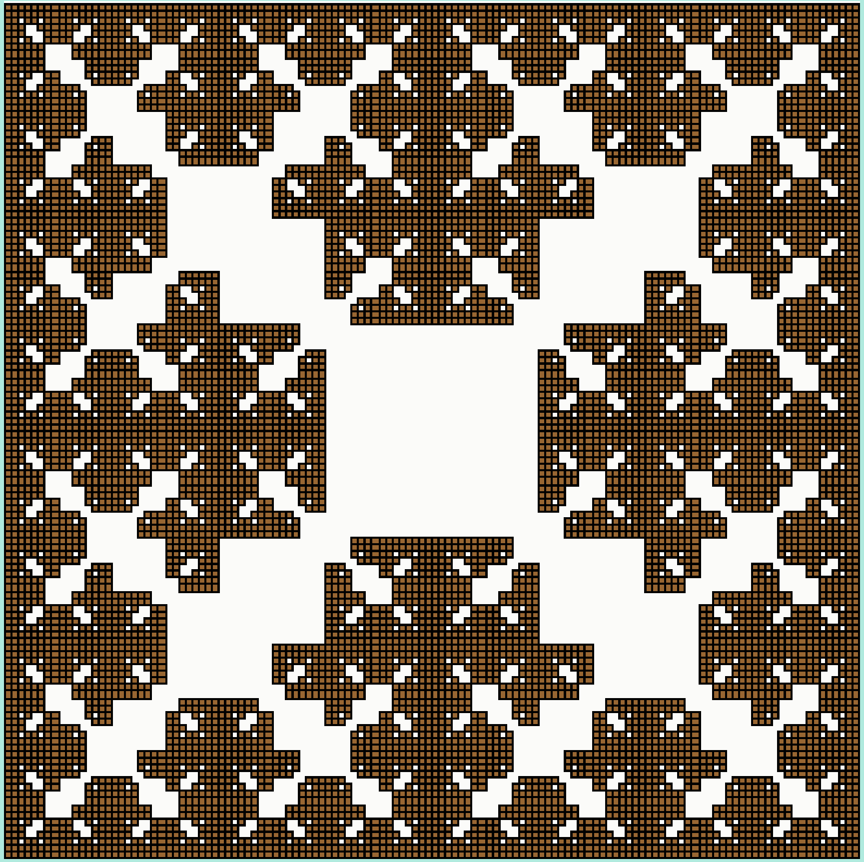
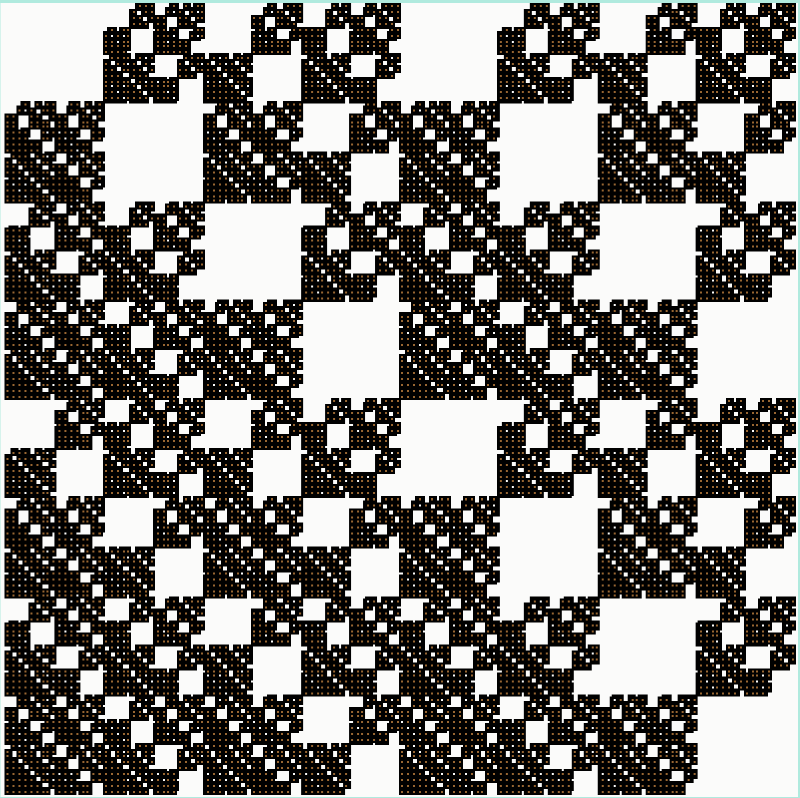
IFS with Memory





