How Many Ways Can We Tile a Rectangular Chessboard With Dominos?

Counting Tilings With Permanents and Determinants

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Abstract

Consider an $m \times n$ rectangular chessboard. Suppose we want to tile this board with dominoes, where a domino is a 2×1 rectangle, and a tiling is a way to place several dominoes on the board so that all of its squares are covered but no dominos overlap or lie partially off the board. Is such a tiling possible? If so, how many are there? The first question is simple, yet the second question is quite difficult! We will answer it by reformulating the problem in terms of perfect matchings in bipartite graphs. Counting these matchings will be achieved efficiently by finding a particularly helpful matrix that describes the edges in a matching, and then finding the determinant of that matrix. Remarkably, there is even a closed-form solution!

(Note: This talk is adapted from a Chapter in Jiří Matoušek's book *Thirty-three Miniatures: Mathematical and Algorithmic Applications of Linear Algebra* [1].)

1 Introduction
- The Problem
- Explorations
- Generalizations
- Applications

2 Reformulation
- Definitions
- Matrices and Permutations
- Permanents and Determinants
- Kasteleyn Signings

3 The Main Theorem
- Graph Properties
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- Lemma 1 (and Proof)
- Lemma 2 (and Proof)
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4 Results & Conclusions
- Summary of Method
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Chessboards & Dominoes

Consider an $m \times n$ rectangular chessboard and 2×1 dominoes.

A **tiling** is a placement of dominoes that covers all the squares of the board perfectly (i.e. no overlaps, no diagonal placements, no protrusions off the board, and so on).

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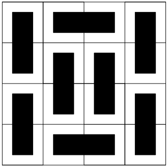

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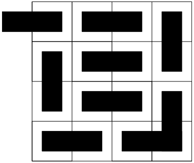

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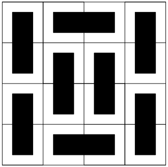

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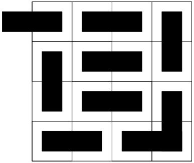

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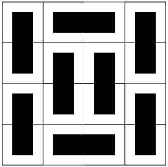

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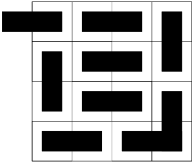

(i) For which m, n do there *exist* tilings?
(ii) If there are tilings, how *many* are there?

(i) Existence of tilings: A fundamental fact

Fact: Tilings exist $\iff$ m, n are not *both* odd (i.e. mn is even)

Proof.

WOLOG m is even. Place $\frac{m}{2}$ dominoes vertically in 1st column. Repeat across n columns. □

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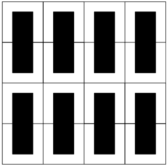
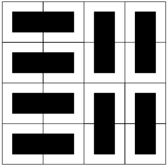
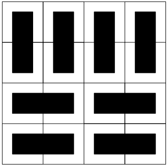


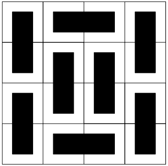
Tiling 2



Tiling 3



Tiling 4



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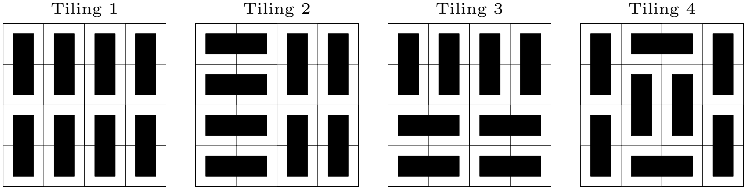

Note: 2 and 3 are *isomorphic*. We won't account for this. (Too hard!)

(ii) Counting tilings: A fundamental example

Consider $m = 2$. A recurrence for $T(2, n)$ is given by

$$T(2, n) = T(2, n-1) + T(2, n-2)$$

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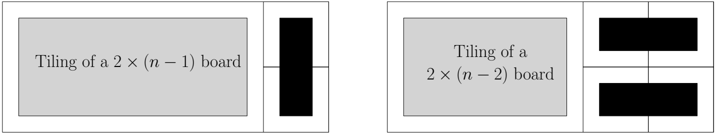
because a tiling of a $2 \times n$ board consists of (a) a tiling of a $2 \times (n-1)$ board with a vertical domino or (b) a tiling of a $2 \times (n-2)$ board with two horizontal dominoes:



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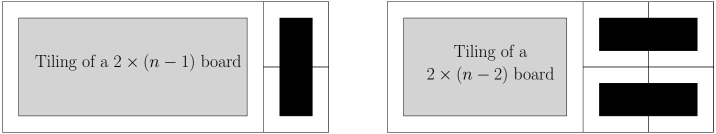

Since $T(2, 1) = 1$ and $T(2, 2) = 2$ (recall: isomorphisms irrelevant) we have $T(2, n) = F_{n-1}$. It's the Fibonacci sequence!

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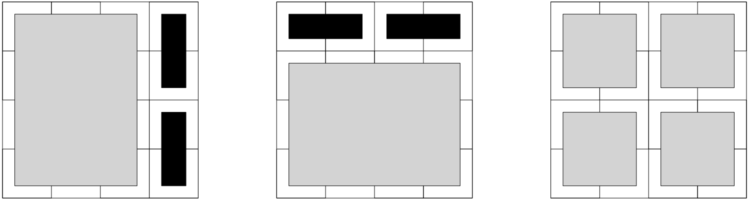

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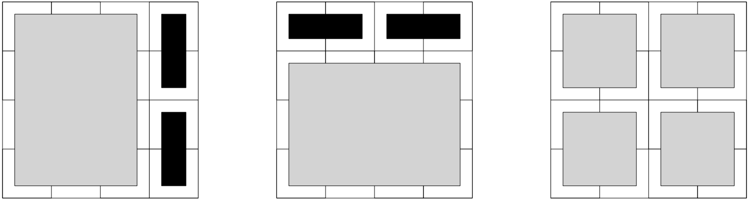


This is too difficult, in general! ☹

Recursion: it's not all bad

One can prove, for example that

$$T(3,2n) = 4T(3,2n-2) - T(3,2n-4)$$

Proof.

Exercise for the reader.

