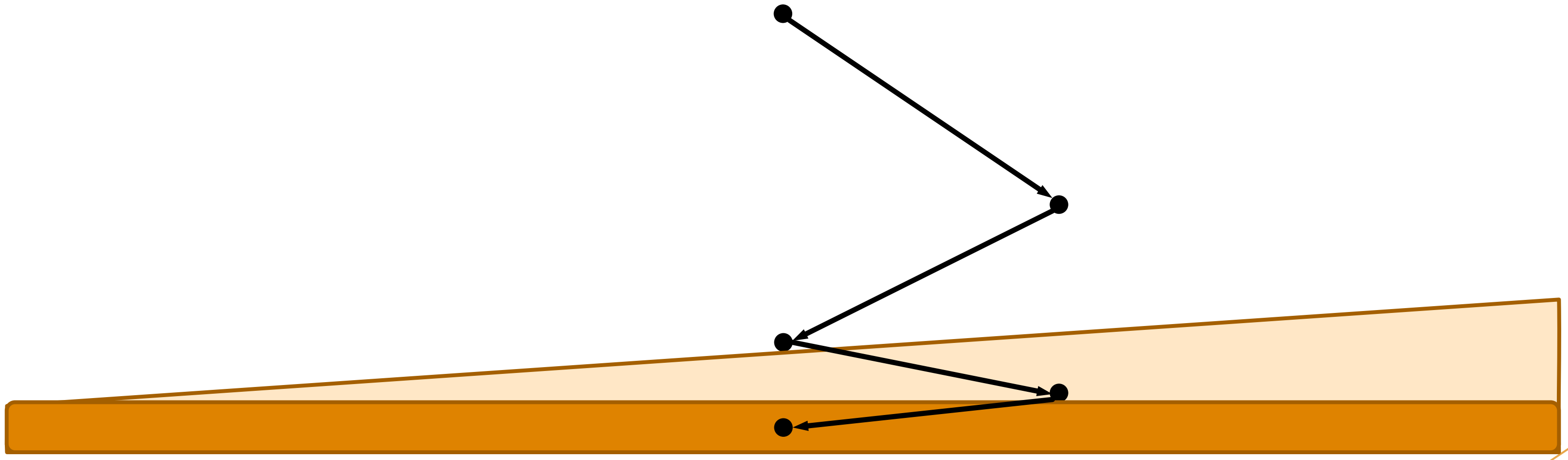


Problem with *Centroid*



Problem with *Centroid*

- ALG unbounded
- $OPT = O(1)$
- Not competitive
- Diameter constant



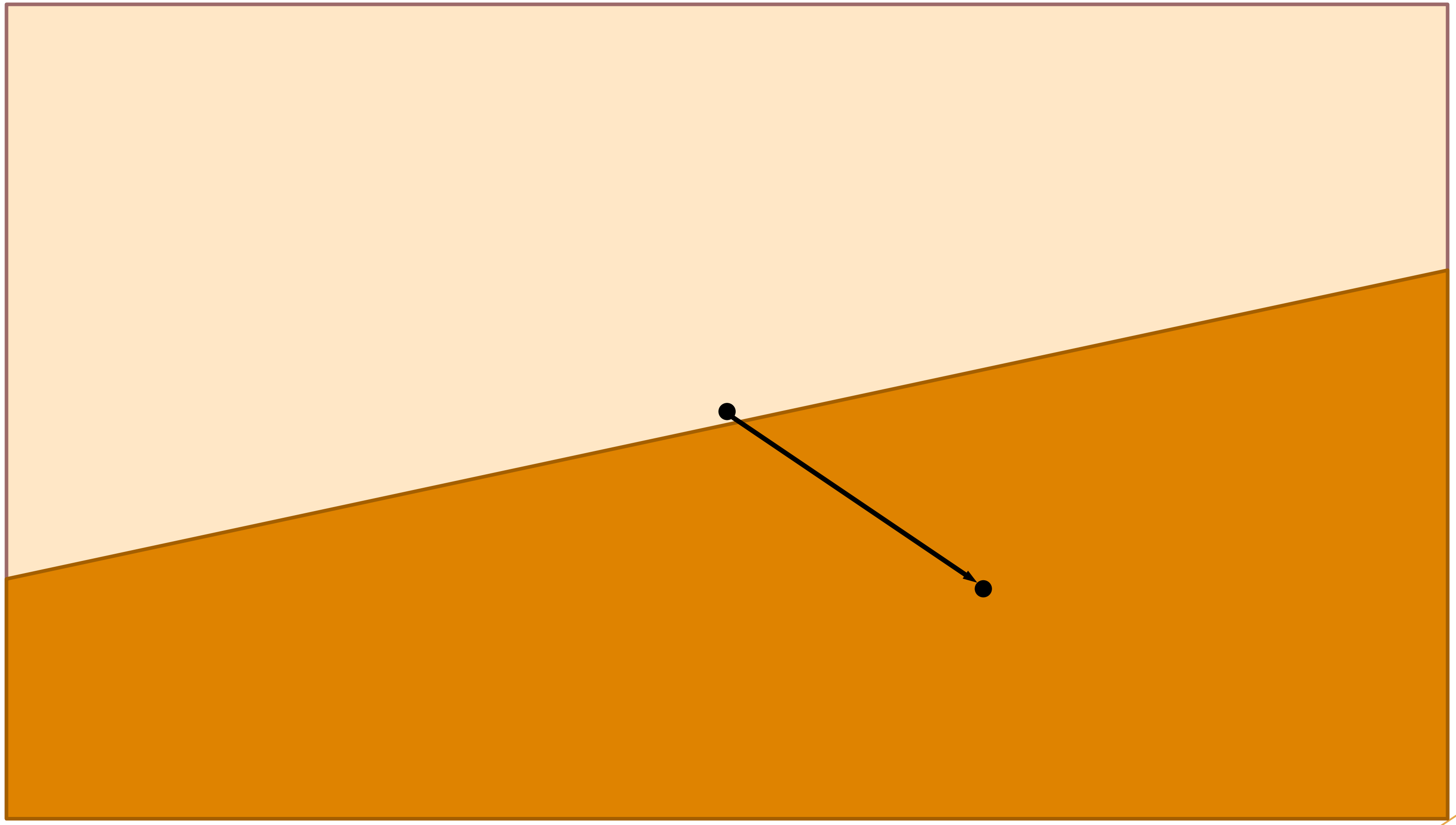
Recursive Centroid (Nested)

[Argue, Bubeck, Cohen, Gupta, Lee 18]

- Recursion on “skinny subspace”
  - Small steps
  - Hyperplane separation $\Rightarrow$ cut parallel to skinny subspace
    - Shrink diameter

Recursive Centroid (Nested)

[Argue, Bubeck, Cohen, Gupta, Lee 18]



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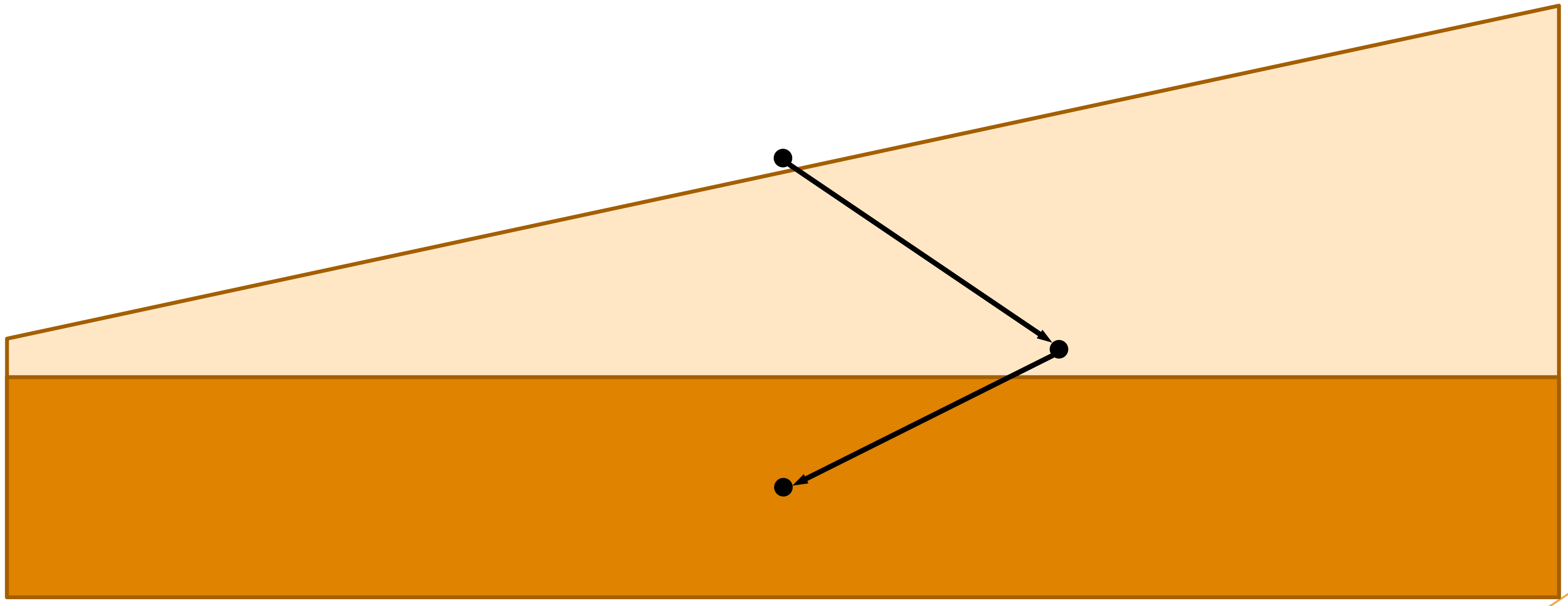

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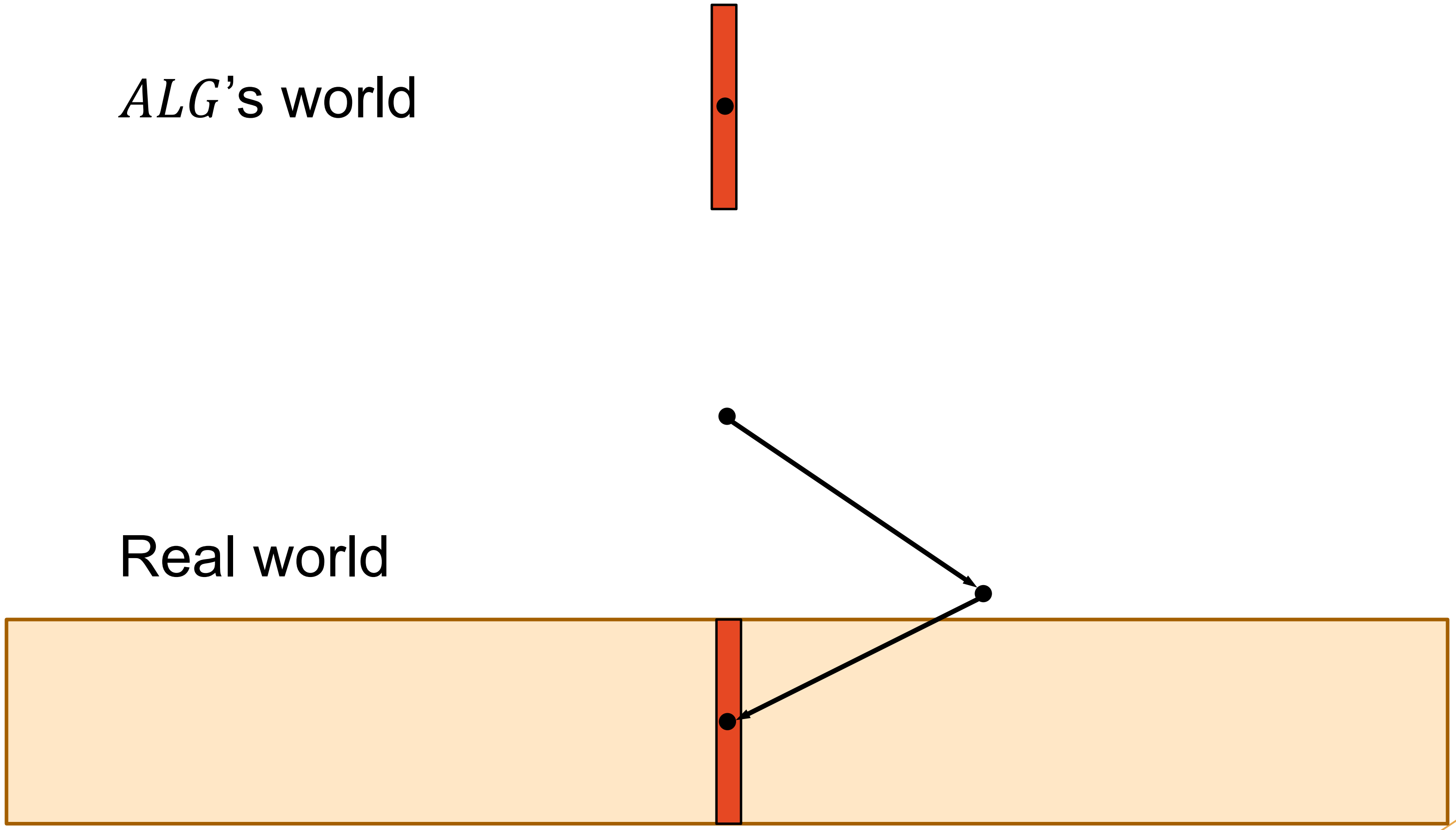

