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Articles

HIGHER BLOCK IFS 2: RELATIONS BETWEEN IFS WITH DIFFERENT LEVELS OF MEMORY

RICHARD BEDIENT,[*] MICHAEL FRAME,[†,§] KEITH GROSS,[*]
JENNIFER LANSKI[*] and BRENDAN SULLIVAN[‡]

[*]*Mathematics Department, Hamilton College*
Clinton, NY 13323, USA

[†]*Mathematics Department, Yale University*
New Haven, CT 06520-8282, USA

[‡]*Mathematics Department, Carnegie-Mellon University*
Pittsburgh, PA 15213, USA
[§]*michael.frame@yale.edu*

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Abstract

We continue the program of Bedient *et al.*[1] by investigating some of the ways of embedding IFS with 1-step memory into IFS with 2-step memory, and 1- and 2-step memory into IFS with 3-step memory. This reveals a hierarchy of attractors of m-step memory IFS as subsets of attractors of n-step memory IFS.

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[§]Corresponding author. Partially supported by NSF DMS 0203203.

1. INTRODUCTION

An IFS has *memory* if only some compositions of the IFS transformations are allowed, all others being forbidden. Recall an IFS has *m-step memory* if there is a collection $\mathcal{F}$ of compositions, all of length $\le m+1$, satisfying two properties.

- Every forbidden composition contains some element of $\mathcal{F}$.
- At least one forbidden composition of length $m+1$ does not contain a forbidden composition of length j for all $j, 1 \le j \le m$.

Such a collection $\mathcal{F}$ is called a *generating set* of forbidden sequences. We call an IFS with m-step memory an m-IFS; a standard (memoryless) IFS is called a 0-IFS.

We build most IFS from four transformations, $I = \{T_1, T_2, T_3, T_4\}$, where

$$\begin{aligned} T_1(x,y) &= \left(\frac{x}{2}, \frac{y}{2}\right) + (0,0), \\ T_2(x,y) &= \left(\frac{x}{2}, \frac{y}{2}\right) + \left(\frac{1}{2}, 0\right), \\ T_3(x,y) &= \left(\frac{x}{2}, \frac{y}{2}\right) + \left(0, \frac{1}{2}\right), \\ T_4(x,y) &= \left(\frac{x}{2}, \frac{y}{2}\right) + \left(\frac{1}{2}, \frac{1}{2}\right). \end{aligned} \tag{1}$$

Denote by $I(\mathcal{F})$ the m-IFS with these transformations and forbidden compositions $\mathcal{F}$.

As a 0-IFS, the transformations (1) produce the filled-in unit square S. Forbidding some compositions of the T_i can generate remarkably intricate fractals. Here we study some relations between fractals generated by IFS using different lengths of memory.

Conditions under which a 1-IFS has the same attractor as a 0-IFS were derived in Ref. 2. A transformation T_i is a *rome* if, for all j, the composition $T_i \circ T_j$ is allowed. The main result of Ref. 2 is that the attractor of a 1-IFS also is the attractor of a 0-IFS if and only if

(1) the 1-IFS has at least one rome, and
(2) for each transformation T_{i_1}, there is an allowed composition $i_n \cdots i_1$, with T_{i_n} a rome.

Moreover, if these are no arbitrarily long allowed sequences of non-romes, the 0-IFS need have only a finite collection of transformations.

Recall that the allowed compositions of a 1-IFS can be encoded in a *transition matrix* $[m_{ij}]$, where $m_{ij} = 0$ if $T_i \circ T_j$ is forbidden, and $m_{ij} = 1$ if $T_i \circ T_j$ is allowed. Romes are recognized easily in the transition matrix: T_i is a rome if every entry of the ith row is 1.

Adapting Proposition 2.3.9 of Ref. 3, in Ref. 1 we show that the attractor of every m-IFS can be realized as the attractor of a 1-IFS. Denote by $\mathcal{F}$ a collection of strings of length $m+1$, so $I(\mathcal{F})$ is an m-IFS. The mth *higher block IFS* $I^{[m]}(\mathcal{F})$ has transformations

$$J = \{T_{i_m} \circ \cdots \circ T_{i_1} : i_n \dots i_1 j \notin \mathcal{F} \text{ for at least one of } j = 1,2,3, \text{ or } 4\}.$$

The allowed transitions are $T_{i_m} \circ \cdots \circ T_{i_1}$ follows $T_{j_m} \circ \cdots \circ T_{j_1}$ if and only if

$$i_{m-1} \cdots i_1 = j_m \cdots j_2, \quad \text{and} \tag{2}$$

$$i_m \cdots i_1 j_1 \notin \mathcal{F}. \tag{3}$$

Denoting by $\mathcal{F}'$ the forbidden pairs of transformations from J, we define

$$I^{[m]}(\mathcal{F}) = J(\mathcal{F}').$$

Then

Theorem 1.1. *The m-IFS $I(\mathcal{F})$ and the 1-IFS $I^{[m]}(\mathcal{F})$ have the same attractor.*

In Ref. 1 this theorem was used to compute the dimensions of attractors of m-IFS by using the formula of Ref. 4

$$\rho[m_{ij} r_j^d] = 1, \tag{4}$$

where $\rho(M)$ denotes the spectral radius (maximum magnitude of the eigenvalues) of the matrix M, and r_j is the contraction factor of the similarity transformation T_j. In the special case that all the $r_j = 1/2$, Eq. (4) simplifies to

$$1 = \rho([m_{ij}(1/2)^d]) = (1/2)^d \rho([m_{ij}]). \tag{5}$$

For use in Sec. 3 recall these results from Ref. 1.