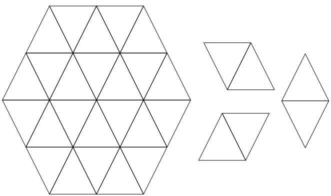
Hint: First prove

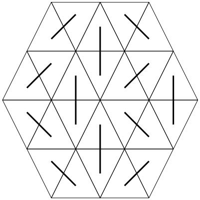
$$T(3,2n+2) = 3T(3,2n) + 2\sum_{k=0}^{n} T(3,2k)$$

□

Hexagonal Tilings

Consider a regular hexagon made of equilateral triangles, and rhombic tiles made of two such triangles.

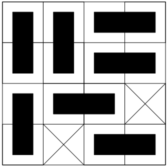




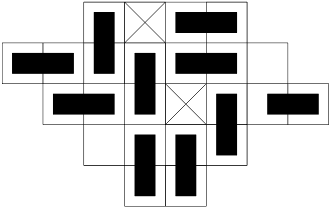
Ask the same questions of (i) existence and (ii) counting.

Altered Rectangles

What if we remove squares from the rectangular boards?



What about other crazy shapes?



Tilings, Perfect Matchings, and The Dimer Model

- **Tilings**: Popular recreational math topic. Great exercises! Tilings of the plane appear in ancient art, and reflect some deep group theoretic principles.
- **Perfect Matchings**: Useful in computer science. Algorithms for finding matchings of various forms in different types of graphs are studied for their computational complexity.
- **The Dimer Model**: Simple model used to describe thermodynamic behavior of fluids. It was the original motivation for this problem, solved in 1961 by P.W. Kasteleyn [2] and independently by Temperley & Fisher [3].

Graph Theory & Linear Algebra

We will take a seemingly roundabout approach to find $T(m, n)$. We will reformulate the problem in terms of **graphs** and then use **linear algebra** to assess properties of particular graphs. This will solve the problem!

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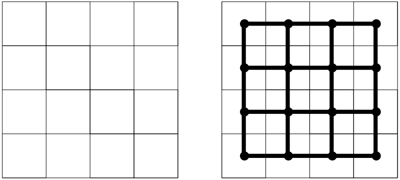
Fundamental idea: A domino tiling corresponds (uniquely) to a perfect matching in the underlying grid graph of the board.

Restatement: A domino tiling is characterized by which squares are covered by the same domino. We merely need to count the ways to properly assign these so that it *is* a tiling.

Example illustration

Represent the board with a dot (**vertex**) in each square and a line (**edge**) between adjacent squares (non-diagonally).

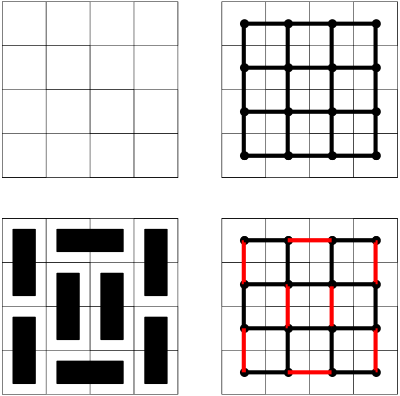


Example illustration

Represent the board with a dot (**vertex**) in each square and a line (**edge**) between adjacent squares (non-diagonally).

A tiling corresponds to a selection of these edges (and *only* these allowable edges) that *covers* every vertex.

In other terminology, this is a **perfect matching**.



Graph terminology

Definition

A **bipartite** *graph is one whose vertices can be separated into two parts, so that edges only go between parts (i.e. no internal edges in a part).*

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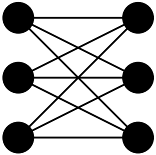
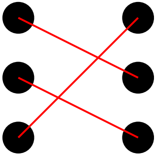
Example: $K_{3,3}$, the *complete* bipartite graph.

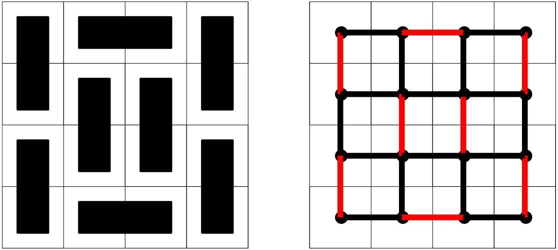




(Note: In general, a perfect matching *requires* an even number of vertices.)

Relevancy to our problem: perfect matchings

Observation: A domino tiling is a perfect matching in the underlying grid graph.



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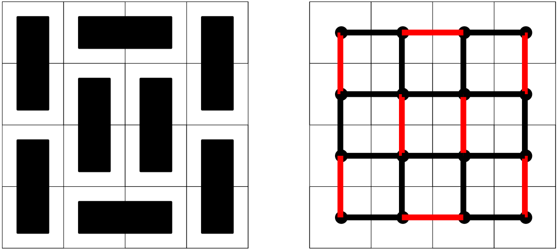
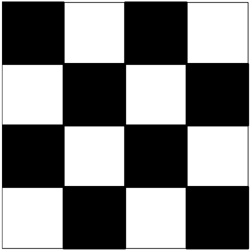
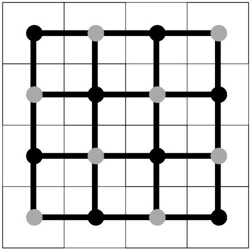


Reason: Edges represent *potential* domino placements (adjacent squares) and all squares must be covered by *exactly* one domino.

Relevancy to our problem: bipartite graphs

Observation: The underlying grid graph is bipartite.





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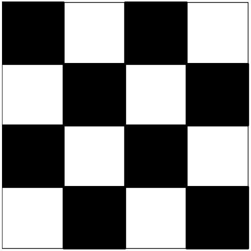

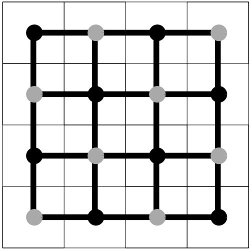

Reason: Color the squares like a chessboard. Take the two vertex parts to be the **black** squares and **white** squares.

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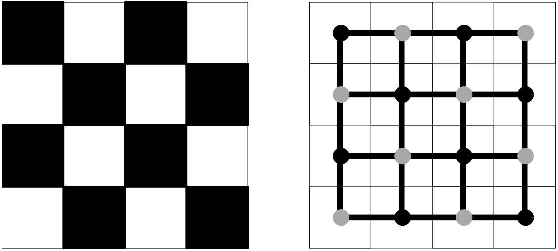

Reason: Color the squares like a chessboard. Take the two vertex parts to be the **black** squares and **white** squares. Edges only connect squares of *opposite* colors, since squares of the *same* color lie along *diagonals*.