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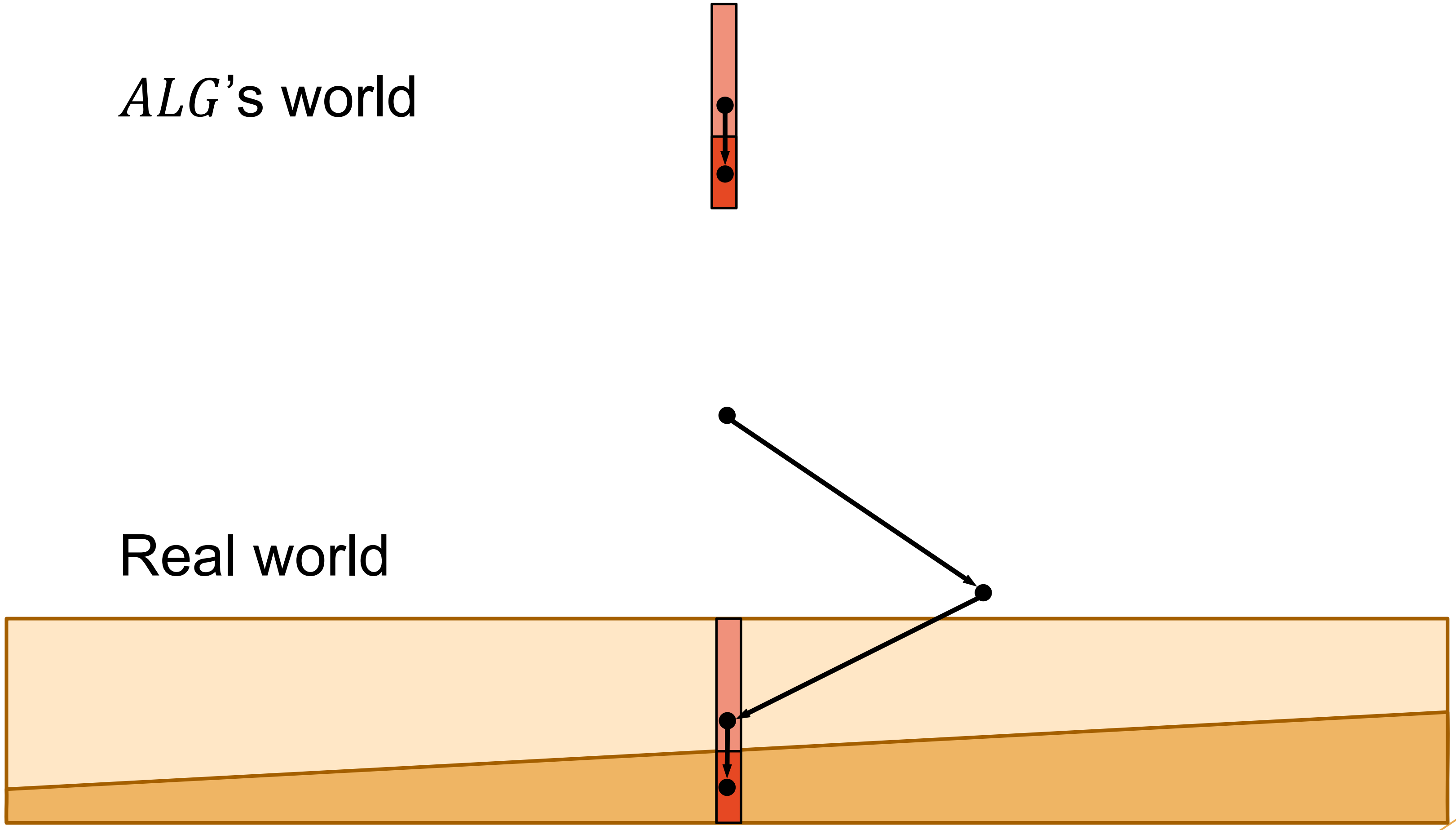
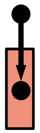


Recursive Centroid (Nested)

[Argue, Bubeck, Cohen, Gupta, Lee 18]

- $O(d \log d)$-competitive
- ALG's world

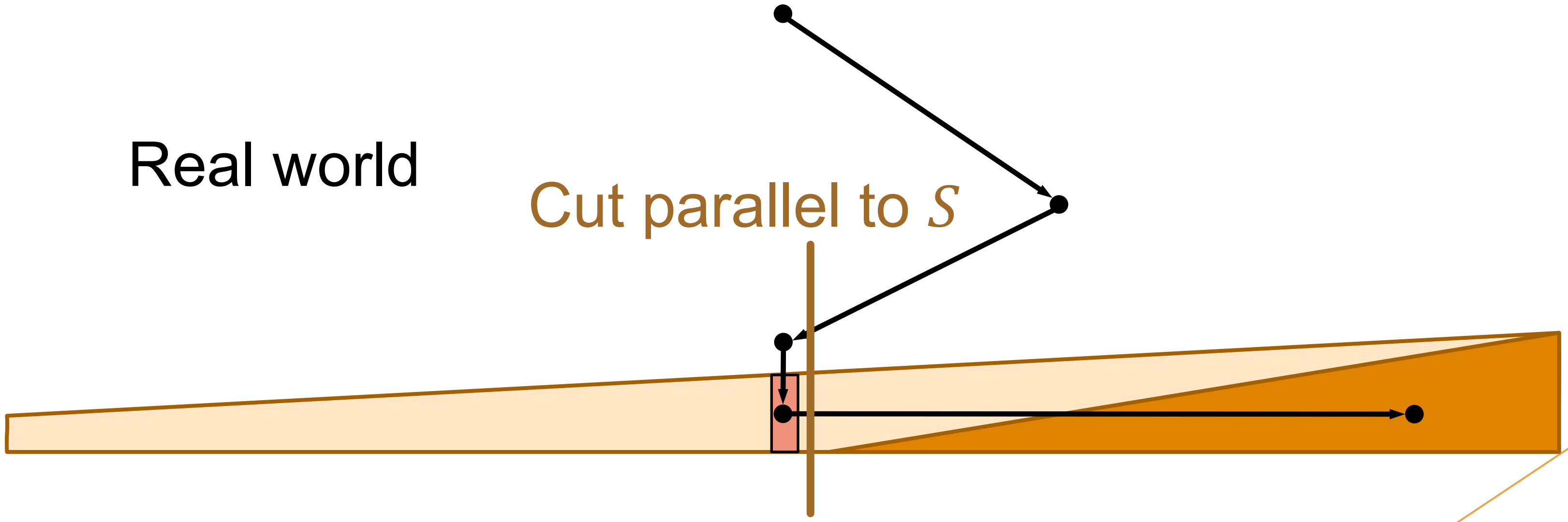


- Improved to $O\left(\sqrt{d \log d}\right)$

  [Bubeck, Lee, Li, Sellke 18]

Real world

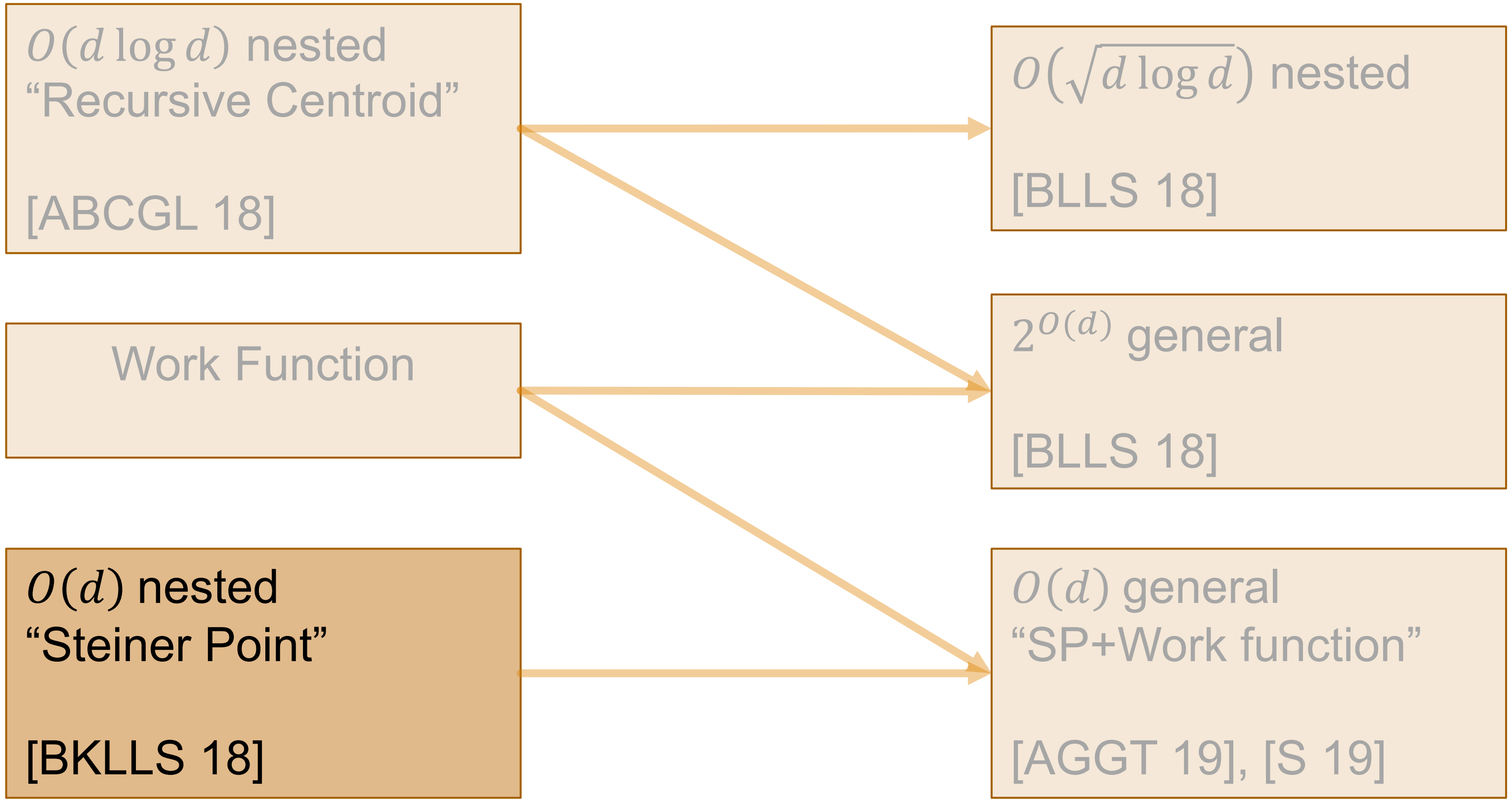
Cut parallel to S



Recap of Centroid

- Reduction to bounded case
- Recursive centroid
  - Move in ‘skinny’ directions
- Optimal for nested (up to log factor)

Part 2 – Steiner Point

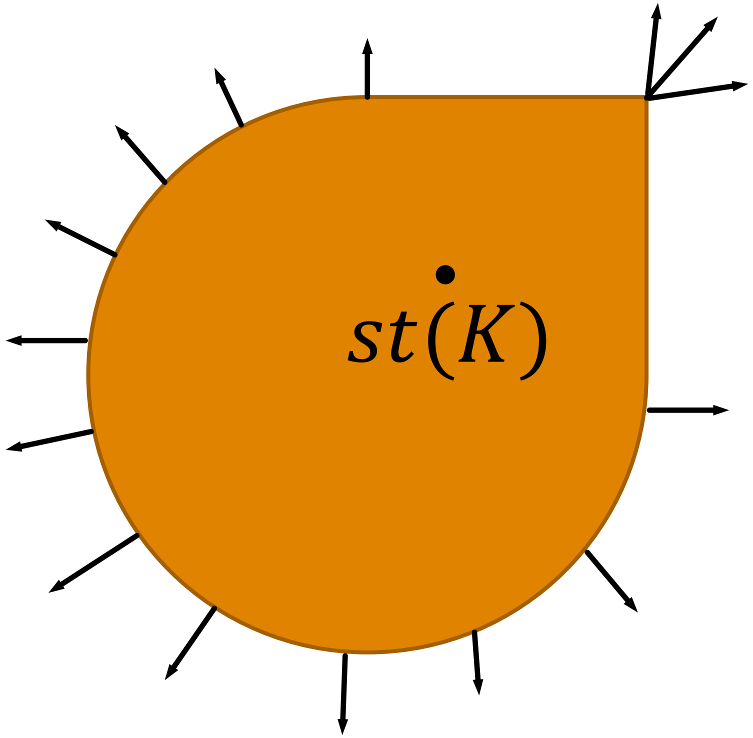


Steiner Point

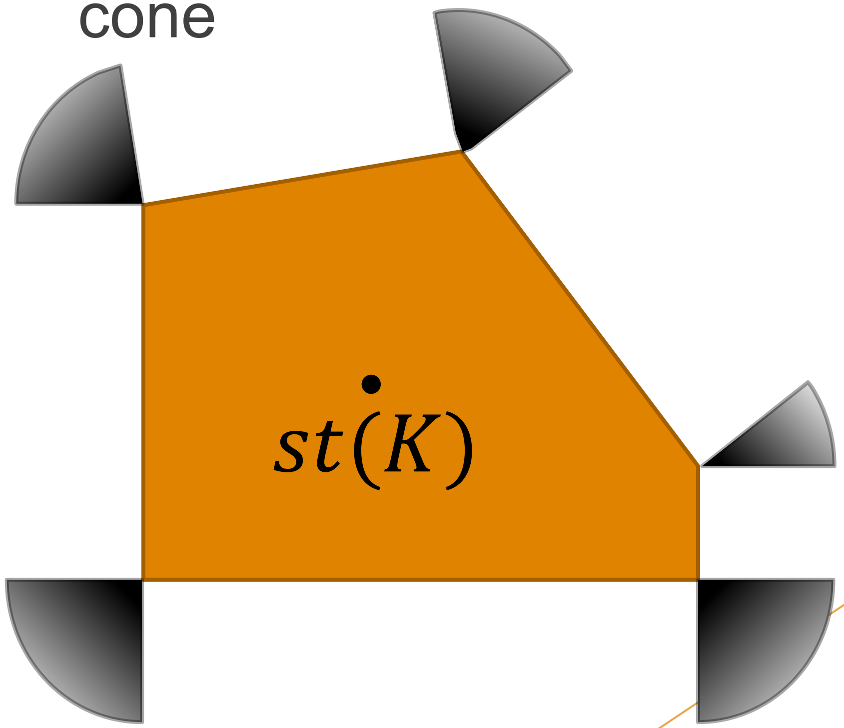
- Alternate “center” of convex body
- Introduced by Steiner in 1826
- Long history in convex geometry
- Lipschitz w.r.t. Hausdorff Distance
  - Natural metric on bounded sets