Corollary 1.1. *The region $A_{i_q \dots i_1}$ is empty if and only if some substring of $i_q \cdots i_1$ belongs to some $\mathcal{F}$ determining this IFS.*

Lemma 1.1. *If* $\mathrm{int}(S_{i_n \cdots i_2 *}) = \emptyset$ *for* $* = 1,2,3,$ *and* 4, *then* $\mathrm{int}(S_{i_n \cdots i_2}) = \emptyset$.

Proof. Because $\mathrm{int}(S_{i_n \dots i_2 *}) = \emptyset$, the four compositions $T_{i_n} \circ \cdots \circ T_{i_2} \circ T_*$ are forbidden. Then the common edge of $S_{i_n \cdots i_2 1}$ and $S_{i_n \cdots i_2 2}$ is empty, as are the other three common edges of the $S_{i_n \cdots i_2 *}$, and so $\mathrm{int}(S_{i_n \cdots i_2}) = \emptyset$. ☐

Corollary 1.1 has a geometric characterization that emphasizes Mandelbrot's dictum "A fractal is as easily described by what has been removed as by what remains." For an m-IFS with attractor A let

$$\mathcal{E}(A) = \{(i_n \cdots i_1) : \mathrm{int}(S_{i_n \cdots i_1}) \cap A = \emptyset, \quad 1 \le n < \infty\}, \tag{6}$$

where $S_{i_n \cdots i_1} = T_{i_n} \circ \cdots \circ T_{i_1}(S)$, a subsquare of the filled-in unit square S. Then Corollary 1.1 can be restated as

$$\mathcal{E}(A) = \{(i_n \cdots i_1) : \text{a substring of } i_n \cdots i_1 \text{ belongs to } \mathcal{F}\},$$

where $\mathcal{F}$ is a generating set of forbidden sequences of the m-IFS.

If A is the attractor of an m-IFS and B is the attractor of an n-IFS, both built from the transformations (1), then certainly

$$A = B \quad \text{if and only if } \mathcal{E}(A) = \mathcal{E}(B). \tag{7}$$

2. SOME EXAMPLES OF 1-IFS AND 2-IFS

Recall that in the transition matrix representing a 1-IFS, the column index can be labeled as 'from' and the row index as 'to,' and recall that i is a rome if the ith row of the matrix contains no 0s. We recall an example of Ref. 2 to illlustrate this point.

Example 2.1. Four types of memory structures.

Figure 2 shows four 1-IFS attractors, determined by the transition matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Reading the allowed compositions is more easily done with *transition graphs*. In the language of subshifts,[3] ours are vertex graphs: each T_i is associated with a vertex of the graph, and an arrow between vertices represents an allowed composition. The transition graphs for these matrices are given in Fig. 1. In transition graphs, T_i is a rome if there is an arrow to vertex i from every vertex, including i.

In the first matrix, 1 and 4 are romes. There is only one sequence consisting entirely of nonrome states: $2 \to 3$, so the left side of Fig. 2 is the attractor of a 0-IFS with a finite number of transformations. In fact, eight are needed. These are the smallest set of compositions giving the *full addresses*, scaled copies of the attractor, that comprise the attractor. Here we see

$$A = T_1(A) \cup T_4(A) \cup T_2T_1(A) \cup T_2T_4(A) \cup T_3T_1(A) \cup T_3T_4(A) \cup T_3T_2T_1(A) \cup T_3T_2T_4(A),$$

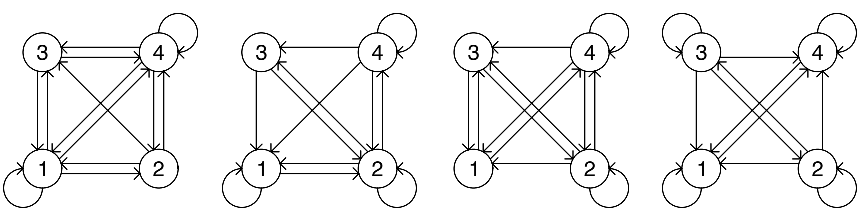

Fig. 1 Transition graphs for the 1-IFS of Example 2.1.

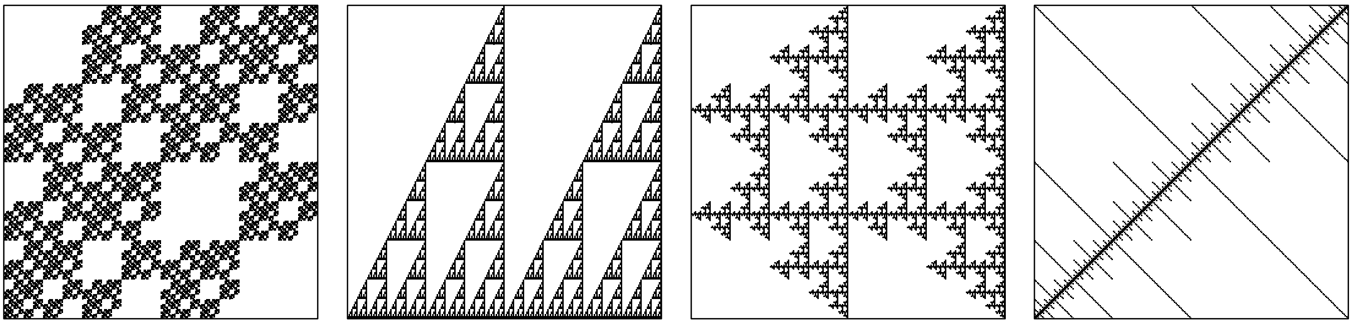

Fig. 2 Attractors of the 1-IFS determined by the matrices of Example 2.1.

that is, all allowed compositions beginning with T_1 or T_4 and containing only one factor of T_1 or T_4.