Notation

We will use B and W to represent the two vertex parts.

Given m, n the grid graph has mn vertices, so each part has $N := \frac{mn}{2}$ vertices.

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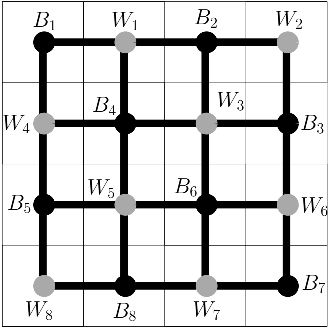
We will number the vertices in each part, from 1 to N. Order is irrelevant, but the convention is to snake from the top-left:



Why bother with this formulation?

We can conveniently represent the grid graph as a matrix and exploit its properties.

Definition

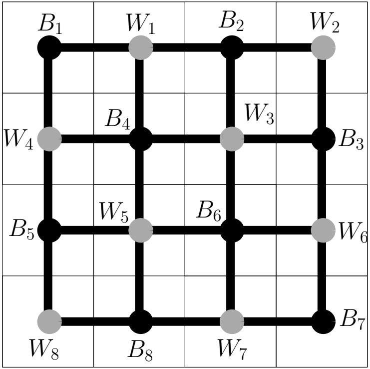
Consider a grid graph G, with $N := \frac{mn}{2}$ vertices in each part. The **adjacency matrix** *A is the $N \times N$ matrix given by*

$$a_{ij} = \begin{cases} 1 & \textit{if } \{b_i, w_j\} \textit{ is an edge in } G \\ 0 & \textit{otherwise} \end{cases}$$

This encodes all of the possible domino placements, so exploring its properties should yield some insight to our problem.

An example adjacency matrix

Recall the 4×4 board and grid graph and construct its corresponding adjacency matrix:



$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

What does a perfect matching look like in A?

Since B and W each have N labeled vertices, a perfect matching is completely characterized by a **permutation** of $\{1, 2, \ldots, N\}$.

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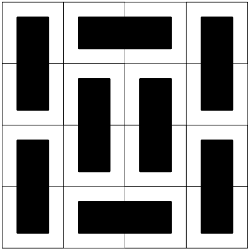
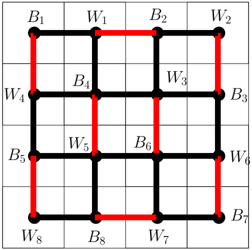
Example: Recall this tiling/matching in the 4×4 board:





This corresponds to the permutation $(4, 1, 2, 5, 8, 3, 6, 7)$ on $\{1, 2, \ldots, 8\}$. It encodes which w_j is adjacent to each b_i.

This does **not** work the other way!

An arbitrary permutation on $\{1, 2, \ldots, N\}$ does **not** necessarily represent a perfect matching in G, though.

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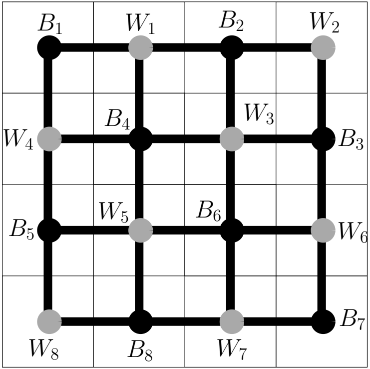

Example:
$(1, 2, 6, 4, 3, 7, 8, 5)$

Notice that $\{b_5, w_3\}$ and $\{b_7, w_8\}$ are not edges in G (those squares are far apart on the board) so this is not a perfect matching and, thus, not a tiling.

Counting tilings via permutations

Recall that S_N is the set of all permuations of $\{1, 2, \ldots, N\}$. (In fact, it is the *symmetric group* on N elements.)

Given $\pi \in S_N$, does π correspond to a perfect matching in G? *Only* if all of the necessary edges represented by π are, indeed, present in G.

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This requires $a_{1,\pi(1)} \cdot a_{2,\pi(2)} \cdots a_{N,\pi(N)} = 1$.

If any such edge is not present, its entry in A will be 0, so the product will be 0.

Counting tilings via the adjacency matrix

Accordingly,

$$T(m,n) = \sum_{\pi \in S_N} a_{1,\pi(1)} \cdot a_{2,\pi(2)} \cdots a_{N,\pi(N)}$$

A permutation that corresponds to a matching in G contributes a 1 to the sum, a permutation that does not correspond to a matching contributes a 0.

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Does this look familiar ...?

Definition

Given an $N \times N$ matrix A, the **permanent** *of A is*

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and the **determinant** *of* A *is*

$$\det(A) = \sum_{\pi \in S_N} \text{sgn}(\pi) \cdot a_{1,\pi(1)} \cdot a_{2,\pi(2)} \cdots a_{N,\pi(N)}$$

where $\text{sgn}(\pi)$ *is* ± 1*, depending on its parity (the number of transpositions required to return* π *to the Identity).*

So . . . are we done?

Given m, n, simply find A and compute $\text{per}(A)$.

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Even when the entries are just 0/1 (like we have), computing the permanent is **#P-complete**.

Computational complexity

NP problems are *decision* problems whose proposed answers can be evaluated in polynomial time. For example:

- Given a set of integers, is there a subset whose sum is 0?
- Given a conjuctive normal form formula, is there an assignment of Boolean values that makes the statement evaluate to True?

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#P problems are the *counting* versions of those decision problems in **NP**. Of course, these problems are *harder* to solve!

- Given a set of integers, how many subsets have sum 0?
- Given a conjuctive normal form formula, how many Boolean assignments make the statement True?

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However, "How *many* perfect matchings are there?" is **#P-complete**. It is *hard*.

This was proven in 1979 by Valiant. In his paper, he introduced the terms **#P** and **#P-complete**.

Computational complexity

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This is because the determinant has some nice algebraic properties that the permanent does not share.

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New goal: Find a matrix $\hat{A}$ such that $|\det(\hat{A})| = \text{per}(A)$, then compute $\det(\hat{A})$. As long as this is done in polynomial-time, we will have solved the overall problem in polynomial-time.