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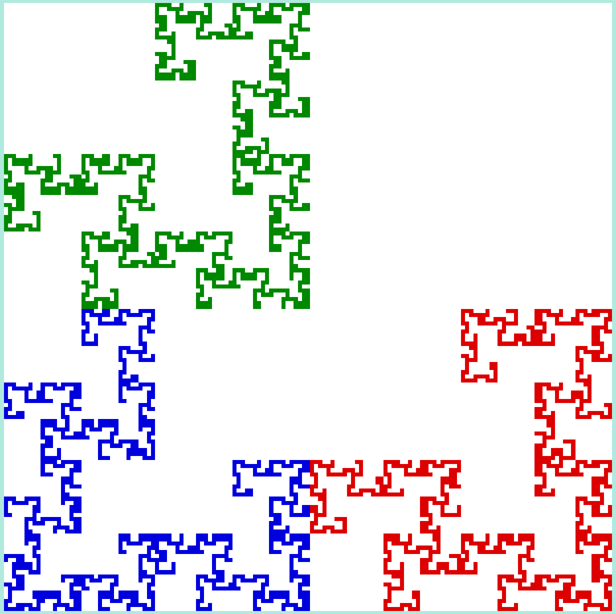
Iterated Function Systems (IFS)

- Set of transformations from $\mathbf{R}^2$ to $\mathbf{R}^2$
  - Contractions, rotations, translations

	r	s	θ	φ	e	f
T_1	½	½	0	0	0	0
T_2	½	½	90	90	1	0
T_3	½	½	90	90	½	½



Uniqueness of Attractors

- For every set of transformations *T*, there exists a unique nonempty, compact subset of $\boldsymbol{R}^2$ that is fixed by *T*
  - This is called the *attractor* of *T*
- Random IFS converges to the same attractor at infinity
  - Varying probabilities only affects the “rate” of convergence

What Convergence Means

- Continually applying each transformation to the previous generation yields the attractor (“at ∞”)

$$
\begin{aligned}
&B, \\
&\bigcup_{i=1}^{n} T_i(B), \\
&\bigcup_{j=1}^{n} \bigcup_{i=1}^{n} T_j(T_i(B)), \\
&\bigcup_{k=1}^{n} \bigcup_{j=1}^{n} \bigcup_{i=1}^{n} T_k(T_j(T_i(B))) \\
&\ldots
\end{aligned}
$$

- Let’s see an animation of convergence:

http://classes.yale.edu/fractals/IntroToFrac/IFS/GasketCat.html#GCAnchor

What Else Can We Do?

- We know we can get different fractals by changing the transformations
  - We can also change the # of transformations
- What if we keep the same set of transformations but restrict the order in which they can be applied?
  - Can we get new fractals?
  - Does this add anything to the big picture?

Our Standard IFS

- 4 transformations:
- *T* fixes the unit square, *S*
  - *S* is the *attractor* of *T*
- Applying transformation *k* = "being in state *k*"
- A sequence of transitions is written as, for example:

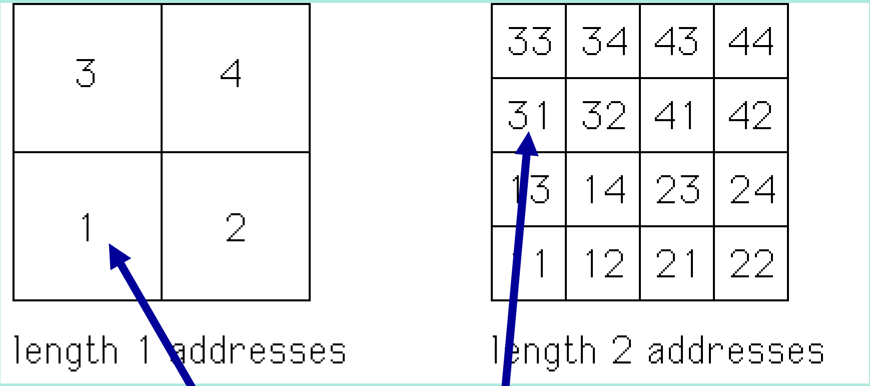
  $1 \rightarrow 3 \rightarrow 3 \rightarrow 4 \rightarrow 2$