Steiner Point

- Average of extreme points in all directions



- Average of extreme points weighted by size of normal cone



Steiner Point Definitions

Visually intuitive

$$st(K) = \int_{\|\theta\|=1} \nabla s_K(\theta)\, d\theta$$

$$\nabla s_K(\theta) \coloneqq \operatorname*{argmax}_{x \in K} \langle \theta, x \rangle$$

Useful for one proof

$$= d \cdot \int_{\|\theta\|=1} s_K(\theta) \cdot \theta\, d\theta$$

$$s_K(\theta) \coloneqq \max_{x \in K} \langle \theta, x \rangle$$

Useful for another proof

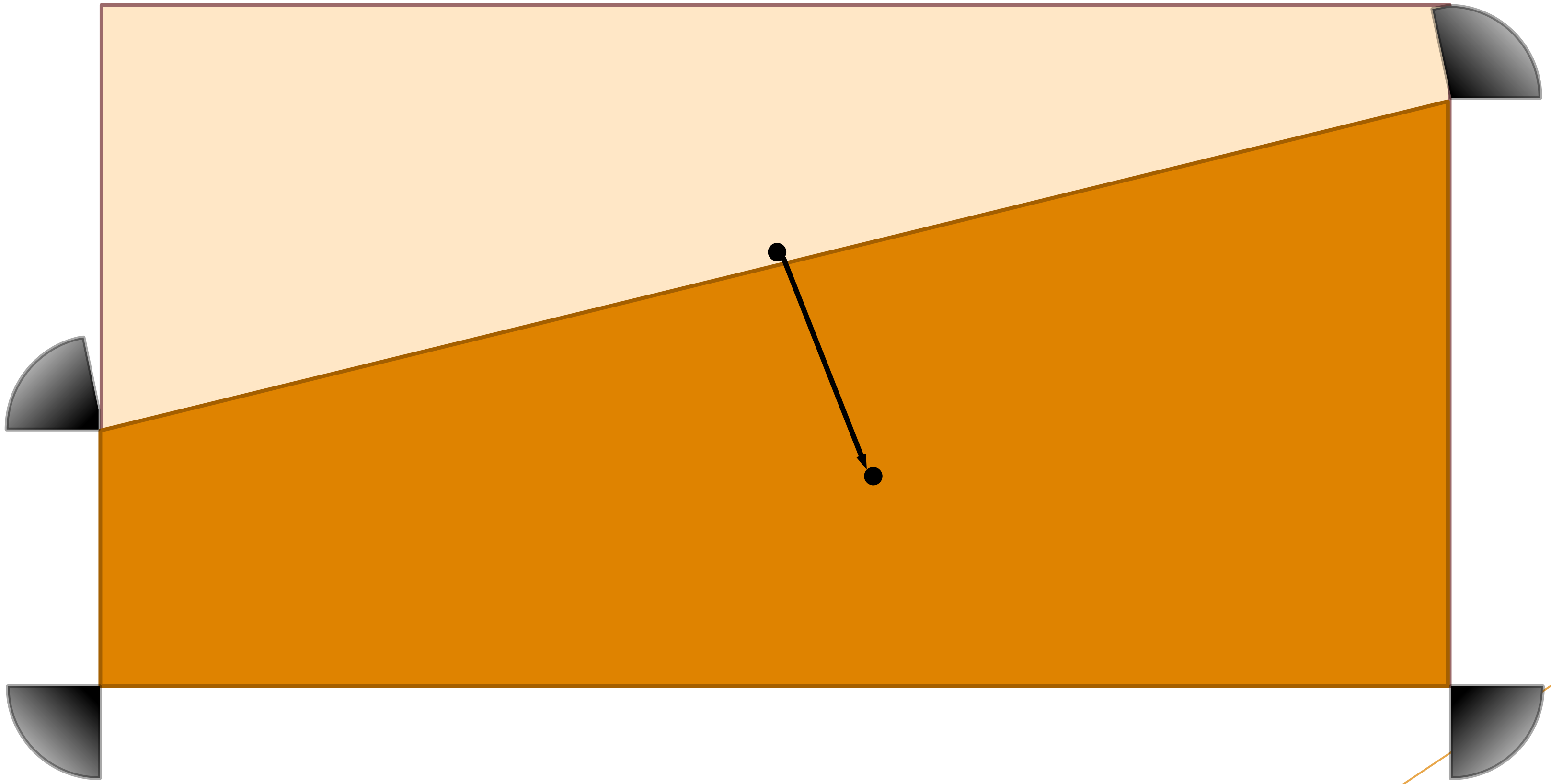
$$= \lim_{\gamma \to \infty} cg(K + \gamma B)$$

$$B = B(0,1)$$

Steiner Point (Nested)

[Bubeck, Klartag, Lee, Li, Sellke 18]

- $x_t = st(K_t)$



Steiner Point (Nested)

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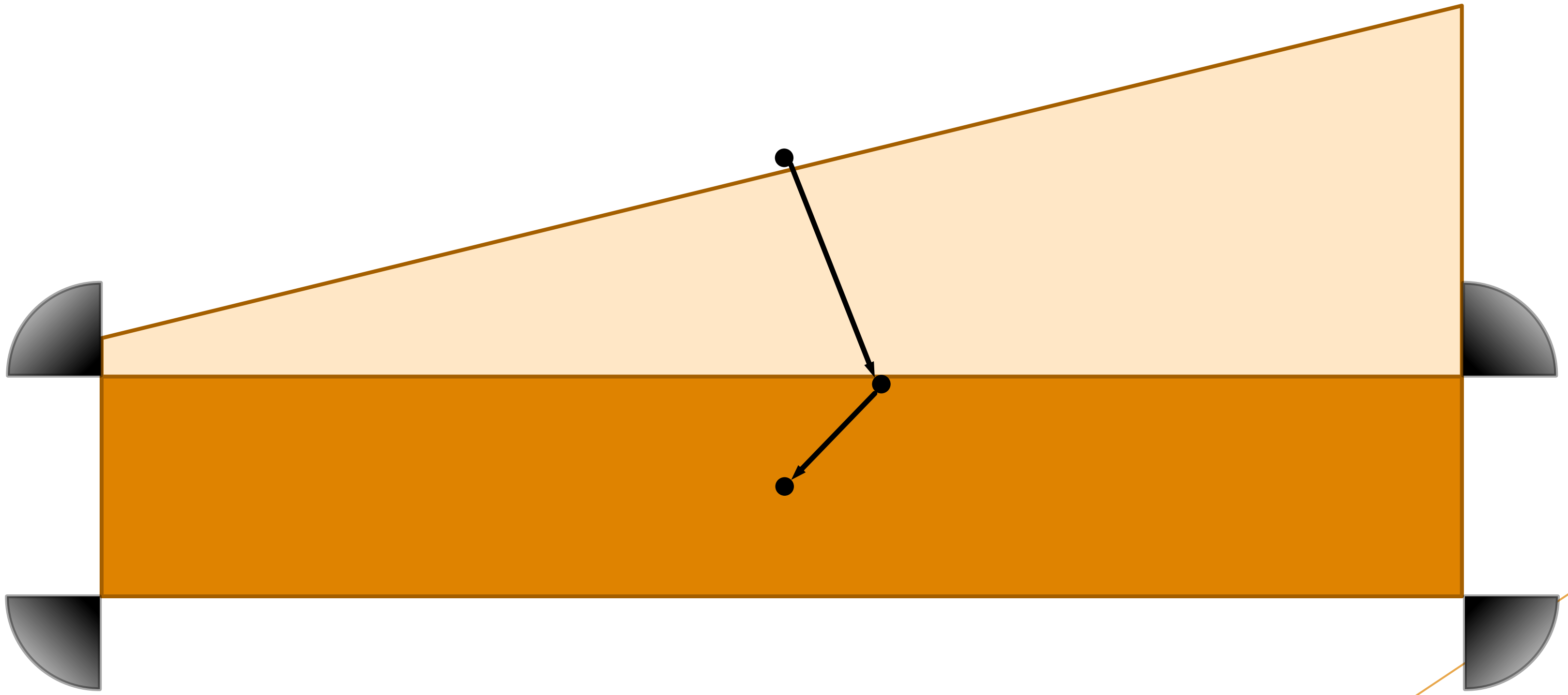

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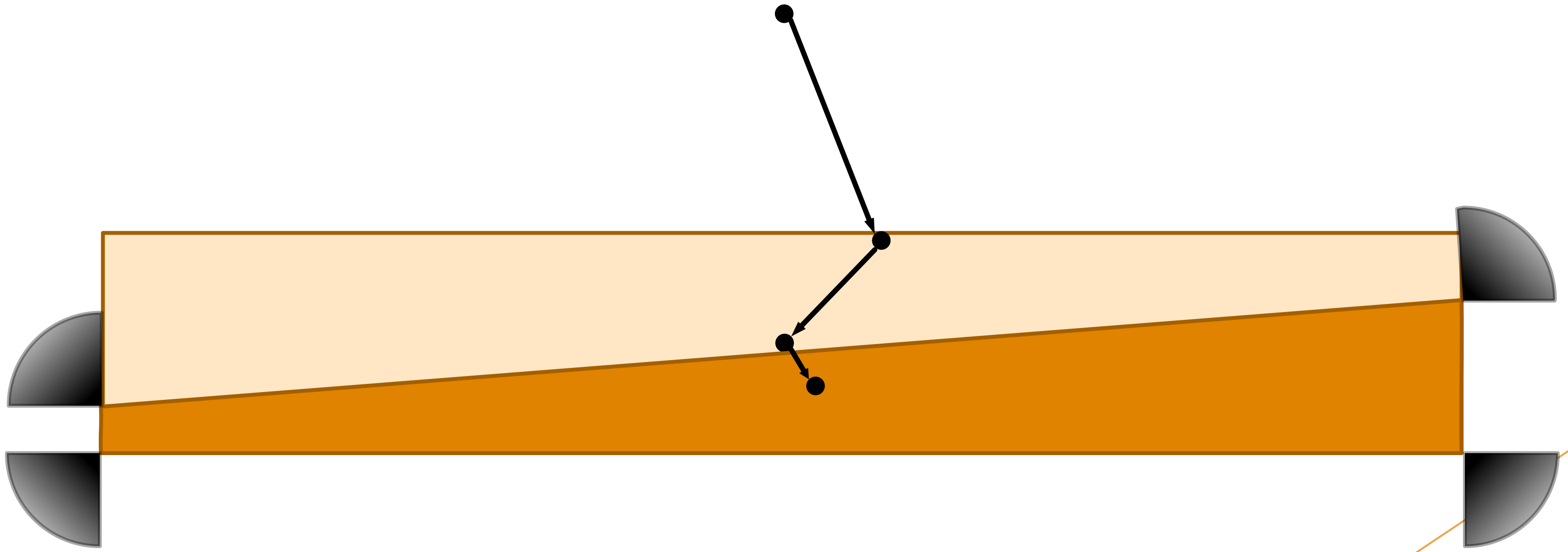
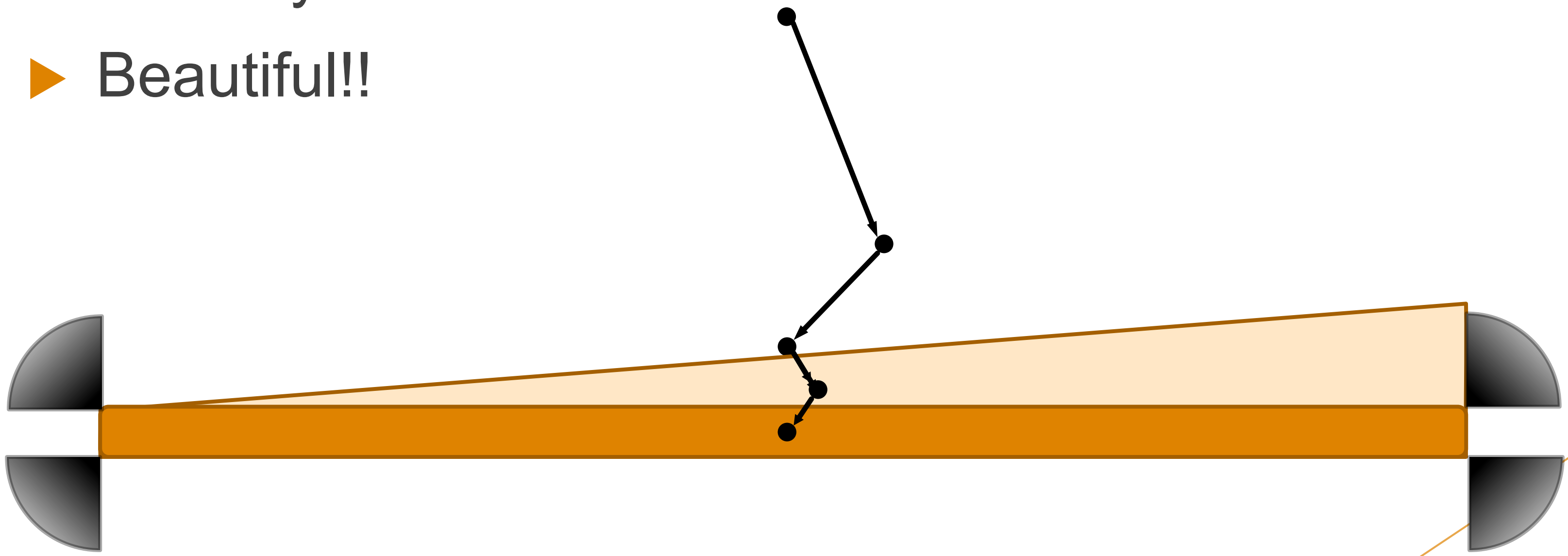


Steiner Point (Nested)

[Bubeck, Klartag, Lee, Li, Sellke 18]

- $x_t = st(K_t)$
- “Smoother version of recursive centroid”
- $O(d)$ competitive
- Memoryless
- Beautiful!!