Definition: weighting the edges

Definition

A **signing** *of G is an assignment of ± 1 weights to the edges:*

$$\sigma : E(G) \to \{-1, +1\}$$

The **signed adjacency matrix** A^σ *is given by*

$$a_{ij}^\sigma = \begin{cases} \sigma\left(\{b_i, w_j\}\right) & \textit{if } \{b_i, w_j\} \textit{ is an edge in } G \\ 0 & \textit{otherwise} \end{cases}$$

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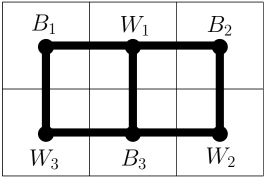
If such a σ satisfies the equation $\operatorname{per}(A) = |\det(A^\sigma)|$, then we say σ is a **Kasteleyn signing** *of G.*

An example: the 2×3 grid graph



$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Notice $\text{per}(A) = \underbrace{1}_{123} + \underbrace{0}_{132} + \underbrace{0}_{213} + \underbrace{0}_{231} + \underbrace{1}_{312} + \underbrace{1}_{321} = 3.$

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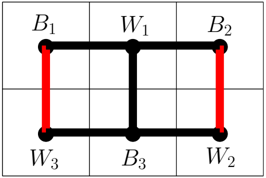

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Weight $\{b_1, w_3\}$ and $\{b_2, w_2\}$ with -1, all others +1. Then,

$A^\sigma = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $\det(A^\sigma) = \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -3$

A non-example: $K_{3,3}$

Fact: There is *no* Kasteleyn signing of $K_{3,3}$.

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Notice that $\text{per}(A) = 6$ here, because all entries are 1, and

$$\begin{aligned}\det(A^\sigma) = &\; a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}\end{aligned}$$

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Let's make these all, say, +1. WOLOG $a_{11} = +1$. Then either a_{22}, a_{33} both +1 or both -1.

If both +1, we get a_{23}, a_{32} and a_{12}, a_{21} and a_{13}, a_{31} have opposite signs. ※

If both -1, we get a_{23}, a_{32} have opposite while a_{12}, a_{21} and a_{13}, a_{31} have same signs. ※ □

Informal statement and proof strategy

Theorem

Every grid graph arising from an $m \times n$ rectangular board has a Kasteleyn signing and we can find one efficiently.

More formal statement forthcoming.

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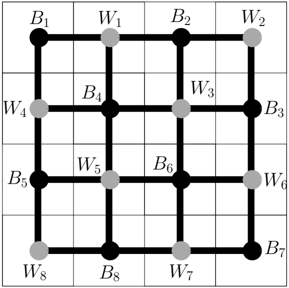
Proof strategy: Lemma 1 provides a *sufficient* condition for a signing to be Kasteleyn. Lemma 2 provides a more specific version of this condition that applies to our grid graphs. The Theorem follows from these two and an algorithm for *building in* the condition of Lemma 2 to a signing.

2-connectivity and planarity

A graph is **planar** if it can be drawn on the plane with no edges crossing.

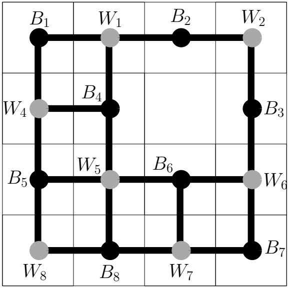
Notice our grid graphs are planar because the rectangular boards are, too. We can just draw the graph on the board!



2-connectivity and planarity

A graph is **2-connected** if it is connected and the removal of any vertex does *not* disconnect the graph.

Notice our grid graphs are 2-connected because even after removing a square, we can connected any two squares with a path of alternating colors; we just might have to "go around" the hole.



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Let G be a bipartite, planar, 2-connected graph. Then G has a Kasteleyn signing that can be found in polynomial-time (in N).

Corollary

$T(m,n)$ can be computed in polynomial-time (in mn).

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Note: The *bipartite* and *2-connected* assumptions can be removed, with effort, but *planarity* is essential.

Definitions: cycles and signs

Definition

A **cycle** *in G is a sequence of vertices and edges that returns to the same vertex. (It does not need to use all vertices in G.)*

A cycle C is **evenly-placed** *if G has a perfect matching outside of C (i.e. with all edges and vertices of C removed.)*

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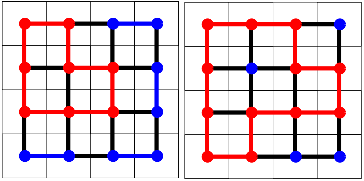

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Given σ on G, a cycle C is **properly-signed** *if its length matches the weights of its edges appropriately:*
If $|C| = 2\ell$, then the number of negative edges on C (call it n_C) should have opposite parity of ℓ, i.e. $n_C \equiv \ell - 1 \pmod 2$.

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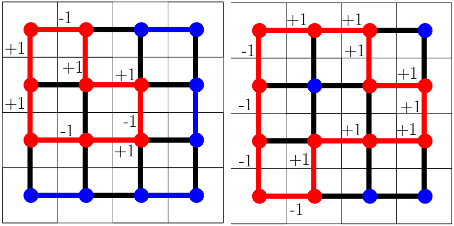

Statement

Lemma 1

If every evenly-placed cycle in G is properly-signed, then σ is a Kasteleyn signing.

Proof strategy: We will define the *sign* of a perfect matching. To make sure σ is Kasteleyn, we require all perfect matchings to have the same sign. The symmetric difference of two matchings is a disjoint union of evenly-placed cycles. Since those are properly-signed, we can make a claim about the signs of the permutations corresponding to matchings.

Proof: the sign of a matching

Take σ and suppose every evenly-placed cycle is properly-signed. For any perfect matching M, define

$$\text{sgn}(M) := \text{sgn}(\pi) a^{\sigma}_{1,\pi(1)} a^{\sigma}_{2,\pi(2)} \cdots a^{\sigma}_{N,\pi(N)} = \text{sgn}(\pi) \prod_{e \in M} \sigma(e)$$

Notice this is the corresponding term in the formula for $\det(A)$.

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Notice this is the corresponding term in the formula for $\det(A)$. To ensure σ is Kasteleyn, we need all matchings to have the *same sign*, so that $\det(A)$ is a sum of all +1s or -1s.

Now, take two arbitrary perfect matchings M, M'.
Goal: Show $\mathrm{sgn}(M) = \mathrm{sgn}(M')$.

Proof: the "product" of two matchings

To achieve this, it suffices to show $\text{sgn}(M)\,\text{sgn}(M') = 1$.