$$\mathcal{T} = \{T_1, T_2, T_3, T_4\}$$
$$T_1(x,y) = (\tfrac{x}{2}, \tfrac{y}{2}) + (0,0)$$
$$T_2(x,y) = (\tfrac{x}{2}, \tfrac{y}{2}) + (\tfrac{1}{2}, 0)$$
$$T_3(x,y) = (\tfrac{x}{2}, \tfrac{y}{2}) + (0, \tfrac{1}{2})$$
$$T_4(x,y) = (\tfrac{x}{2}, \tfrac{y}{2}) + (\tfrac{1}{2}, \tfrac{1}{2})$$

	r	s	θ	φ	e	f
T_1	½	½	0	0	0	0
T_2	½	½	0	0	½	0
T_3	½	½	0	0	0	½
T_4	½	½	0	0	½	½

Addresses & Sequences

- This is how we gather information about the fractals we produce



333	334	343	344	433	434	443	444
331	332	341	342	431	432	441	442
313	314	323	324	413	414	423	424
311	312	321	322	411	412	421	422
133	134	143	144	233	234	243	244
131	132	141	142	231	232	241	242
113	114	123	124	213	214	223	224
111	112	121	122	211	212	221	222

$$\begin{aligned} S_{i_n i_{n-1} \ldots i_1} &= T_{i_n}(S_{i_{n-1} \ldots i_1}) \\ &= T_{i_n} T_{i_{n-1}}(S_{i_{n-2} \ldots i_1}) \\ &= \ldots \\ &= T_{i_n}(T_{i_{n-1}}(\ldots T_{i_1}(S))) \end{aligned}$$