In the second matrix, 1 and 2 are romes, and there is a cycle $4 \to 4$ of nonromes, so the left middle of Fig. 2 is the attractor of a 0-IFS, but with infinitely many transformations:

$$T_1, T_2, T_3T_2, T_4T_2, T_3T_4T_2, T_4^2T_2, T_3T_4^2T_2, \ldots,$$
$$T_4^nT_2, T_3T_4^nT_2, \ldots$$

The third matrix has no romes, so the right middle of Fig. 2 is not the attractor of a 0-IFS.

In the fourth matrix, 1 and 4 are romes, but there is no path from a rome to 2 and 3, the nonromes. This violates the second condition of the memory-reduction result, and we observe the right of Fig. 2 is not the attractor of a 0-IFS. □

For the moment, eschewing a three-dimensional $4 \times 4 \times 4$ representation of a 2-IFS, we use a quadruple of 4×4 matrices in this order

$$[m_{ij1}] \quad [m_{ij2}] \quad [m_{ij3}] \quad [m_{ij4}], \tag{8}$$

where $m_{ijk} = 1$ if and only if the composition $T_i \circ T_j \circ T_k$ is allowed, that is, if the sequence $k \to j \to i$ is allowed. Otherwise, $m_{ijk} = 0$. We use $[m_{ijk}]$ as shorthand notation for Eq. (8) and still call it a transition matrix, although it is an array of matrices.

Example 2.2. A 2-IFS and equivalent 1-IFS.

On the left of Fig. 4 we see the attractor of the 2-IFS determined by these triples

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}. \tag{9}$$

In the middle of Fig. 4 we see the attractor with length 3 address squares superimposed. From this, we see the forbidden triples are

$$\mathcal{F}_4 = \{112, 141, 143, 224, 231, 232, 323, 324, 331, 412, 414, 443\}.$$

All pairs $T_i \circ T_j$ are allowed, so the 2nd higher block IFS has 16 transformations $T_1 \circ T_1, \ldots, T_4 \circ T_4$. The transition matrix is

$$\begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1
\end{bmatrix}. \tag{10}$$

Every 1 in this matrix requires two conditions. Three illustrations (both conditions are satisfied, one is satisfied but not the other) suffice.

Consider the 1 in row 4, column 14. Column 14 corresponds to $T_4 \circ T_2$ and row 4 to $T_1 \circ T_4$. This transition is allowed because the compositions agree on their overlap (the second function of $T_4 \circ T_2$ is the first function of $T_1 \circ T_4$), and the triple 142 is allowed. All the 0s imposed by this overlap condition are shown in Fig. 3. Rows and columns are labeled by the compositions: ij refers to $T_i \circ T_j$.

to \ from	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
11					0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0					0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0					0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0				
21					0	0	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0					0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0					0	0	0	0
24	0	0	0	0	0	0	0	0	0	0	0	0				
31					0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0					0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0					0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0				
41					0	0	0	0	0	0	0	0	0	0	0	0
42	0	0	0	0					0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0	0					0	0	0	0
44	0	0	0	0	0	0	0	0	0	0	0	0				

Fig. 3 The 0s imposed by the overlap condition in a 2nd higher block IFS.

Next, consider the 0 in row 4, column 15. Column 15 corresponds to $T_4 \circ T_3$. While $T_1 \circ T_4$ and $T_4 \circ T_3$ agree on their overlap, the triple 143 is forbidden.

The 0 in row 3, column 14 occurs because row 3 corresponds to $T_1 \circ T_3$, which does not agree with $T_4 \circ T_2$ on the overlap.

Finally, in generating the right image of Fig. 4, note that a 1 in the $T_i \circ T_j$ row and the $T_j \circ T_k$ column means not that $T_i \circ T_j$ follows $T_j \circ T_k$, but that T_i foollows $T_j \circ T_k$. Recall this 1 in the transition matrix corresponds to allowing the triple ijk.

The left side of Fig. 4 shows the attractor of this 2nd higher block IFS, giving another illustration of the method of Ref. 1.

Example 2.3. Another 2-IFS and equivalent 1-IFS.

On the left of Fig. 5 we see the attractor of the 2-IFS determined by these triples

$$\begin{bmatrix}1&0&1&0\\1&1&0&1\\0&0&1&1\\1&1&1&1\end{bmatrix} \quad \begin{bmatrix}0&1&1&1\\1&1&1&1\\1&1&0&1\\1&0&1&1\end{bmatrix}$$
$$\begin{bmatrix}1&1&0&1\\1&0&1&1\\1&1&1&1\\1&1&1&0\end{bmatrix} \quad \begin{bmatrix}1&1&1&1\\1&1&0&0\\1&0&1&1\\0&1&0&1\end{bmatrix} \qquad (11)$$

In the middle of Fig. 5 we see the attractor with length 3 address squares superimposed. From this, we see the forbidden triples are

$$\mathcal{F}_4 = \{112, 121, 133, 141, 223, 231, 234, 244, 311, 321, 332, 324, 414, 422, 434, 443\}.$$