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because any edge common to both contributes $\sigma(e)^2 = 1$, so we only care about the edges belonging to *exactly* one matching.

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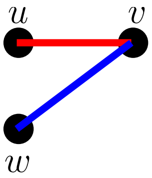
Goal: Show $\operatorname{sgn}(\pi)=\operatorname{sgn}(\pi')\cdot(-1)^L$, so $\operatorname{sgn}(M)\operatorname{sgn}(M')=1$.

Proof: $M \Delta M'$ is a disjoint union of cycles

Take any vertex u. Find its neighbor v in M.
Find the neighbor w of v in M'.



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Take any vertex u. Find its neighbor v in M.
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If $w = u$ then $\{u, v\}$ is an edge in *both* matchings, so $\{u, v\} \notin M\Delta M'$.

Proof: $M \Delta M'$ is a disjoint union of cycles

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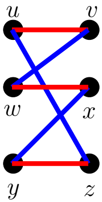
Find the neighbor w of v in M'.

If $w \neq u$, then repeat this process, alternately finding the next neighbor from M and then M'. Since G is finite, this terminates and closes a cycle.



(Note: this cannot close back on itself "internally" since these are *perfect* matchings.)

Repeat on an unused vertex.

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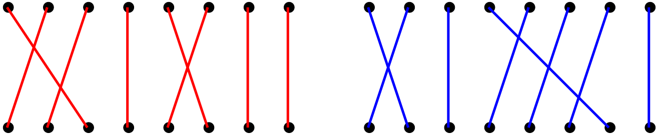
Example:
Consider these two matchings on 8 vertices:



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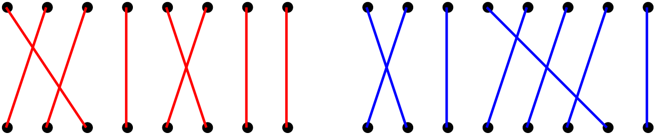
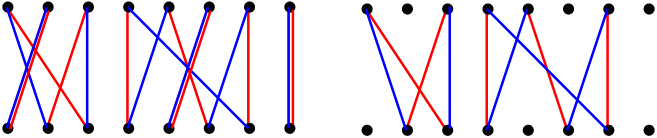


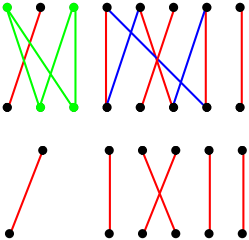
Overlay them and remove common edges to obtain $M \Delta M'$:



Proof: the cycles of $M \Delta M'$ are evenly-placed

Consider removing such a cycle C from the graph. (Note: its *vertices* are removed, too.)

We can use the edges of, say, M that were not removed. That is, $M - (M \Delta M')$ is a perfect matching on $G - C$.



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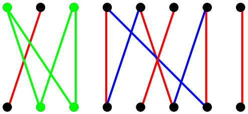

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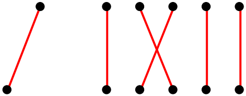

Thus, all such cycles are evenly-placed, so they are **properly-signed**, by assumption.

This information will help us complete the proof.

Proof: the signs on the cycles

Say $M \Delta M'$ has k cycles, with lengths $|C_i| = 2\ell_i$.
Properly-signed $\implies n_{C_i} \equiv \ell_i - 1 \pmod{2}$ (# of neg. edges)

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&= \operatorname{sgn}(\pi)\operatorname{sgn}(\pi') \prod_{i=1}^{k} (-1)^{\ell_i - 1} \\
&= \operatorname{sgn}(\pi)\operatorname{sgn}(\pi') \cdot (-1)^{L} \\
&\qquad \text{where } L := (\ell_1 - 1) + (\ell_2 - 1) + \cdots + (\ell_k - 1)
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Proof: π and π' differ by L transpositions

Claim: We can morph π into π' by considering these cycles and identifying L transpositions.

Take C_i. We will identify $\ell_i - 1$ transpositions that will make π and π' identical on the vertices of C_i.

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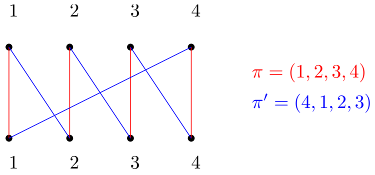
Relabel vertices so π and π' are ordered permutations on $\{1, 2, \ldots, \ell_i\}$. Since C_i is a cycle, *no* positions are identical.



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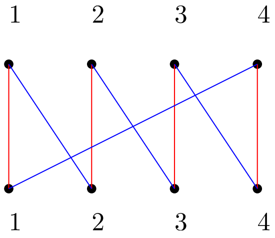

$\pi = (1, 2, 3, 4)$ $\pi(1) = 1$

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This only happens if $\pi(j) = \pi'(k) = t$ and also $\pi(k) = \pi'(j)$. This means $(j, \pi(k), k, \pi(k))$ was a 4-cycle to begin with.

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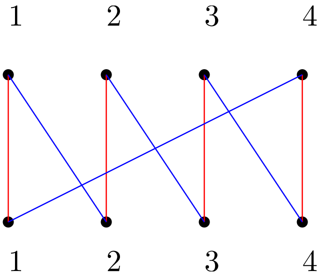
For illustration's sake, here is how that process would play out:



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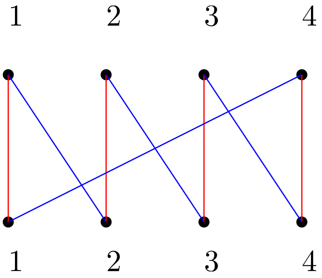

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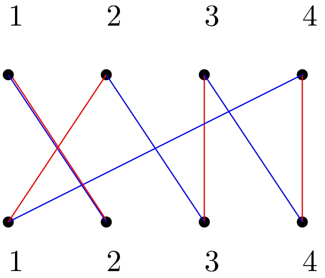

$\pi = (2, 1, 3, 4)$

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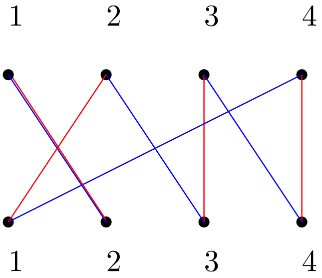