Analysis

$$ALG = \sum_{i=1}^{T-1} \|st(K_i) - st(K_{i+1})\|$$

$$= \sum_{i=1}^{T-1} \left\| d \cdot \int_{\|\theta\|=1} \left(s_{K_i}(\theta) - s_{K_{i+1}}(\theta)\right) \theta \, d\theta \right\|$$

(Jensen) $$\leq d \cdot \int_{\|\theta\|=1} \left(\sum_{i=1}^{T-1} \left| s_{K_i}(\theta) - s_{K_{i+1}}(\theta) \right| \right) \cancel{\|\theta\|}^{1} \, d\theta$$

(Nested) $$= d \cdot \int_{\|\theta\|=1} \left(\sum_{i=1}^{T-1} s_{K_i}(\theta) - s_{K_{i+1}}(\theta) \right) d\theta$$

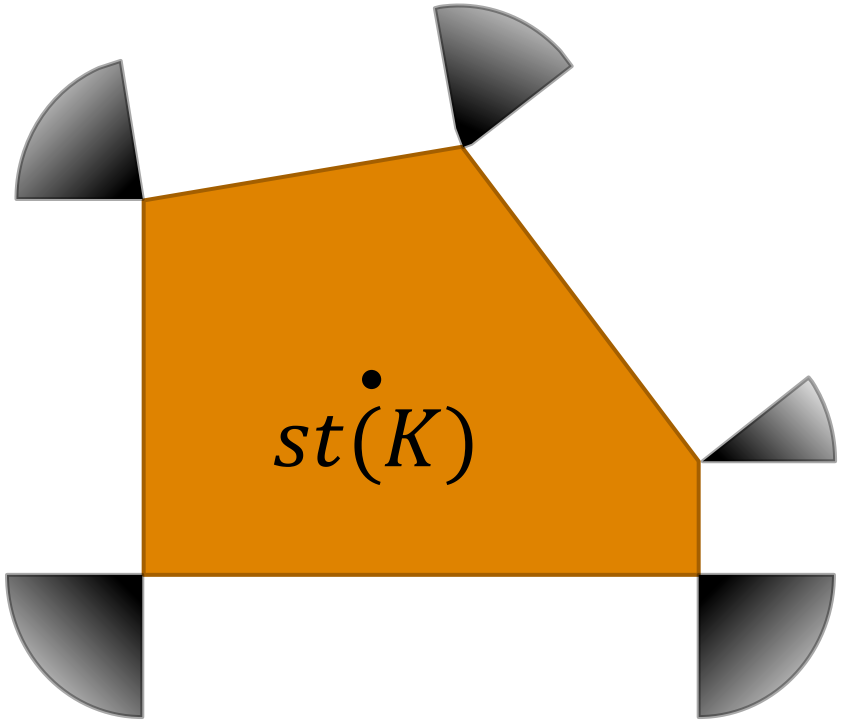
$$= d \cdot \int_{\|\theta\|=1} \left(s_{K_1}(\theta) - s_{K_T}(\theta) \right) d\theta$$

(Bounded) $$\leq d \cdot diam(K_1) \leq O(d) \cdot OPT$$

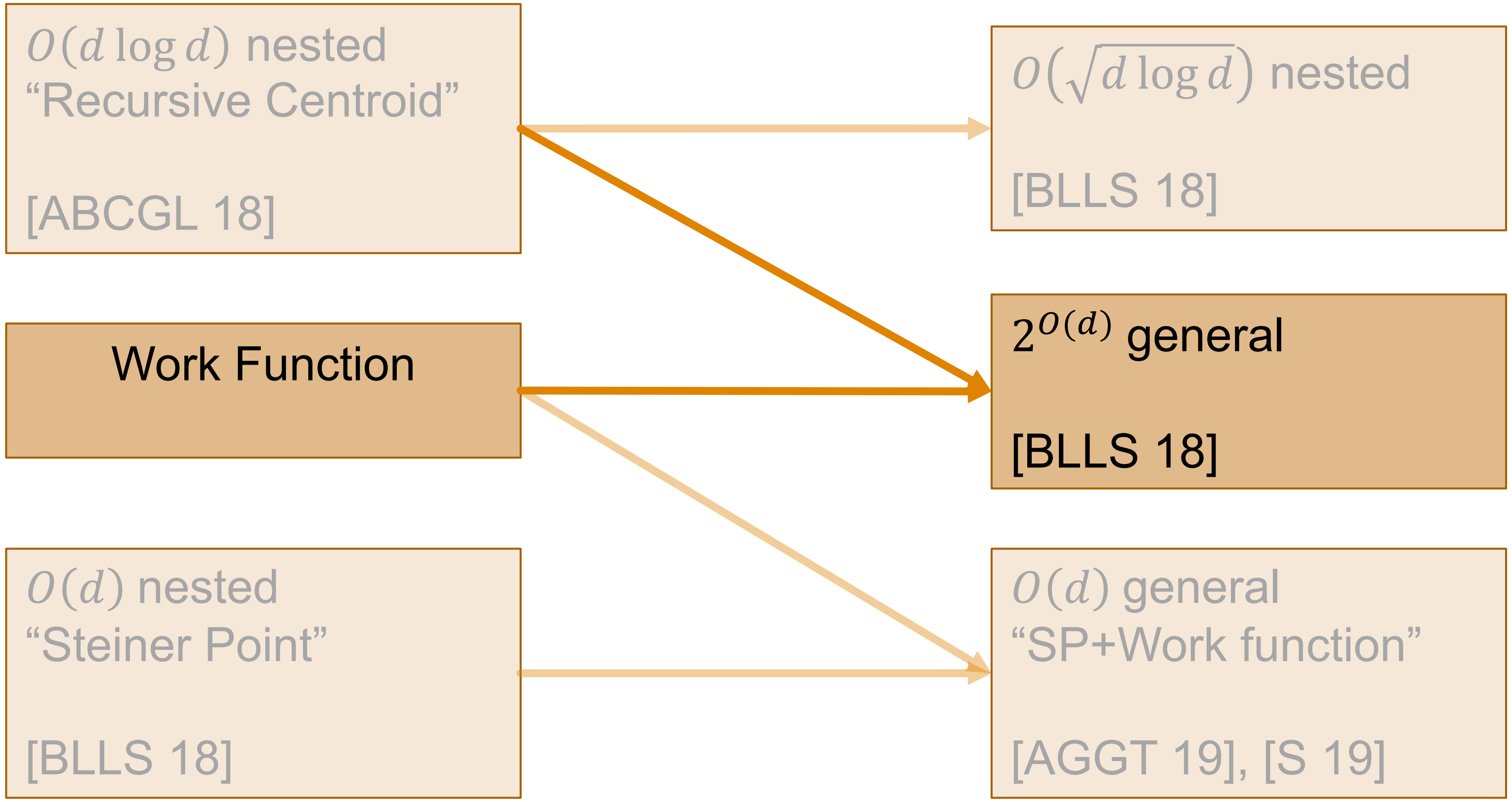
☺

Recap of Steiner Point

- Average of extreme points weighted by normal cone size
- $st(K) = \lim_{\gamma \to \infty} cg(K + \gamma B)$
- "Smoother recursive centroid"
- Elegant and magical!



Part 3 – Work Function



Reduction Framework

- Given:
  - General instance $K_1, \dots, K_T$
  - $f(d)$ competitive nested $NEST$
- Goal: Construct $\Omega_1, \dots, \Omega_T$ so that
  - Ω_t convex and $\Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_T$
  - $NEST(\Omega_1, \dots, \Omega_t) \leq O\big(f(d)\big) \cdot OPT(K_1, \dots, K_T)$
  - $NEST$ outputs points $x_i \in K_i$

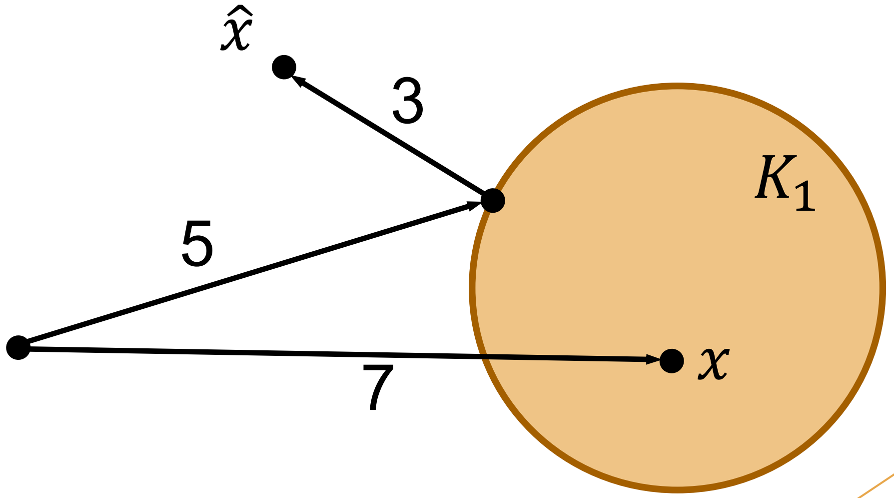
Work Function

- Central to related problems
- $w_t(x) :=$ min cost to satisfy requests $1, \ldots, t$ and end at x

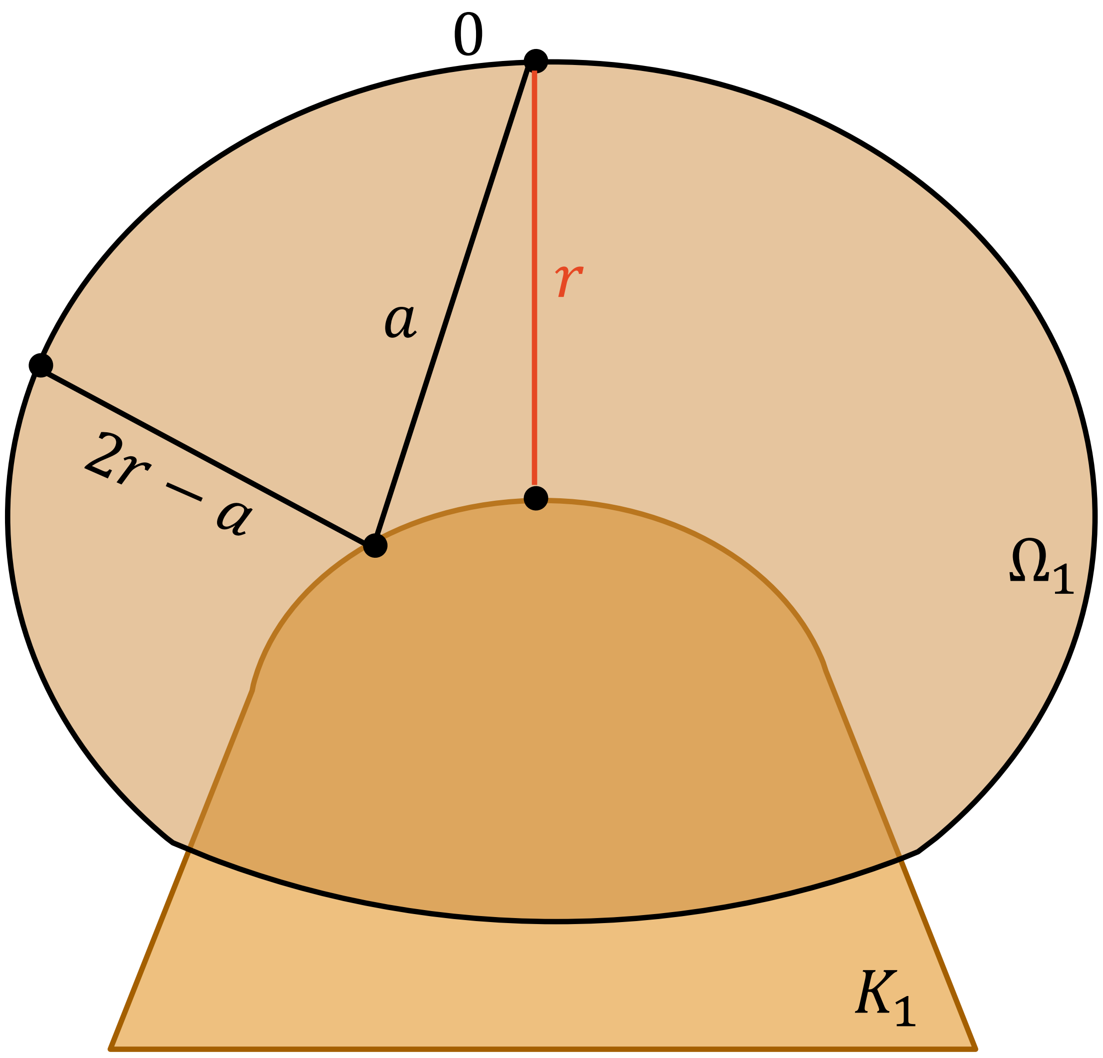
$$= \min_{y_i \in K_i} \sum_{i=1}^{t} \|y_i - y_{i-1}\| + \|y_t - x\|$$