All pairs $T_i \circ T_j$ are allowed, so the 2nd higher block IFS has 16 transformations $T_1 \circ T_1, \ldots, T_4 \circ T_4$. The transition matrix is

$$\begin{bmatrix}
1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0\\
0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1
\end{bmatrix}. \qquad (12)$$

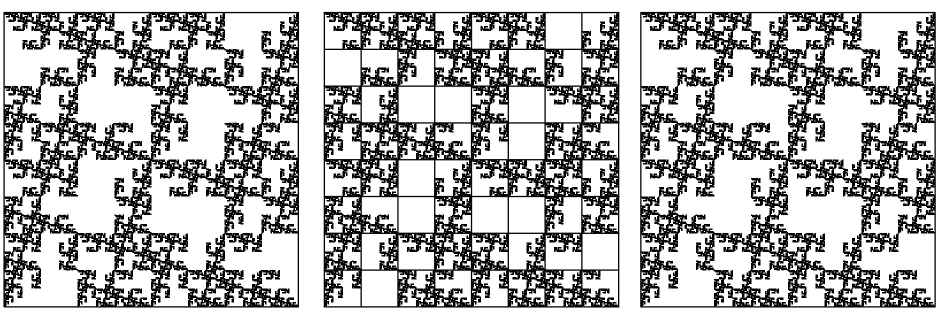

Fig. 4 **Left:** The attractor of the 2-IFS with transitions given by Eq. (9). **Middle:** The attractor with length 3 address squares superimposed. **Right:** The attractor of the equivalent 2nd higher block IFS.

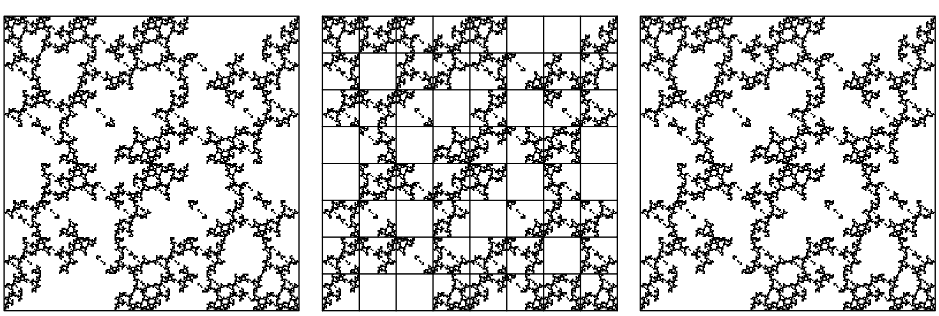

Fig. 5 **Left:** The attractor of the 2-IFS with transitions given by Eq. (11). **Middle:** The attractor with length 3 address squares superimposed. **Right:** The attractor of the equivalent 2nd higher block IFS.

The left side of Fig. 5 shows the attractor of this 2nd higher block IFS, giving another illustration of the method of Ref. 1.

The attractor of Fig. 4 contains two obvious triples of lines, one triple with slope $-1/3$, the other with slope 3. For each triple, there is a 2-IFS with attractor equal to that triple (allow exactly those compositions corresponding to the occupied length 3 address squares), but no member of these triples can be generated without the other two, using 2-IFS with transformations (1).

Example 2.4. A 3 line attractor.

In Fig. 4, examining the length 3 address squares through which the triple with slope $-1/3$ passes, we are led to this 2-IFS.

$$\begin{bmatrix}0&0&1&0\\0&1&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&1&0\\1&1&0&1\\1&1&0&1\\0&0&1&0\end{bmatrix}$$
$$\begin{bmatrix}0&1&0&0\\1&0&1&1\\1&0&1&1\\0&1&0&0\end{bmatrix} \quad \begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&1&0\\0&1&0&0\end{bmatrix}. \tag{13}$$

Indeed, this 2-IFS generates these lines. See the left side of Fig. 6.

This triple of lines cannot be generated by a 1-IFS using transformations (1). The occupied length 2 addresses give this transition matrix

$$\begin{bmatrix}0&1&1&0\\1&1&1&1\\1&1&1&1\\0&1&1&0\end{bmatrix},$$

giving the IFS attractor on the right side of Fig. 6. The triple of lines is superimposed for reference.

On the other hand, Theorem 4.1 of Ref. 1 guarantees that the 2-IFS (13) can be generated by a 1-IFS. From the middle of Fig. 6 we see that the forbidden pairs are

$$11, 14, 41, 44.$$

In additon to these, the forbidden triples are

$$121, 122, 133, 134, 223, 224,$$
$$231, 232, 323, 324, 331, 332,$$
$$421, 422, 433, 434.$$

(Note the triples 211, 214, 241, 244, 311, 314, 341, 344 are forbidden, but these are empty as a consequence of the forbidden pairs.) The 2nd higher block IFS has transformations $S_k = T_i \circ T_j$, where ij is the kth element of

$$\mathcal{W} = \{12, 13, 21, 22, 23, 24, 31, 32, 33, 34, 42, 43\}.$$

and transition matrix

$$\begin{bmatrix}0&0&0&0&1&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&1&0&0&0&0\\1&1&0&0&0&0&0&0&0&0&0&0\\0&0&1&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&1&0&0\\0&0&0&0&0&0&0&0&0&0&1&1\\1&1&0&0&0&0&0&0&0&0&0&0\\0&0&1&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&1&0&0\\0&0&0&0&0&0&0&0&0&0&1&1\\0&0&0&0&1&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&1&0&0&0&0\end{bmatrix}. \tag{14}$$

The 1-IFS using the transformations of $\mathcal{W}$ and the transition matrix (14) generates the three lines seen on the right side of Fig. 6.