$\pi = (2, 1, 3, 4)$ $\pi(1) = 2$

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Swap positions 1 and 3 in π

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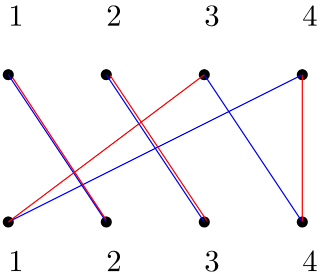

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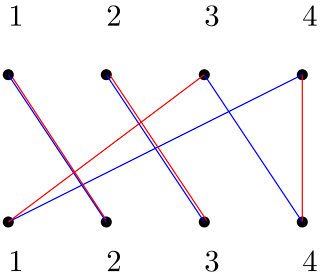

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Swap positions 1 and 4 in π

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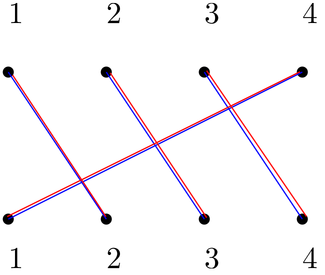

$\pi = (4, 1, 2, 3)$

$\pi' = (4, 1, 2, 3)$

Now $\pi = \pi'$

Proof: wrapping up

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$\text{sgn}(\pi) = \text{sgn}(\pi') \cdot (-1)^L$.

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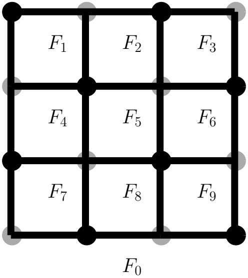
We now have a way of more easily checking if a signing is Kasteleyn. The next Lemma helps us check even *more* easily because it exploits the planarity and 2-connectivity of G.

Planar graphs

A planar drawing of a graph has **vertices**, **edges**, and **faces**.



There is one *outer face*; the rest are *inner faces*.

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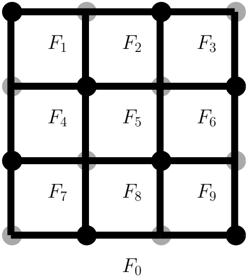

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Euler's Formula: $V + F = E + 2$

Statement and proof strategy

Lemma

Fix a planar drawing of a bipartite, planar, 2-connected graph G, with signing σ. If the boundary cycle of every inner face is properly-signed, then σ is Kasteleyn.

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Fix a planar drawing of a bipartite, planar, 2-connected graph G, with signing σ. If the boundary cycle of every inner face is properly-signed, then σ is Kasteleyn.

Proof strategy: Overall, we invoke Lemma 1. An arbitrary, well-placed cycle C encloses some inner faces. Euler's Formula relates $|C|$ and the lengths of the boundary cycles inside C. The proper-signing of those boundary cycles will tell us C is also properly-signed, so Lemma 1 applies.

Proof: An evenly-placed cycle encloses inner faces

Let C be an evenly-placed cycle in G. Restrict our attention to the vertices and edges inside and on C.

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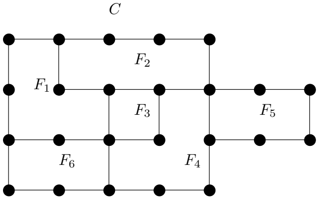
Say we have inner faces $F_1, \ldots, F_k$ with boundary cycles C_i of length $2\ell_i$.



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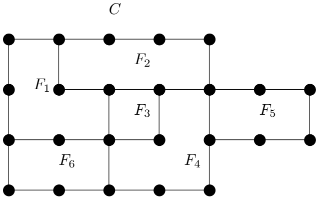

Counting:

- $V = r + 2\ell$ (where r is the number of vertices *inside* C)
- $E = \frac{1}{2}\left(|C| + |C_1| + |C_2| + \cdots + |C_k|\right) = \ell + \ell_1 + \cdots + \ell_k$
- $F = k + 1$ (including the outer face)

Proof: applying Euler's Formula and assumptions

Euler's Formula $\implies$

$$r + 2\ell + k + 1 = \ell + \ell_1 + \cdots + \ell_k + 2$$

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$$2s + \ell - 1 = \ell_1 + \cdots + \ell_k - k$$

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Goal: Use this to show $n_C \equiv \ell - 1 \pmod 2$.

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Overall, then

$$n_C \equiv (\ell_1 - 1) + \cdots + (\ell_k - 1) \equiv \ell_1 + \cdots + \ell_k - k \equiv \ell - 1 \pmod 2$$

so C is properly-signed, as well! Apply Lemma 1. □

Constructing a signing that satisfies Lemma 2

Take our grid graph G and fix a planar drawing. We will describe a method that constructs a signing σ that guarantees every inner face's boundary cycle is properly-signed. It will do this, essentially, one-by-one for each face (whence polynomial-time).

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Eventually, we have G_k with no inner faces.

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When e_i is added back in, it is the boundary of only the inner face F_i in G_i. All the other boundary edges of F_i are present, so we have a definitive choice whether $\sigma(e_i) = \pm 1$ to ensure that boundary cycle is properly-signed. □

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(This can't screw up, because once a boundary cycle is *fixed* to be properly-signed, it won't affect the signing of any other cycle. This fixing happens when its *last* boundary edge is added.)

What have we accomplished?

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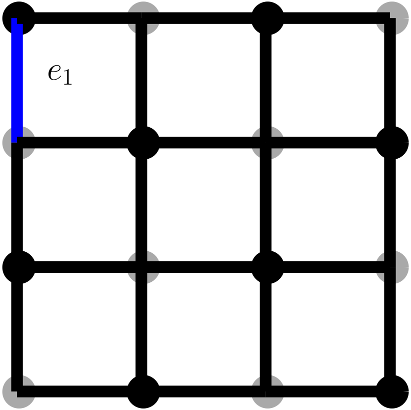
- Compute $\det(A^\sigma)$. (Computationally fast)

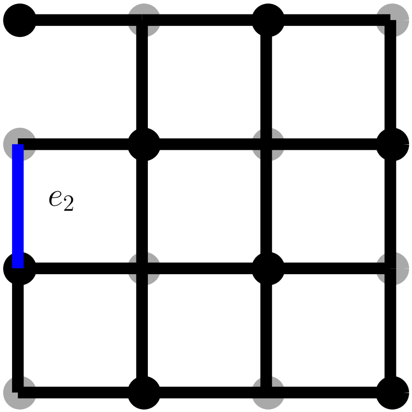
$T(4,4) = ?$

Set $G_1 := G$. Identify e_1 and remove it.