$w_1(x) = 7$

$w_1(\hat{x}) = 5 + 3$



Work Function Sublevel Set



$$\Omega_1 = \{x \mid w_1(x) \leq 2r\}$$

Reduction Framework

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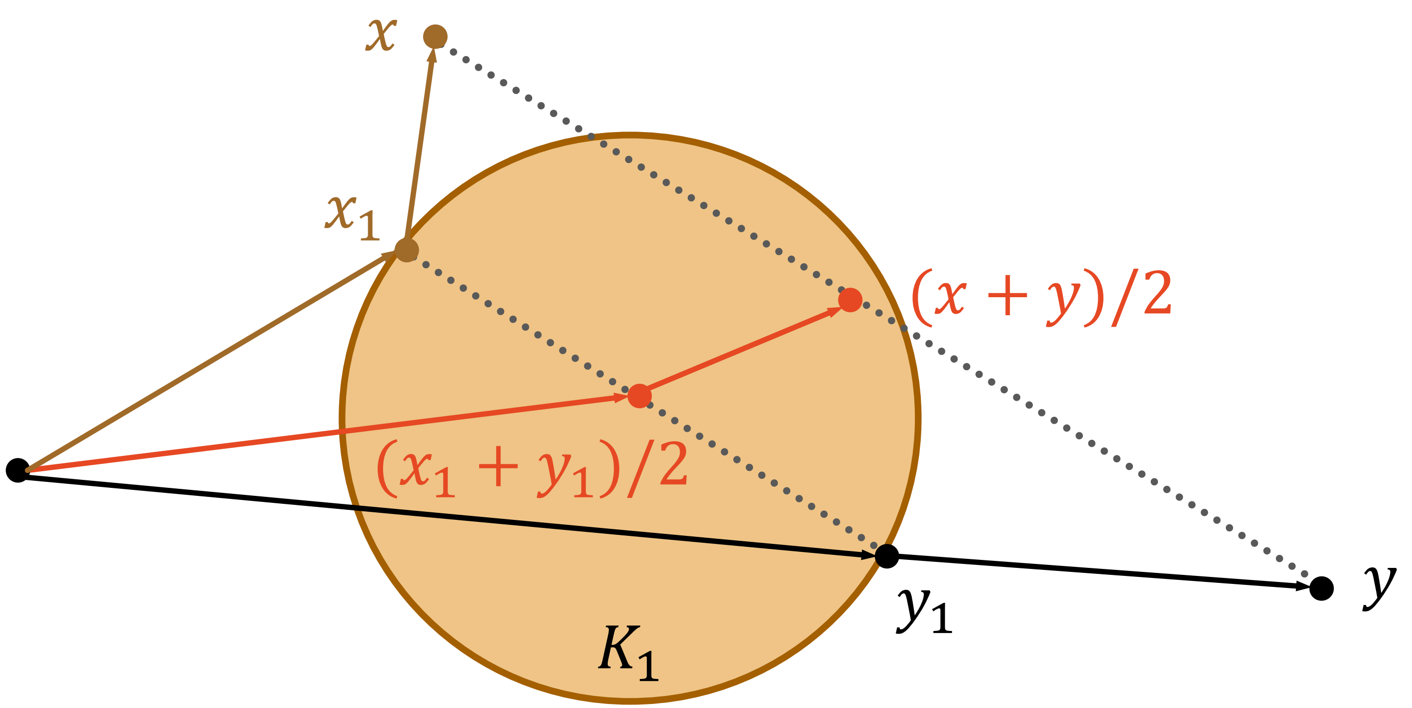
  - $NEST$ outputs points $x_i \in K_i$

Candidate
$\Omega_t = \{x \mid w_t(x) \leq 2r\}$

Convexity

w_t is convex $\quad\Longrightarrow\quad$ $\Omega_t = \{x \mid w_t(x) \leq 2r\}$ is convex



Nested

$$w_t(x) \leq w_{t+1}(x) \quad \Longrightarrow \quad \{x \mid w_t(x) \leq 2r\} \supseteq \{x \mid w_{t+1}(x) \leq 2r\}$$

$$\Omega_t \supseteq \Omega_{t+1}$$

Cost to satisfy requests $1, \ldots, t+1$ and end at x

Cost to satisfy requests $1, \ldots, t$ and end at x

Reduction Framework

- Given:
  - General instance $K_1, \ldots, K_T$
  - $f(d)$ competitive nested $NEST$
- Goal: Construct $\Omega_1, \ldots, \Omega_T$ so that
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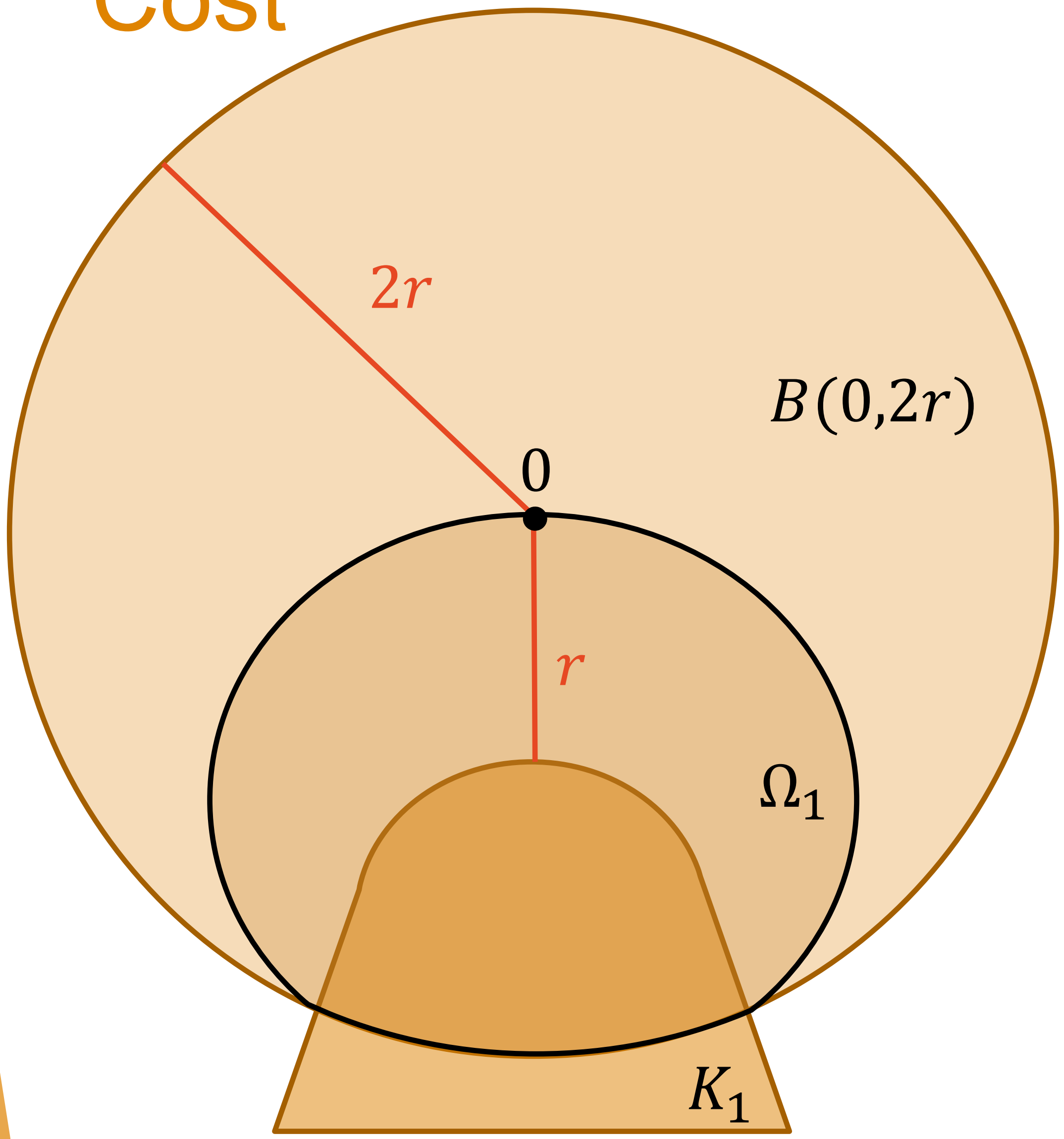
  - $NEST(\Omega_1, \ldots, \Omega_t) \leq O\big(f(d)\big) \cdot OPT(K_1, \ldots, K_T)$
  - $NEST$ outputs points $x_i \in K_i$

Candidate

$\Omega_t = \{x \mid w_t(x) \leq 2r\}$

✓

Cost



$$\Omega_1 = \{ x \mid w_1(x) \le 2r \} \subseteq B(0,2r)$$

$$ALG \le f(d) \cdot diam(\Omega_1)$$
$$\le O\big(f(d)\big) \cdot r$$

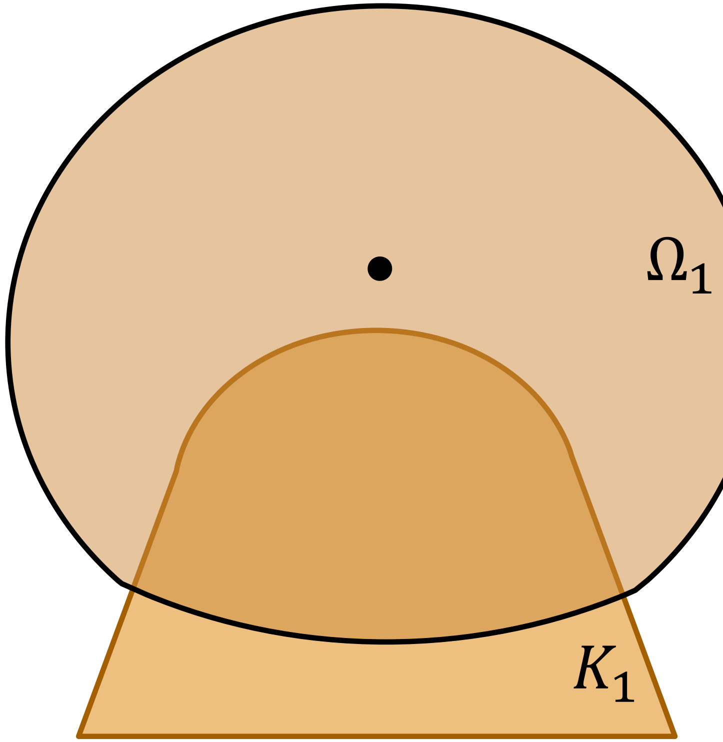
Reduction Framework

- Given:
  - General instance $K_1, \dots, K_T$
  - $f(d)$ competitive nested $NEST$
- Goal: Construct $\Omega_1, \dots, \Omega_T$ so that
  - Ω_t convex and $\Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_T$ ✓
  - $NEST(\Omega_1, \dots, \Omega_t) \leq O\big(f(d)\big) \cdot OPT(K_1, \dots, K_T)$ ✓
  - $NEST$ outputs points $x_i \in K_i$

Candidate
$\Omega_t = \{x \mid w_t(x) \leq 2r\}$

(In)feasibility

- $\Omega_t \nsubseteq K_t$
  - May play infeasible point
- Fix: project onto K_t each step?
  - Must control extra cost



Reduction Framework

- Given:
  - General instance $K_1, \ldots, K_T$
  - $f(d)$ competitive nested $NEST$
- Goal: Construct $\Omega_1, \ldots, \Omega_T$ so that
  - Ω_t convex and $\Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_T$ ✓
  - $NEST(\Omega_1, \ldots, \Omega_t) \leq O\big(f(d)\big) \cdot OPT(K_1, \ldots, K_T)$ ✓
  - $NEST$ outputs points $x_i \in K_i$?