In each of Examples 2.2–2.4, the dimension d of the attractor is the solution of Eq. (5) where M is the transition matrix (10), (12), and (14), respectively. The spectral radii are 3.21432, 2.95748, and

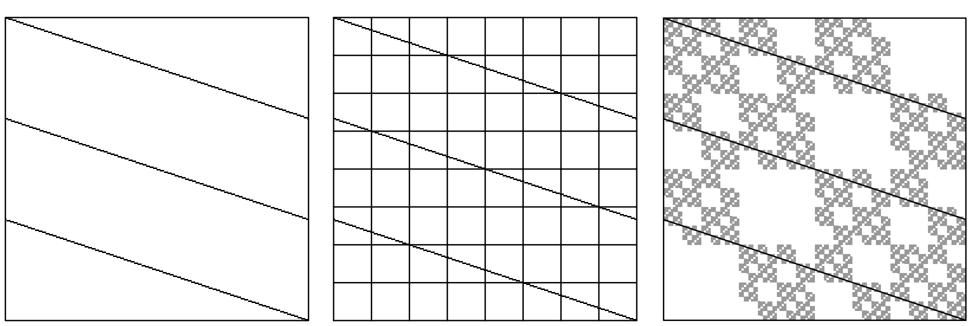

Fig. 6 **Left:** The attractor of the 2-IFS with transitions given by Eq. (13). **Middle:** The attractor with length 3 address squares superimposed. **Right:** The 1-IFS with allowed pairs implied by the triple of lines on the left.

2, giving dimensions

$$d = 1.68451, \quad d = 1.56437, \quad \text{and} \quad d = 1.$$

Certainly, the attractor of Fig. 4 appears to be more filled-in, and so should have a higher dimension, than that of Fig. 5. The attractor of Fig. 6 consists of three straight line segments, so of course has $d = 1$.

3. SOME SIMPLE RELATIONS BETWEEN 2-IFS AND 1-IFS

Now we investigate three ways in which the attractor of a 2-IFS can be related to the attractor of a 1-IFS. These results can be generalized to relations between attractors of m-IFS and $(m-1)$-IFS.

First we note the most elementary way in which a 2-IFS attractor is also a 1-IFS attractor. Put another way, Prop. 3.1 demonstrates that every 1-IFS attractor is also the attractor of some 2-IFS.

Proposition 3.1. *For all $i, j = 1, 2, 3,$ and 4 suppose $a_{ij1} = a_{ij2} = a_{ij3} = a_{ij4}$. Then the 2-IFS with transition matrix $[a_{ijk}]$ has the same attractor as the 1-IFS with transition matrix $[b_{ij}]$, where each $b_{ij} = a_{ij1}$.*

Proof. Suppose A is the attrctor of the 2-IFS with transition matrix $[a_{ijk}]$ and B is the attractor of the 1-IFS with transition matrix $[b_{ij}]$. Following Eq. (7), it suffices to show $\mathcal{E}(A) = \mathcal{E}(B)$.

First, suppose $(i_n \cdots i_1) \in \mathcal{E}(A)$. Then some segment of $(i_n \cdots i_1)$, say $i_{j+1}i_ji_{j-1}$, belongs to $\mathcal{F}(A)$, the set of length 3 forbidden words determined by the transition matrix A. (The corresponding result for shorter sequences follows easily.) Then $a_{i_{j+1}i_ji_{j-1}} = 0$. By hypothesis, $b_{i_{j+1}i_j} = 0$. That is, $(i_{j+1}i_j) \in \mathcal{F}(B)$, the set of length 2 forbidden words determined by the transition matrix B. It follows from Corollary 1.1 that $(i_n \cdots i_1) \in \mathcal{E}(B)$.

Now suppose $(i_n \cdots i_1) \in \mathcal{E}(B)$. Then some segment of $(i_n \cdots i_1)$ belongs to $\mathcal{F}(B)$. That is, for some $j, b_{i_{j+1}i_j} = 0$. Then by hypothesis, $a_{i_{j+1}i_j*} = 0$, where $*$ stands for each of $1, 2, 3,$ and 4. From Lemma 1.1 we see that $\mathrm{int}(S_{i_{j+1}i_j})$ lies in the complement of A. From Corollary 1.1 we deduce that $(i_n \ldots i_1) \in \mathcal{E}(A)$. □

Example 3.1 illustrates Proposition 3.1.

Example 3.1. 1-IFS and 2-IFS having the same attractor.

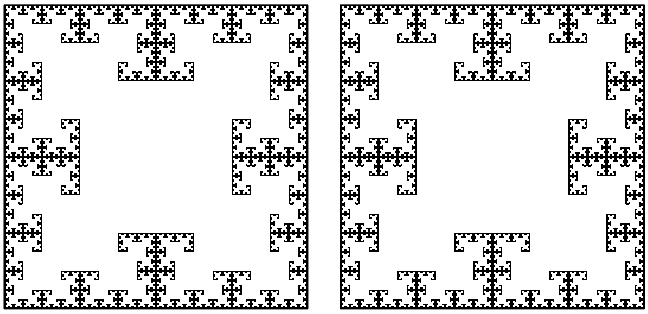

Fig. 7 The attractor of the 2-IFS **(left)** and 1-IFS **(right)** of Example 3.1.