$T(4,4) = ?$

Identify e_2 and remove it.



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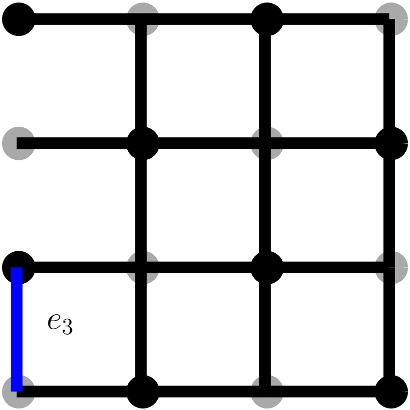

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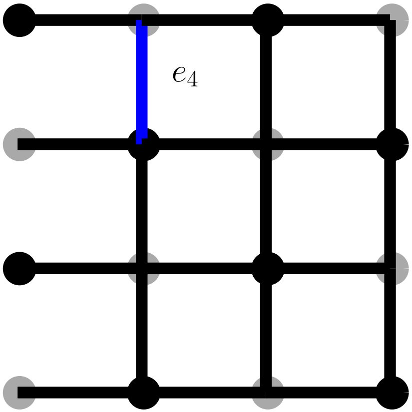
Identify e_4 and remove it.



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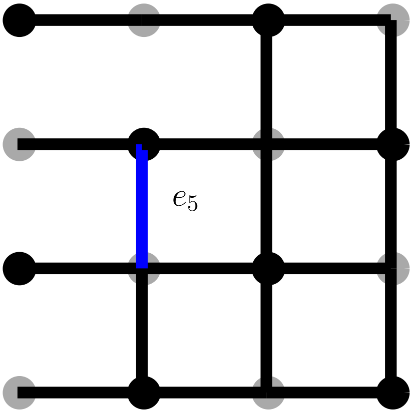
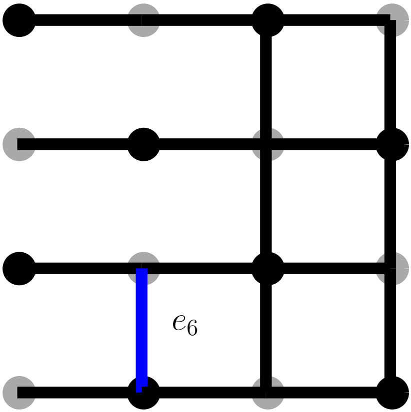
Identify e_5 and remove it.



$T(4,4) = ?$

Identify e_6 and remove it.



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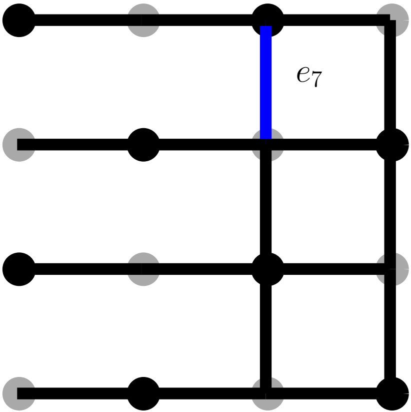
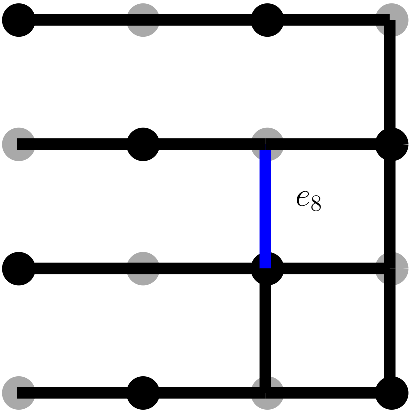
Identify e_7 and remove it.



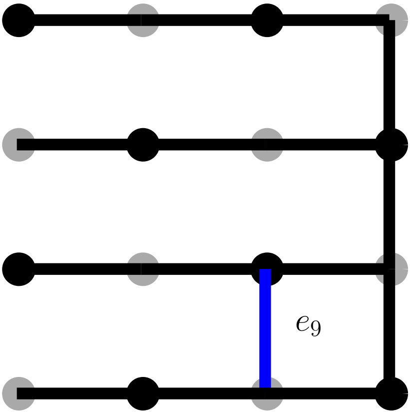
$T(4,4) = ?$

Identify e_8 and remove it.



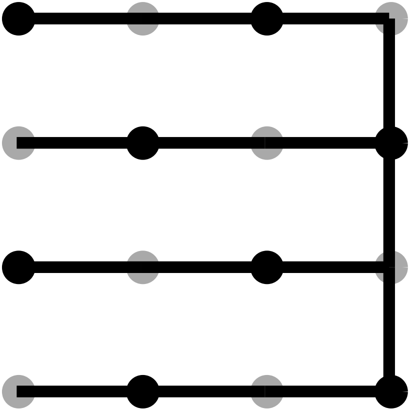
$T(4, 4) = ?$

Identify e_9 and remove it.



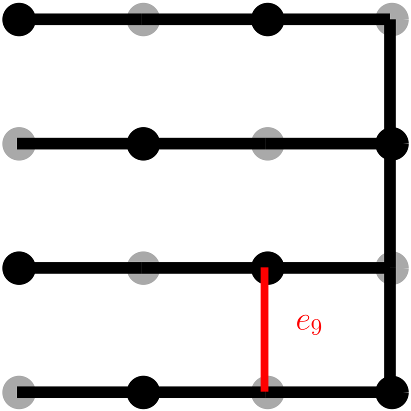
$T(4,4) = ?$

Assign +1 to all remaining edges. (Note: +1 and -1.)



$T(4,4) = ?$

Add e_9 back in. It must be -1.