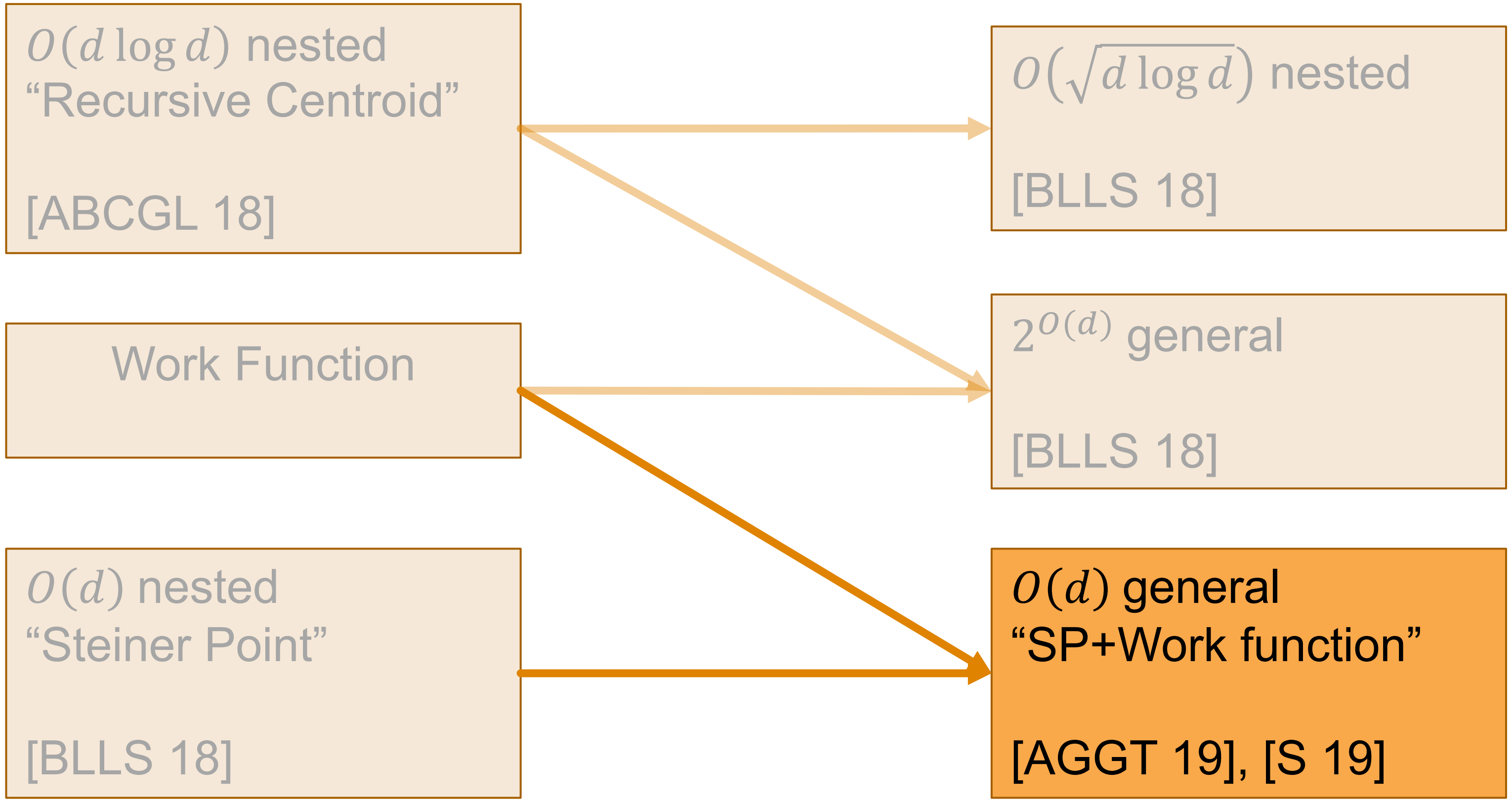
Candidate

$\Omega_t = \{x \mid w_t(x) \leq 2r\}$

Recap of Work Function

- Construct nested instance
  - Asymptotically same cost
- May play infeasible point

Part 4 – Main Theorem



Steiner Point + Work Function

[Argue, Gupta, Guruganesh, Tang 19]

- Given:
  - General instance $K_1, \dots, K_T$
  - $\boldsymbol{NEST}$ = **Steiner Point**
- Goal: Construct $\Omega_1, \dots, \Omega_T$ so that
  - Ω_t convex and $\Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_T$ ✓
  - $NEST(\Omega_1, \dots, \Omega_t) \leq O(d) \cdot OPT(K_1, \dots, K_T)$ ✓
  - $NEST$ outputs points $x_i \in K_i$ ~~?~~ ✓

Candidate

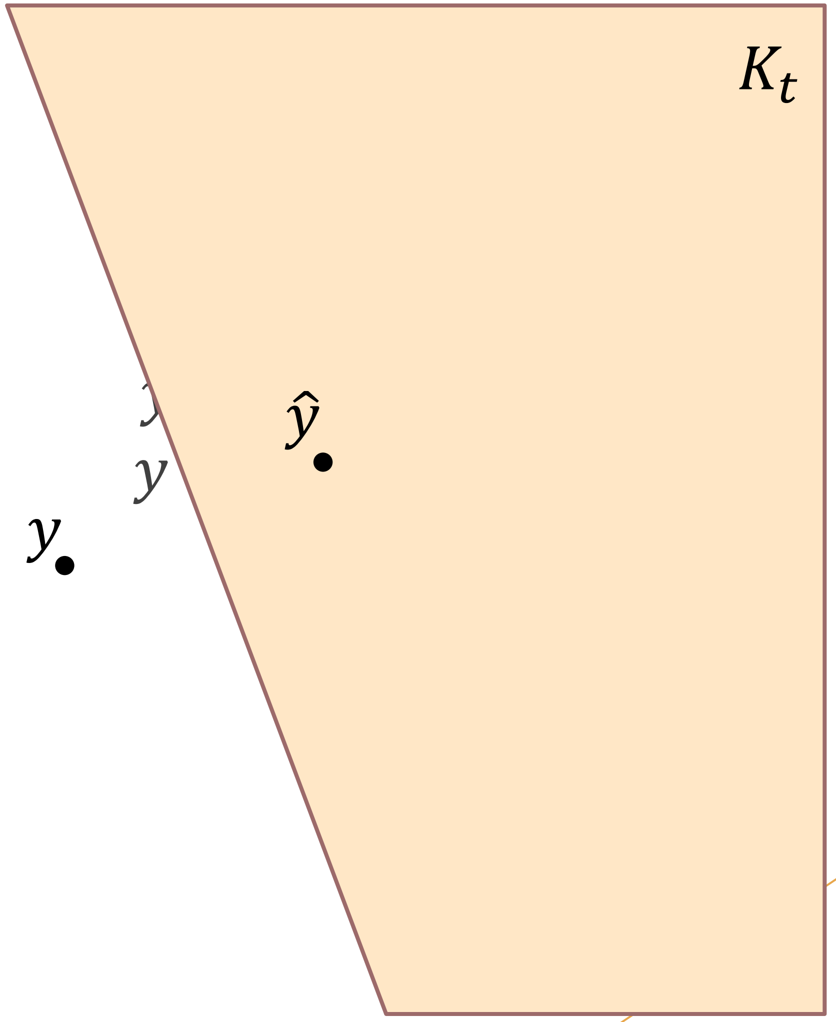
$\Omega_t = \{x \mid w_t(x) \leq 2r\}$

Main Theorem: $O(d)$ competitive general algorithm

Proof of Feasibility Lemma

- $K_t = \{\, x \mid \langle a, x \rangle \geq b \,\}$ (w.l.o.g.)
- Define

$$\hat{y} = \begin{cases} reflect(y) \\ y \end{cases}$$



Goal: $st(\Omega_t) \in K_t$

$\Omega_t = \{x \mid w_t(x) \leq 2r\}$

$st(\Omega_t) = \lim_{\gamma \to \infty} cg(\Omega_t + \gamma B)$

Proof of Feasibility Lemma

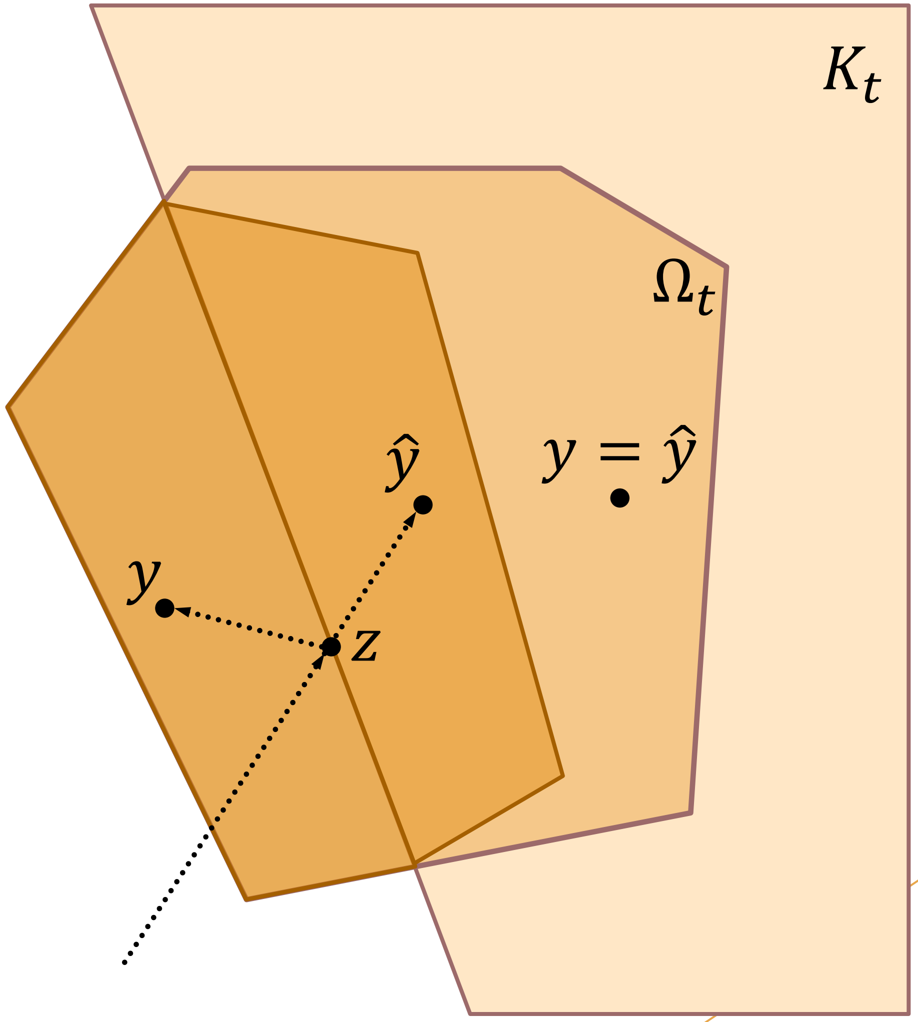
Claim: If $\boldsymbol{y} \in \boldsymbol{\Omega}_t$ then $\hat{\boldsymbol{y}} \in \boldsymbol{\Omega}_t$