Figure 7 shows the attractor of the 2-IFS with this transition matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}. \tag{15}$$

Note the forbidden triples are $41*, 32*, 23*,$ and $14*$. This attractor also is produced by the 1-IFS with this matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}. \tag{16}$$

Note the forbidden pairs are 41, 32, 23, and 14, illustrating Prop 3.1. Because the compositions $T_1 \circ T_4, T_2 \circ T_3, T_3 \circ T_2$ and $T_4 \circ T_1$ are forbidden, the 2nd higher block IFS has 12 transformations. The 0s imposed by the overlap condition are the only 0s in the transition matrix M. This is a consequence of the particular form of the equivalence of the 2-IFS and the 1-IFS. The eigenvalues of M are -1, eight 0s, 1, 1, and 3, so by Eq. (5), the dimension of the attractor is $\log_2(3)$. □

The information of a 1-IFS can be encoded into a 2-IFS in two additional ways. These are the topics of the next two examples.

Proposition 3.2. *For all $j, k = 1, 2, 3,$ and 4 suppose $a_{1jk} = a_{2jk} = a_{3jk} = a_{4jk}$. Let A denote the attractor of the 2-IFS with transition matrix $[a_{ijk}]$ and B the attractor of the 1-IFS with transition matrix $[b_{jk}]$, where each $b_{jk} = a_{1jk}$. Then*

$$A = T_1(B) \cup T_2(B) \cup T_3(B) \cup T_4(B).$$

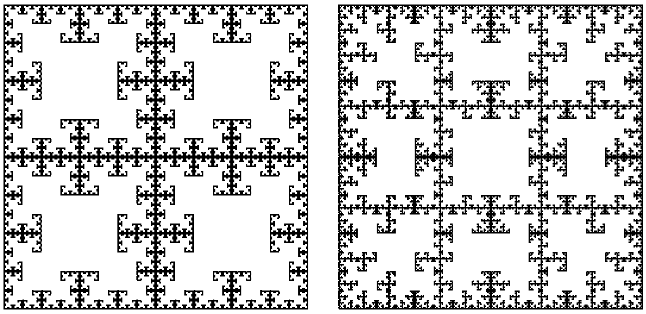

Fig. 8 The attractors of Examples 3.2 **(left)** and 3.3 **(right)**.

Proof. This is straightforward: the forbidden pairs jk determine B, and copies of the corresponding B_{jk} with empty interiors are placed in each of A_1, A_2, A_3, and A_4 by the forbidden triples $*jk$. That is, $(jk) \in \mathcal{E}(B)$ if and only if $(*jk) \in \mathcal{E}(A)$. □

Example 3.2 illustrates Proposition 3.2.

Example 3.2. A 2-IFS with attractor consisting of four copies of the attractor of a 1-IFS.

The left side of Fig. 8 shows the attractor of the 2-IFS with this matrix. Note the forbidden triples are $*41, *32, *23$, and $*14$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}. \tag{17}$$

This attractor consists of four copies of the 1-IFS attractor shown on the right side of Fig. 7. Note the forbidden pairs are $41, 32, 23$, and 14, illustrating Proposition 3.2. Because all compositions $T_i \circ T_j$ are allowed, the 2nd higher block IFS has 16 transformations. The eigenvalues of the transition matrix are -1, twelve 0s, $1, 1$, and 3, so the dimension of the attractor is $\log_2(3)$, hardly a surprise because the attractor of Example 3.2 is just four copies of the attractor of Example 3.1. □

Thinking of the 3-IFS transition matrix as a $4 \times 4 \times 4$ cubical array, Example 3.1 can be viewed as building that array by placing four copies of the 4×4 1-IFS matrix parallel to the front face of the cubical array. In Example 3.2 the 4×4 array is placed parallel to the top face of the cube. The last possibility, shown in Example 3.3, is with the 4×4 array placed parallel to the left face of the cube. See Fig. 9.

Example 3.3. The third relationship between a_{ijk} and b_{ij}.

The right side of Fig. 8 shows the attractor of the 2-IFS with this matrix. Note the forbidden triples are $4*1, 3*2, 2*3$, and $1*4$,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{18}$$

Because the forbidden pairs of the 1-IFS are split by the $*$, there is no clear relation between the attractors of the right sides of Figs. 7 and 8. Because all compositions $T_i \circ T_j$ are allowed, the 2nd higher block IFS has 16 transformations. The eigenvalues of the transition matrix are -1, twelve 0s, $1, 1$, and 3, so the dimension of the attractor is $\log_2(3)$,

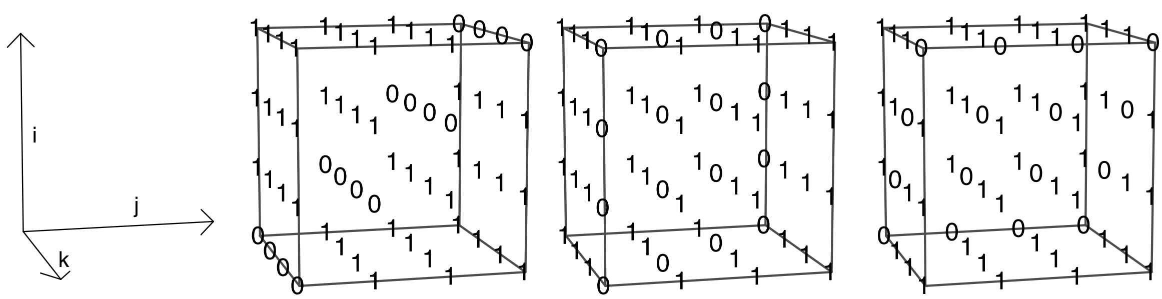

Fig. 9 **Left:** Orientation of the transition cube for the composition $T_i \circ T_j \circ T_k$. **Left middle, right middle, and right:** cubical representations of the transition matrices of Examples 3.1–3.3, respectively.