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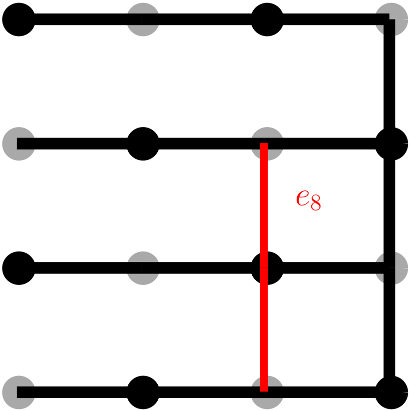

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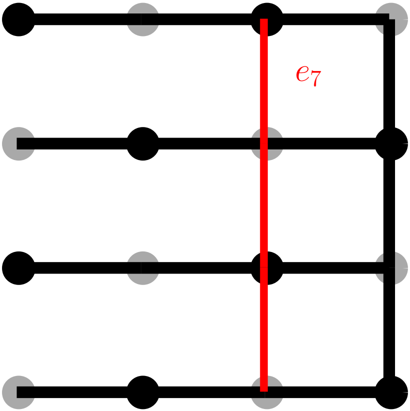

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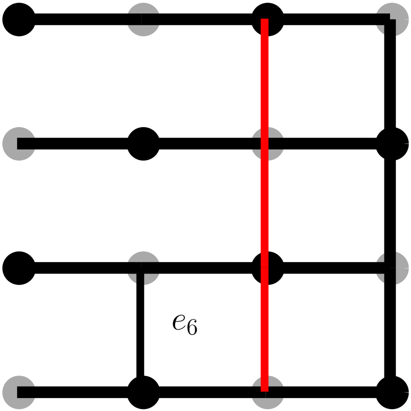

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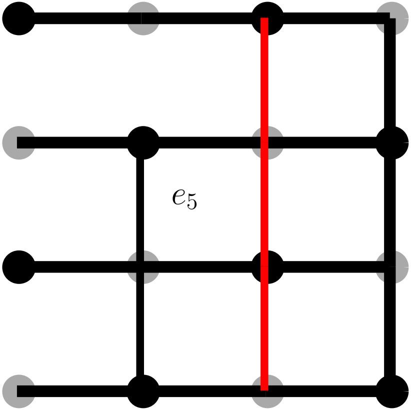

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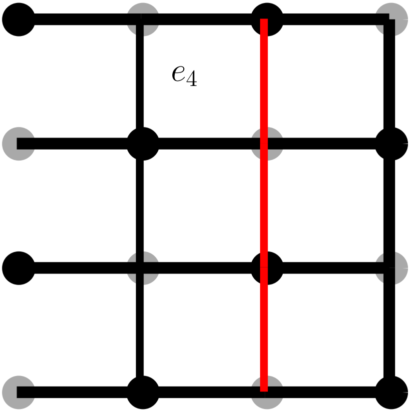

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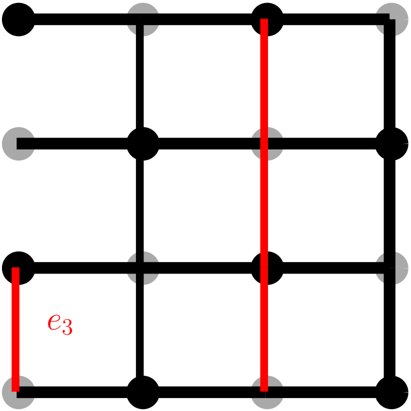
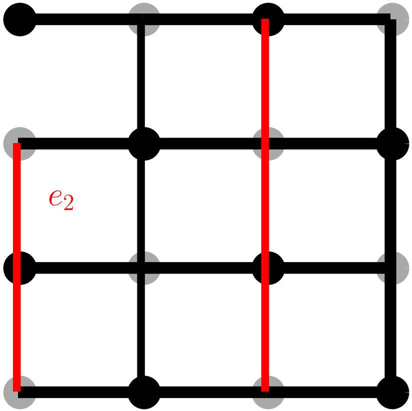


$T(4,4) = ?$

Add e_2 back in. It must be -1.



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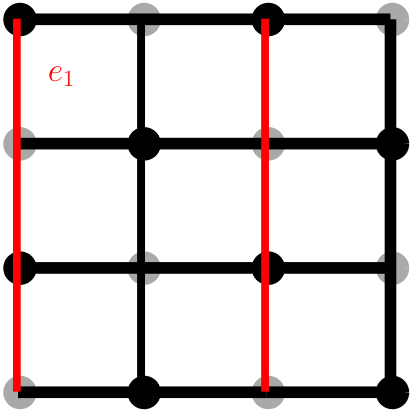
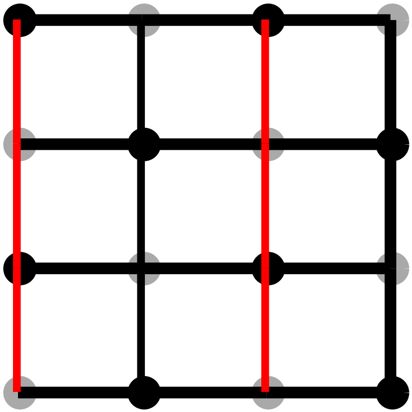


$T(4,4) = ?$

This is a Kasteleyn signing of G.



$T(4,4) = 36$

$$A^\sigma = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad \det(A^\sigma) = 36 = T(4,4)$$

Crazy product out of nowhere

Amazingly, there is a closed-form solution:

$$T(m,n) = \prod_{k=1}^{m} \prod_{\ell=1}^{n} \left| 2\cos\frac{k\pi}{m+1} + 2\mathrm{i}\cos\frac{\ell\pi}{n+1} \right|$$
$$= \prod_{k=1}^{m} \prod_{\ell=1}^{n} \left(4\cos^2\frac{k\pi}{m+1} + 4\cos^2\frac{\ell\pi}{n+1} \right)^{1/2}$$

Having this shortens the computation time required, of course. Deriving it involves several extra steps.

Cartesian products and eigenvalues

One can show that the adjacency matrices of grid graphs are actually adjacency matrices of the **Cartesian product** of two graphs: a $1 \times n$ row graph and an $m \times 1$ column graph.

The eigenvalues of those matrices are “easily” comptuable.

The determinant of a matrix is the product of its eigenvalues.

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This is explored through a series of problems, whose solutions are also available online [5].

This "ruins the fun" of finding $T(m, n)$ by hand, and doesn't belie any inherent structure/pattern to the problem.

Areas that are being/should be explored

- Hexagonal tilings: closed-form, patterns, etc.
- Random tilings: any regularity?
- Counting perfect matchings in *any* planar graph (Kasteleyn, the Pfaffian method)
- Applications to theoretical physics
- Accounting for isomorphic tilings
- Computational complexity of determinants and permanents
- Enumeration of tilings
- Analyzing closed form: patterns, asymptotics, etc.

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THANK YOU

☺