If $\langle a, y \rangle \geq 0$ then $\hat{y} = y$

Else, $\langle a, y \rangle < 0$

$$w_t(y) = \min_{z \in K_t} \|y - z\| + w_{t-1}(z)$$

$$\Rightarrow \; w_t(\hat{y}) \leq w_t(y) \leq 2r \qquad ☺$$

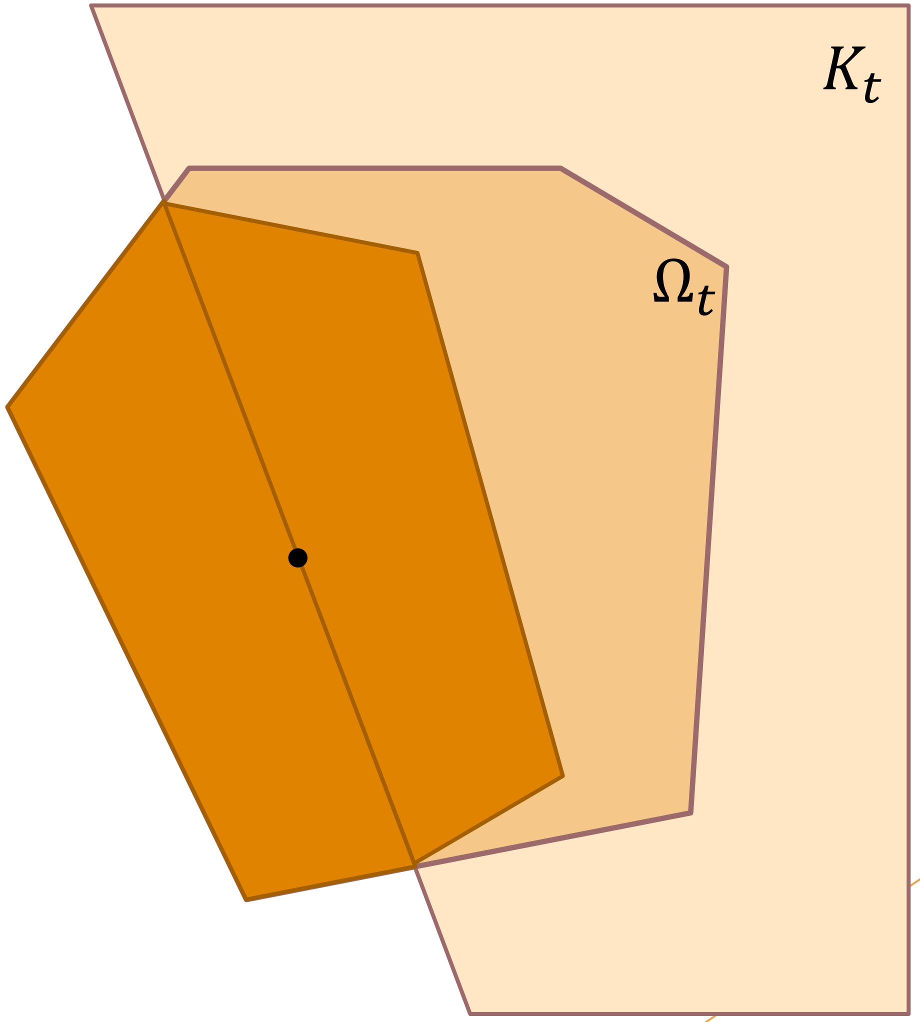


Goal: $st(\Omega_t) \in K_t$

$\Omega_t = \{x \mid w_t(x) \leq 2r\}$

$st(\Omega_t) = \lim_{\gamma \to \infty} cg(\Omega_t + \gamma B)$

Proof of Feasibility Lemma



Goal: $st(\Omega_t) \in K_t$

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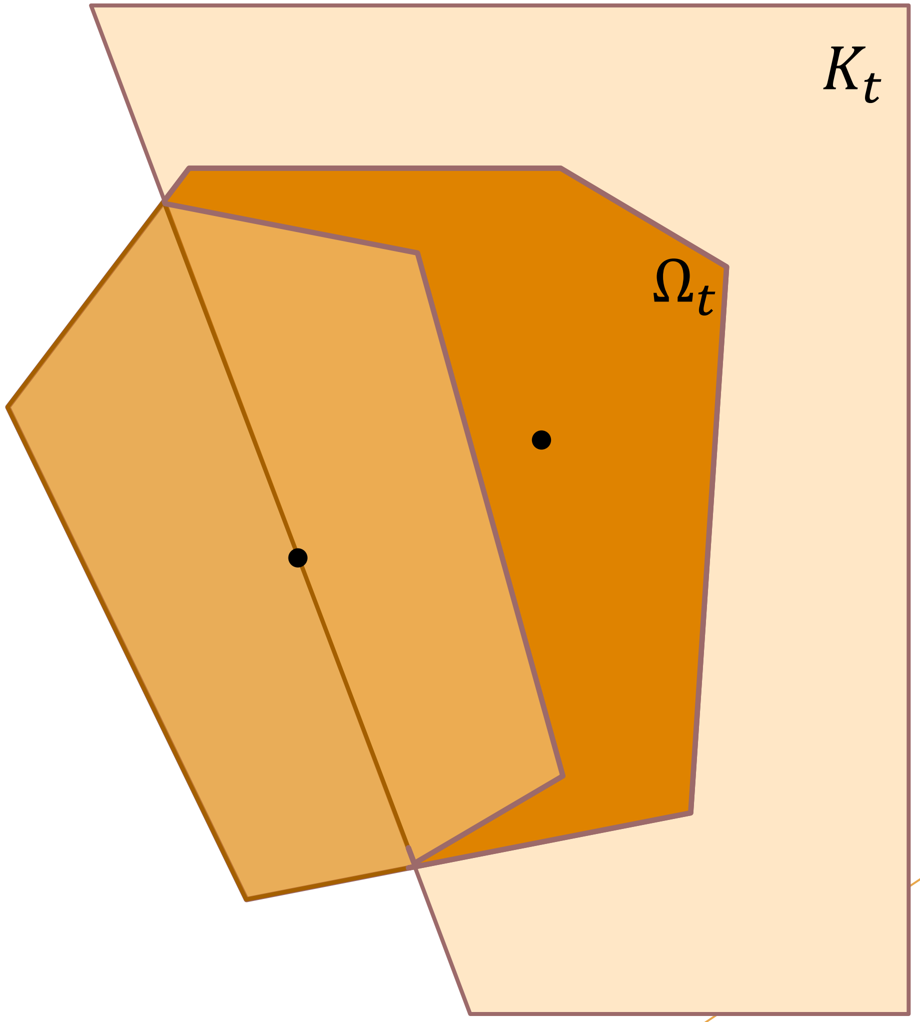
Proof of Feasibility Lemma



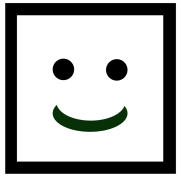
Goal: $st(\Omega_t) \in K_t$

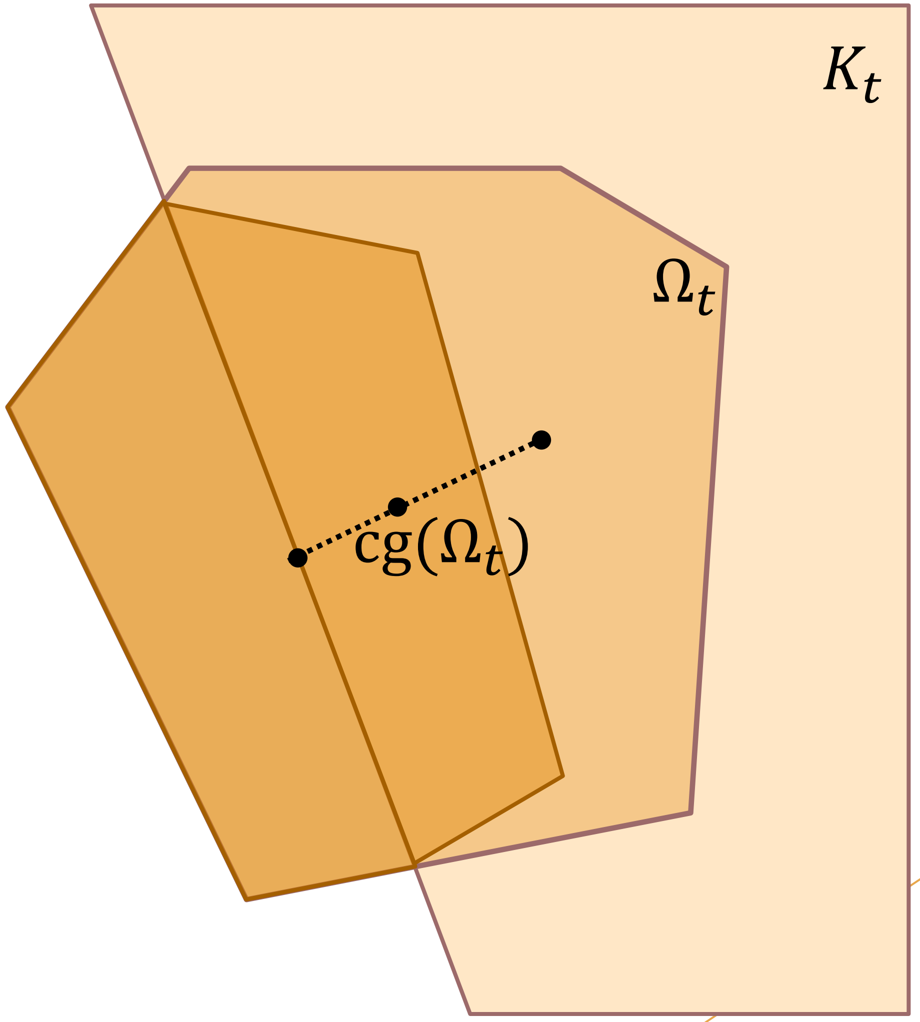
$\Omega_t = \{x \mid w_t(x) \leq 2r\}$

$st(\Omega_t) = \lim_{\gamma \to \infty} cg(\Omega_t + \gamma B)$

Proof of Feasibility Lemma

$\mathrm{cg}(\Omega_t) \in K_t$



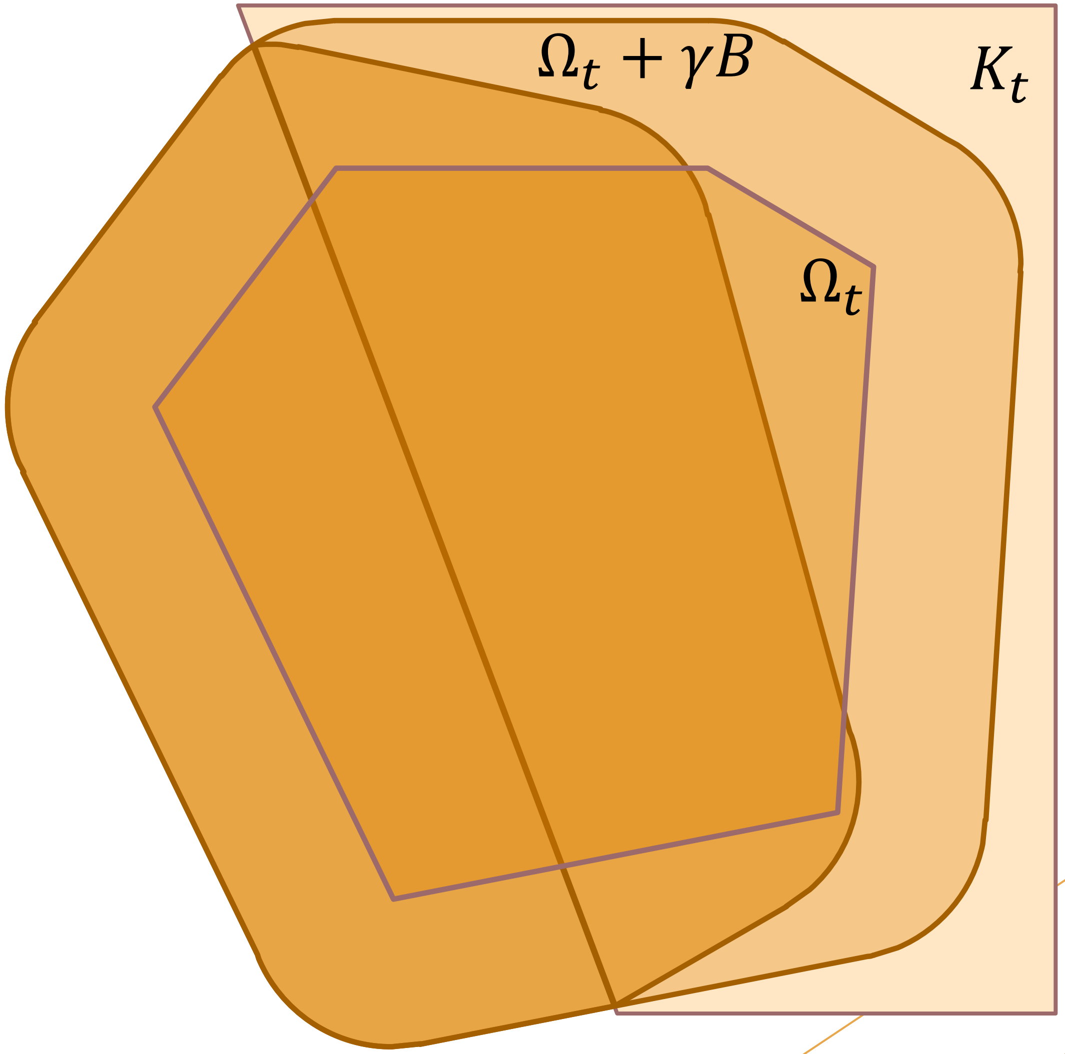


Goal: $st(\Omega_t) \in K_t$

$\Omega_\mathrm{t} = \{x \mid w_t(x) \leq 2r\}$

$st(\Omega_t) = \lim_{\gamma \to \infty} cg(\Omega_t + \gamma B)$

Proof of Feasibility Lemma



Goal: $st(\Omega_t) \in K_t$
$\Omega_t = \{x \mid w_t(x) \leq 2r\}$
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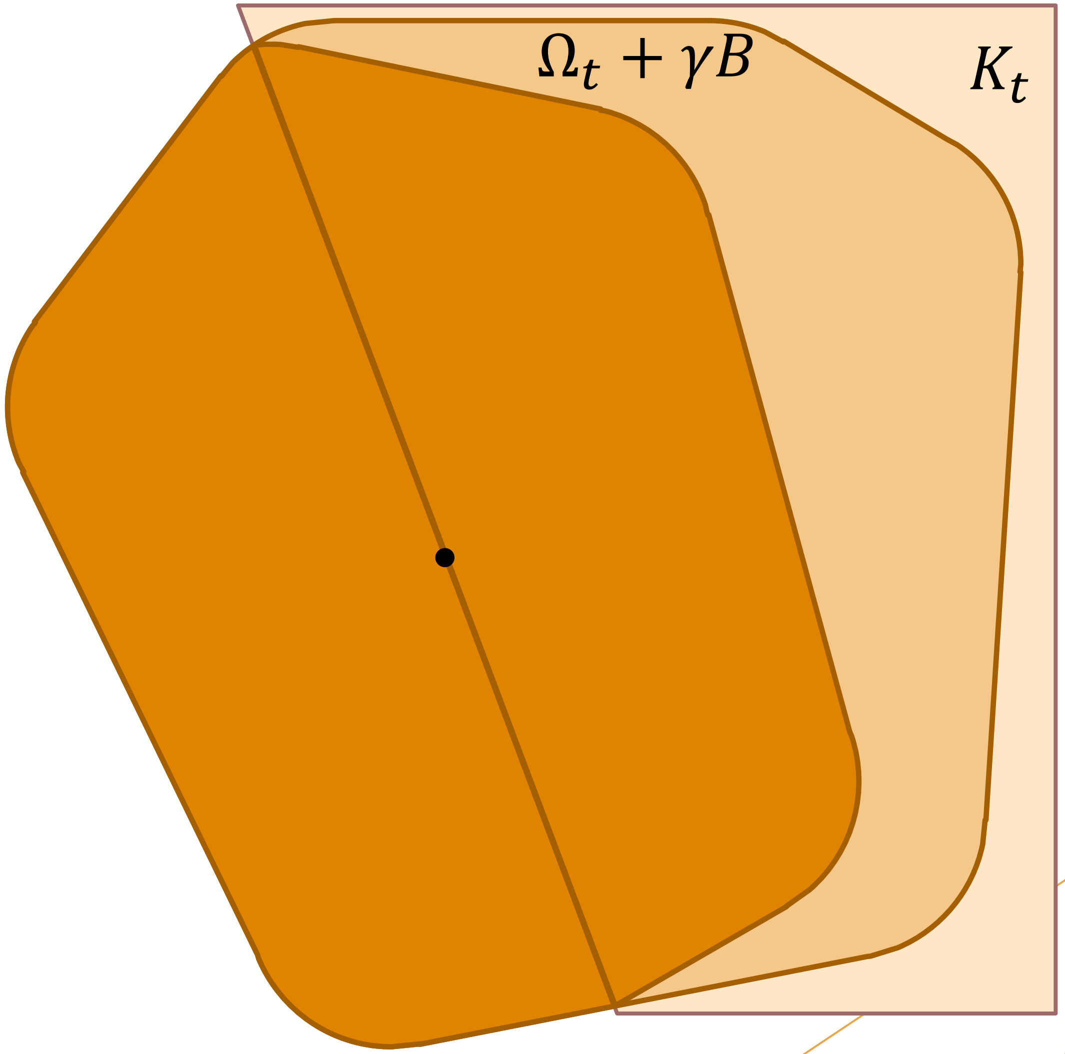
Proof of Feasibility Lemma