a bit less obvious than in comparing Examples 3.1 and 3.2. □

4. REPRESENTING 3-IFS, AND SOME EXAMPLES

A 3-IFS using transformations (1) can be represented by a $4 \times 4 \times 4 \times 4$ array, which we display as a 4×4 array of 4×4 arrays,

$$\begin{matrix} [m_{ij11}] & [m_{ij12}] & [m_{ij13}] & [m_{ij14}] \\ [m_{ij21}] & [m_{ij22}] & [m_{ij23}] & [m_{ij24}] \\ [m_{ij31}] & [m_{ij32}] & [m_{ij33}] & [m_{ij34}] \\ [m_{ij41}] & [m_{ij42}] & [m_{ij43}] & [m_{ij44}] \end{matrix}, \tag{19}$$

where $m_{ijkl} = 1$ if and only if the composition $T_i \circ T_j \circ T_k \circ T_l$ is allowed, otherwise $m_{ijkl} = 0$.

With each additional step of memory, a larger collection of attractors can be produced. Figure 10 shows two examples. The transition arrays can be deduced from the empty length 4 address squares.

The memory-reduction method of Ref. 1 can be applied here. Note the overlap condition is more demanding: $T_a \circ T_b \circ T_c$ can follow $T_i \circ T_j \circ T_k$ if $b = i, c = j$, and $abck$ is an allowed quadruple. The 3rd block IFS requires as many as 64 transformations $T_i \circ T_j \circ T_k$, but because of the overlap condition, at most 16×16 entries of the 64×64 transition matrix are non-zero.

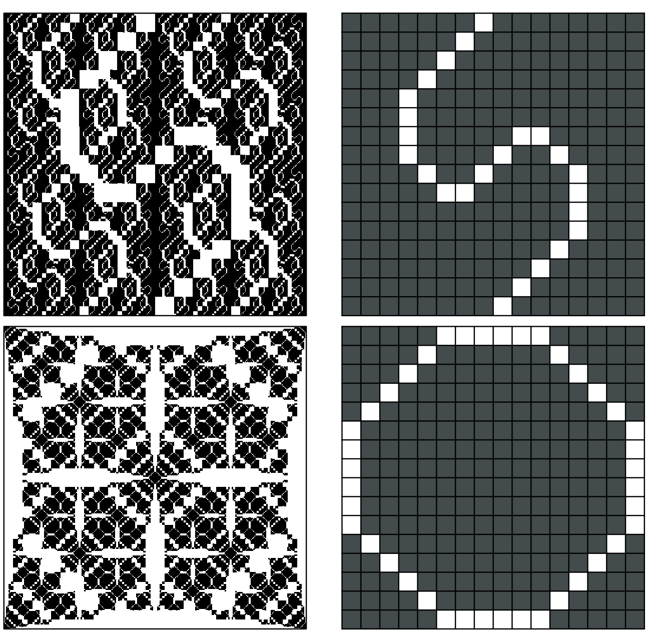

Fig. 10 Two examples of 3-IFS **(left)**, with the empty length 4 address squares indicated **(right)**.

5. SOME SIMPLE RELATIONS BETWEEN 3-IFS AND 1- AND 2-IFS

Representing a 2-IFS transition array as a $4 \times 4 \times 4$ cube, in Sec. 3 we saw a 1-IFS transition matrix can be embedded in a $4 \times 4 \times 4$ array in three obvious ways, copies parallel to each of the three faces of the cube.

Similarly, a 3-IFS transition array can be represented as a $4 \times 4 \times 4 \times 4$ hypercube, and a 2-IFS transition array can be embedded in a $4 \times 4 \times 4 \times 4$ array in four obvious ways, copies parallel to the cubical faces of the hypercube. For one embedding, the 3-IFS attractor is identical to that of the 2-IFS, for another, the 3-IFS attractor consists of four copies of the 2-IFS attractor; the other two placements are more complicated.

For example, Fig. 11 shows these four embeddings of the attractor of the 2-IFS determined by the transition array a_{ijk} given by Eq. (9). At the top left the transition array of the 3-IFS is a_{ijk*}, top right a_{ij*k}, bottom left a_{i*jk}, and bottom right a_{*ijk}.

As expected, the a_{ijk*} embedding produces the same attractor as the 2-IFS, and the a_{*ijk} embedding produces four copies of the 2-IFS attractor.

Similarly, 1-IFS can be embedded in 3-IFS in six obvious ways because there are six placements of

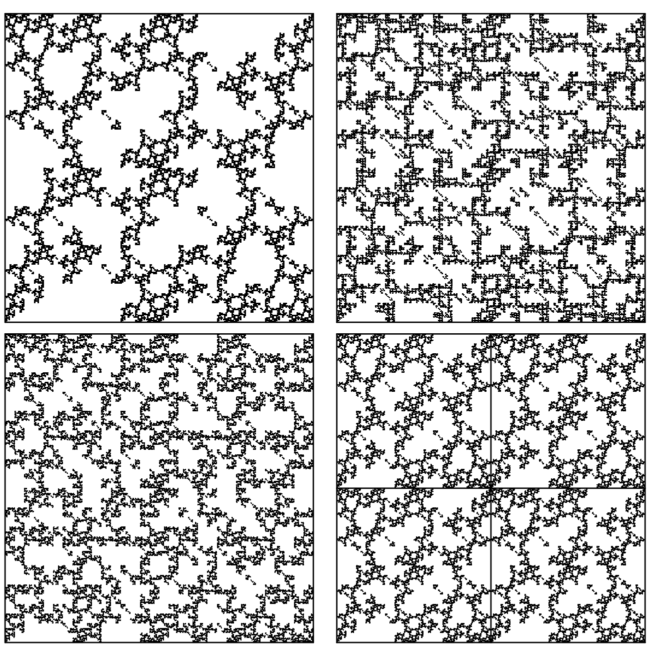

Fig. 11 Four embeddings of a 2-IFS into a 3-IFS. The lines superimposed on the lower right figure show it consists of four copies of the upper left figure.