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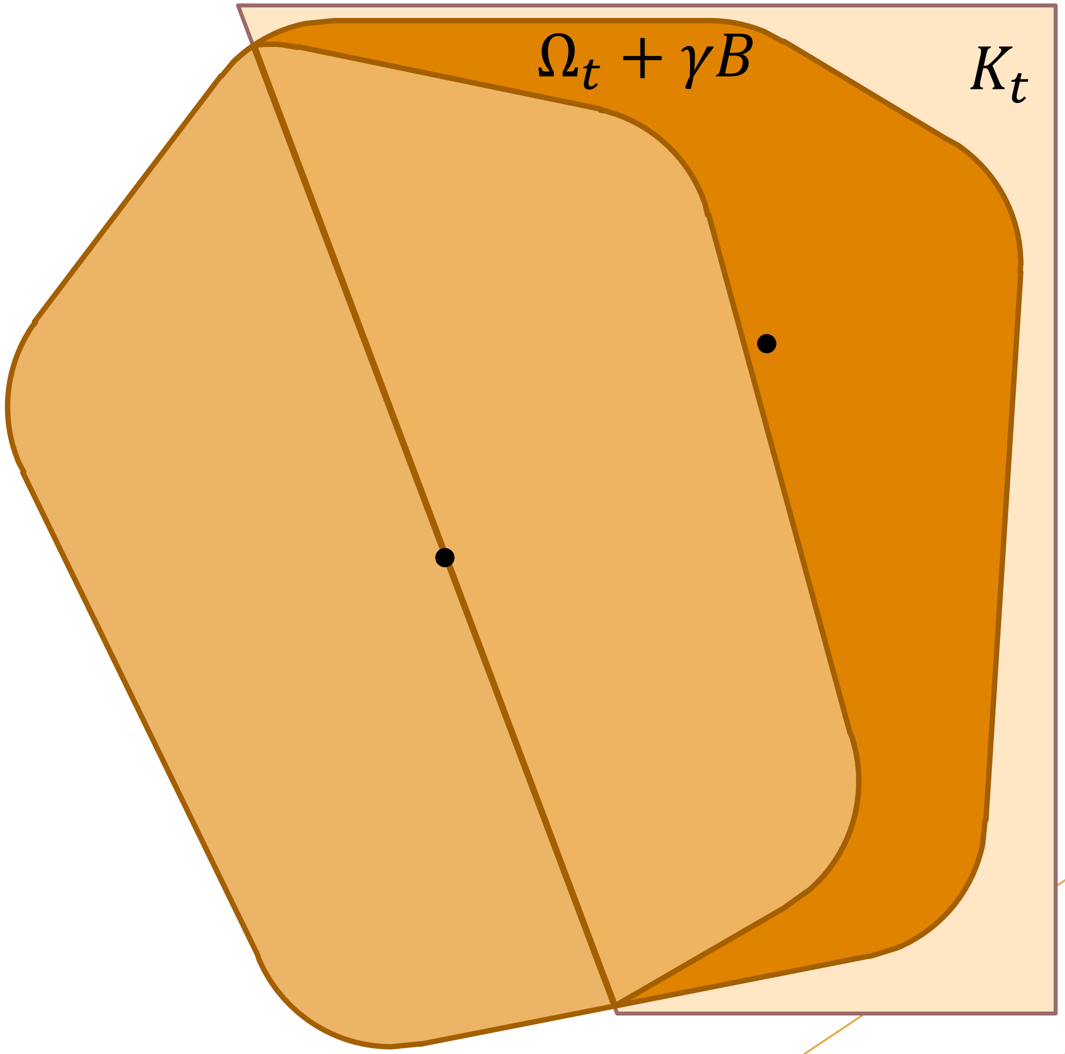
$\Omega_t = \{x \mid w_t(x) \leq 2r\}$

$st(\Omega_t) = \lim_{\gamma \to \infty} cg(\Omega_t + \gamma B)$

Proof of Feasibility Lemma



Goal: $st(\Omega_t) \in K_t$

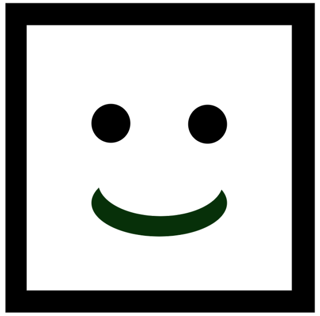
$\Omega_t = \{x \mid w_t(x) \leq 2r\}$

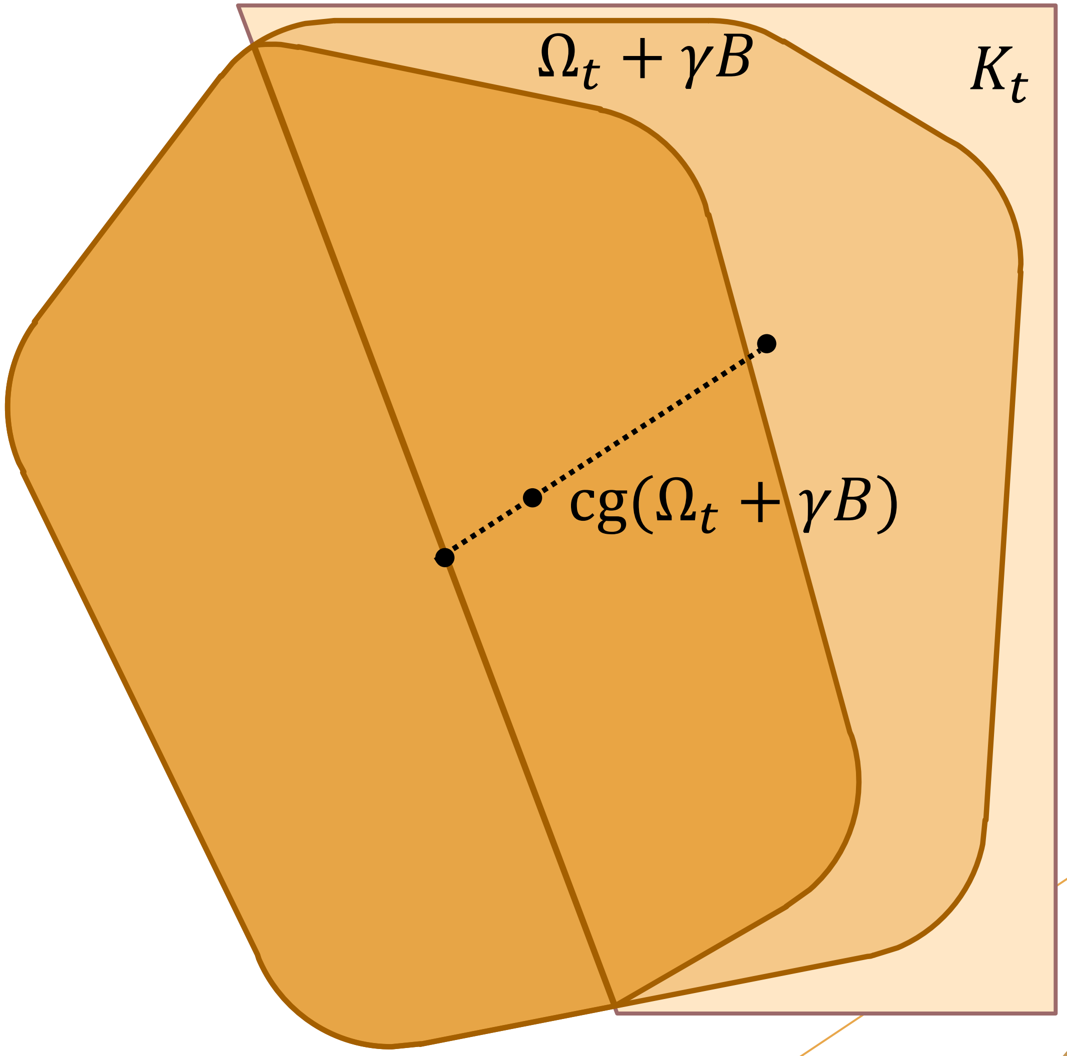
$st(\Omega_t) = \lim_{\gamma \to \infty} cg(\Omega_t + \gamma B)$

Proof of Feasibility Lemma

$\mathrm{cg}(\Omega_t + \gamma B) \in K_t$
for all $\gamma \geq 0$

$\mathrm{st}(\Omega_t) \in K_t$

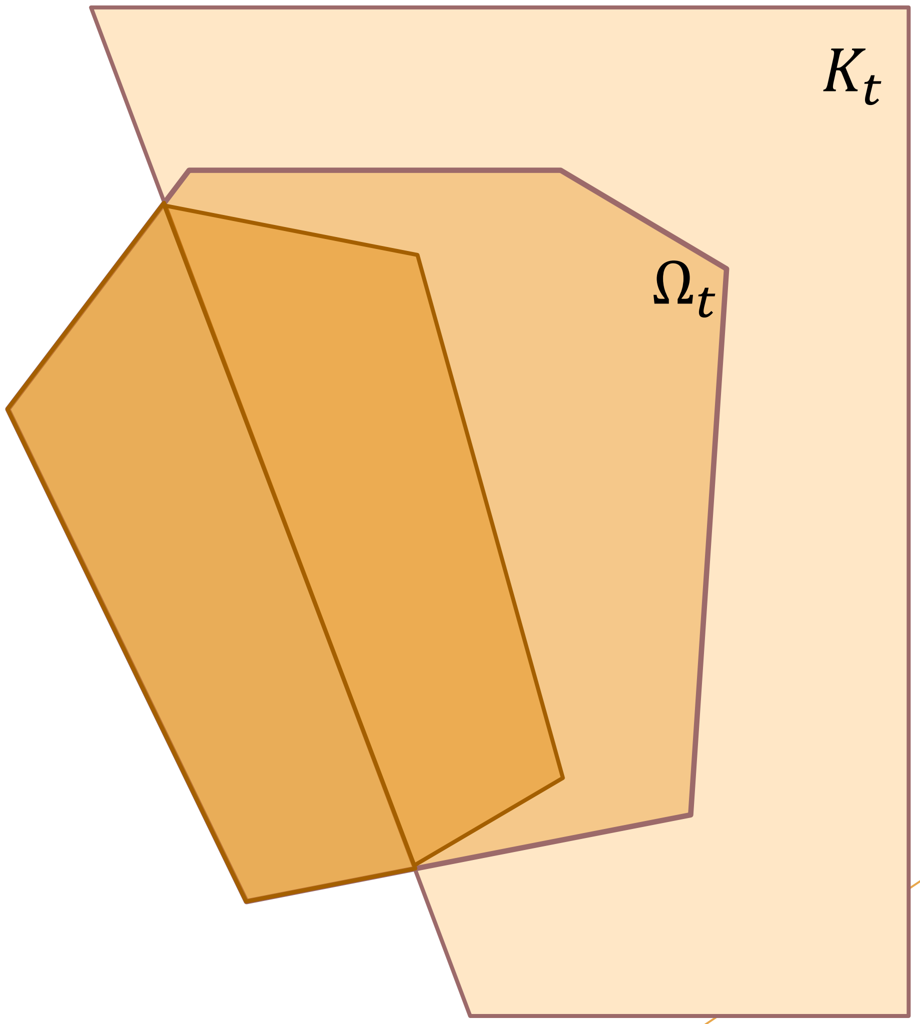




Goal: $st(\Omega_t) \in K_t$
$\Omega_\mathrm{t} = \{x \mid w_t(x) \leq 2r\}$
$st(\Omega_t) = \lim_{\gamma \to \infty} cg(\Omega_t + \gamma B)$

Recap of Main Theorem

- Algo: $x_t = st(\Omega_t)$
  - $\Omega_t = \{x \mid w_t(x) \leq 2r\}$
- $O(d)$ competitiveness
  - Ω_t convex, $\Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_T$
  - Feasibility: $x_t \in K_t$
  - $ALG \leq O(d) \cdot r \leq O(d) \cdot OPT$



Steiner Point of a Convex Function

[Sellke 19]

- $st(f) := \int_{\theta \in B^*} \nabla f^*(\theta) d\theta$
  - $f^*(\theta) =$ Fenchel dual
  - $B^* :=$ dual space unit ball
- Algorithm: $x_t = st(w_t)$
  - Analysis similar to nested Steiner point
  - Arbitrary norm

Open questions

- $O(\sqrt{d})$-competitive general chasing
- Applications to related problems
  - Paging
  - MTS
  - k-server

Coach the ARML Team!

[Shameless plug]

- Talented 6^{th}-12^{th} graders
- Awesome topics
- Pizza during practice
- Teach 2-3 times / semester
- Sundays 4-6:30pm
- Talk to me or Alex Rudenko

Thank you!

Questions?

References

- "Chasing Convex Bodies with Linear Competitive Ratio"
  Argue, Gupta, Guruganesh, Tang, *SODA '20* *[This talk]*
- "A Nearly-Linear Bound for Chasing Nested Convex Bodies"
  Argue, Bubeck, Cohen, Gupta, Lee, *SODA '19*
- "Chasing Nested Convex Bodies Nearly Optimally,"
  Bubeck, Klartag, Lee, Li, Sellke, *SODA '20*
- "Competitively Chasing Convex Bodies"
  Bubeck, Lee, Li, Sellke, *STOC '19*
- "Chasing Convex Bodies and Functions"
  Friedman, Linial, *Discrete and Computational Geometry '93*
- "Chasing Convex Bodies Optimally"
  Sellke, *SODA '20* *[Similar results to this talk]*