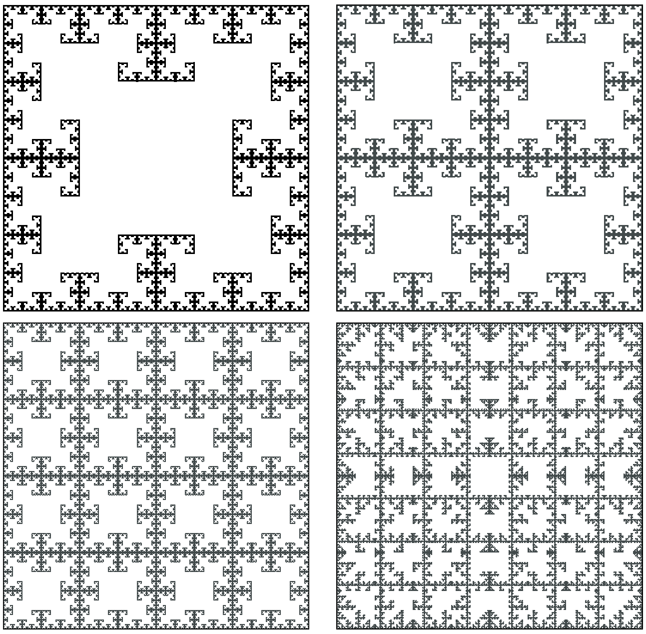

Fig. 12 Four embeddings of a 1-IFS into a 3-IFS.

two free addresses among the four that specify a 3-IFS. For example, Fig. 12 shows four embeddings of the attractor of the 1-IFS determined by the transition array a_{ij} given by (16). At the top left the transition array of the 3-IFS is a_{ij**}, top right a_{*ij*}, bottom left a_{**ij}, and bottom right a_{i**j}.

As expected, the a_{ij**} embedding produces the same attractor as the 1-IFS, the a_{*ij*} embedding produces four copies of the 1-IFS attractor, and the a_{**ij} embedding produces 16 copies.

These examples are instances of this more general result. Suppose an m-IFS has attractor A and generating set $\mathcal{F}$ of forbidden strings of length $m+1$. For eveny $n > m$ and for non-negative a and b with $n - m = a + b$, denote by

$$(*^a)\mathcal{F}(*^b) = \{(*^a)\mu(*^b) : \mu \in \mathcal{F}\}$$

the generating set of forbidden strings for an n-IFS. The attractor of this n-IFS consists of 4^a copies of A, each scaled by a factor of 2^a.

What can be achieved by other embeddings, along hypercube diagonals instead of edges?

6. CONCLUSION

We have investigated some relations between the attractors of IFS with different lengths of memory. For example, recalling an n-IFS is determined by an $(n+1)$-dimensional array,

$$m_{i_{n+1}\cdots i_1} = \begin{cases} 1 & \text{if } T_{i_{n+1}} \circ \cdots \circ T_{i_1} \text{ is allowed,} \\ 0 & \text{if } T_{i_{n+1}} \circ \cdots \circ T_{i_1} \text{ is forbidden,} \end{cases}$$

the attractor of the n-IFS with

$$m_{*\cdots *i_{k+1}\cdots i_1 *\cdots *} = m_{i_{k+1}\cdots i_1}$$

consists of 4^j copies of the attractor of the k-IFS with transition array $[m_{i_{k+1}\cdots i_1}]$, where j is the number of $*$s to the left of i_{k+1}. Other embeddings give more complex variations of the lower-dimensional attractor, perhaps a kind of geometric convolution, though we have not yet explored this direction.

Besides answering a topological question, why should we be interested in reducing the memory of an IFS? One example can be found in parsing driven IFS. For a function $f : [0,1] \to [0,1]$ having a single critical point whose iterates form a finite set C, dividing $[0,1]$ into subintervals determined by the points of C gives a Markov partition of $[0,1]$ with respect to f. Construct an IFS with the number of transformations equal to the number of subintervals in the partition of $[0,1]$ and satisfying the open set condition. Then select the order of IFS transformations by the order in which the iterates $f^n(x_0)$ visit the subintervals gives a *driven IFS*. Under these conditions, the driven IFS is a 1-IFS. Now an experimentally determined data set $\{x_1, \ldots, x_N\}$, sequential in time or some other natural ordering, has range $[\min(x_i), \max(x_i)]$ that can be divided into subintervals in many ways. Without knowledge of the underlying dynamics, a natural choice of subintervals may be elusive, discouragingly so if the goal of the driven IFS analysis is to deduce properties of the dynamical process that generated the data set. Arbitrary subdivision usually gives rise to driven IFS with longer memory effects. In Ref. 5 we apply the memory-reduction technique developed here to data-driven IFS, with the goal of uncovering some properties of the dynamics generating the data.

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