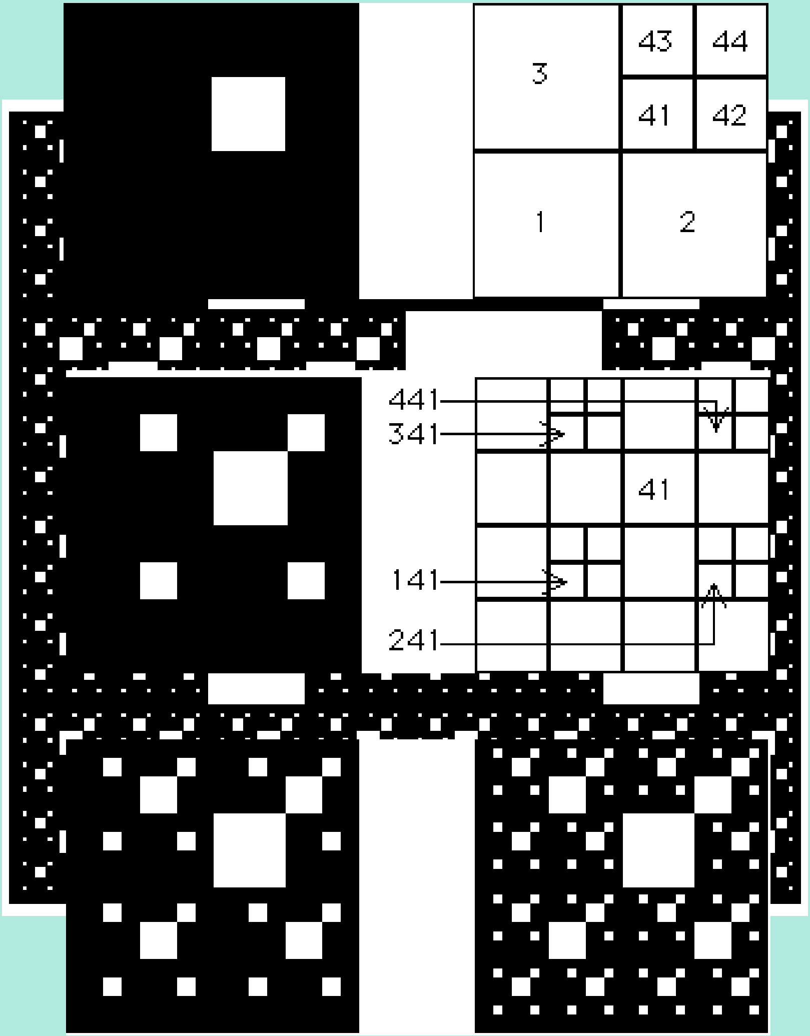
$$T_1(S) = S_1$$

$$i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_{n-1} \rightarrow i_n$$

Forbid a Pair of Transformations

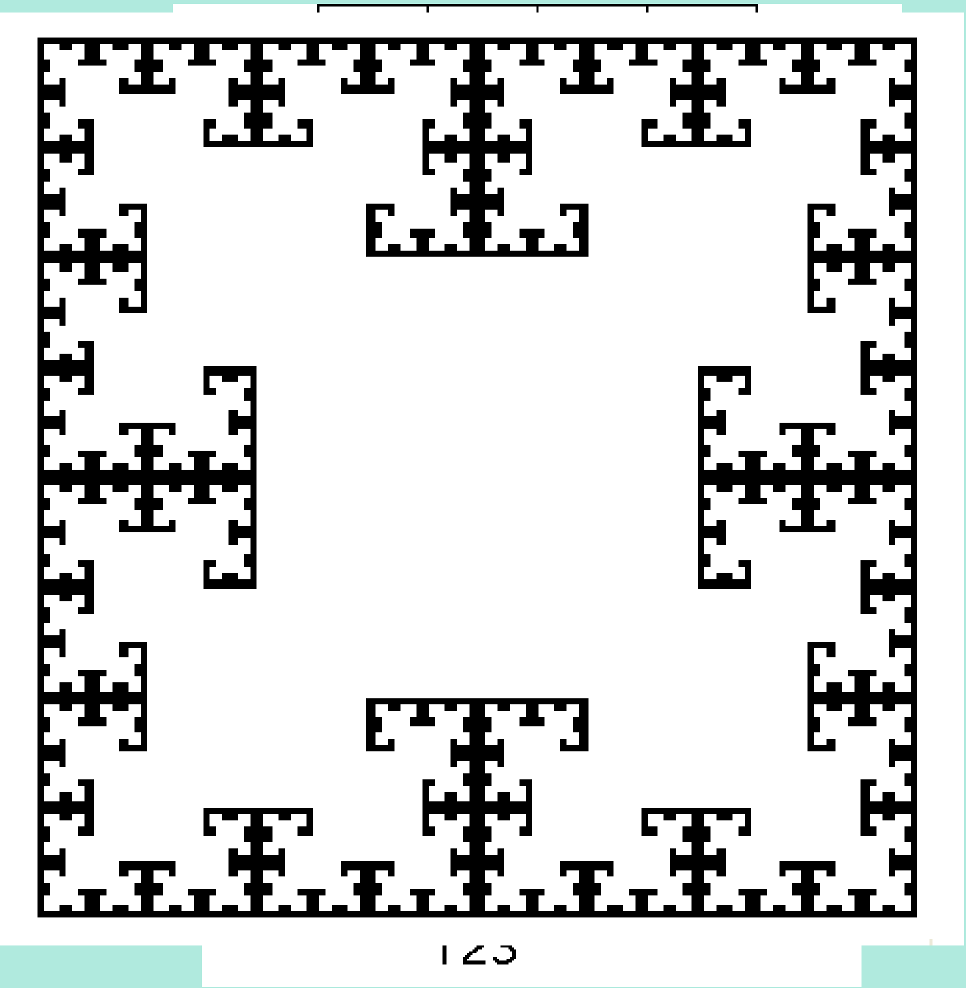
- For example: T_4 never immediately follows T_1
  - If we're in state 1, we can't enter state 4
- This is akin to restricting the allowed sequences of transitions so that we never see:

  .... $\rightarrow 1 \rightarrow 4 \rightarrow$ ....
- Equivalent statement about addresses:
  - Any box with address … 41… is empty



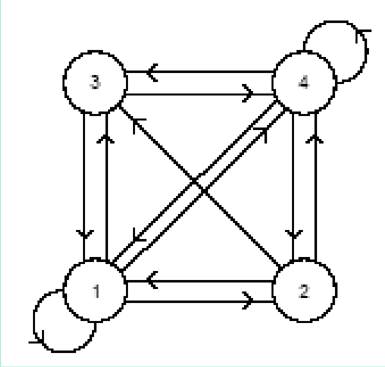
Forbid Multiple Pairs

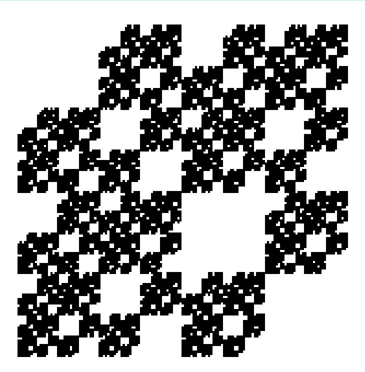
- For example:
  - T_1 never immediately follows T_4
  - T_2 never immediately follows T_3
  - T_3 never immediately follows T_2
  - T_4 never immediately follows T_1

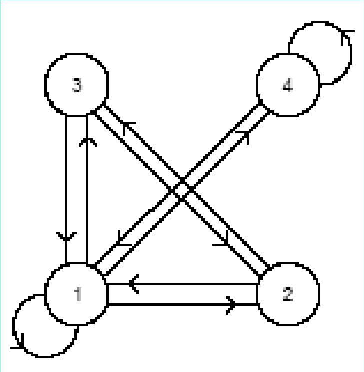


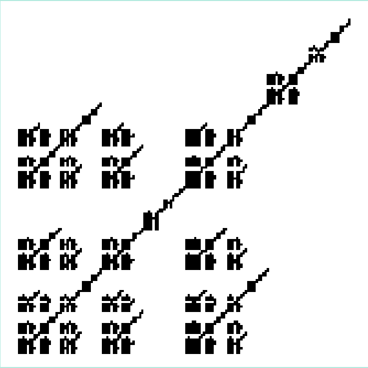
Transition Graph

- This is a way to visualize the allowed transitions
  - Vertices represent each state
  - Directed edges represent the allowed transitions
- For example:





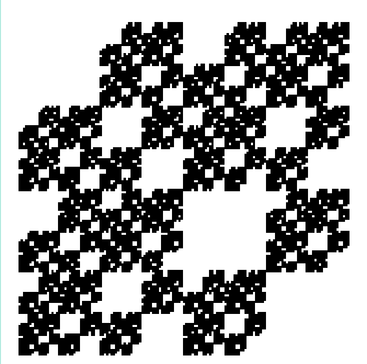


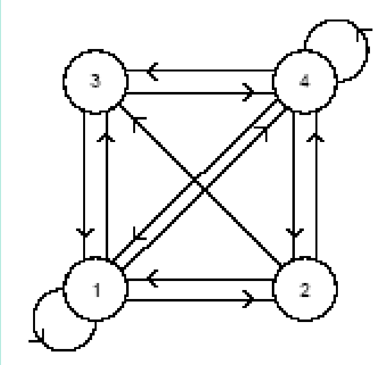


Probability Matrix

- This is another way of encoding which transitions are allowed and which are forbidden
- *P* is an *n* x *n* matrix for *T* with *n* transformations
  - j → k is allowed iff $P_{jk} > 0$
- This can be simplified by knowing that the probabilities do not matter
  - Will converge to the same attractor as the deterministic rIFS
  - We only care whether P_{jk} is 0 or not

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/2 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$





Classifying Attractors

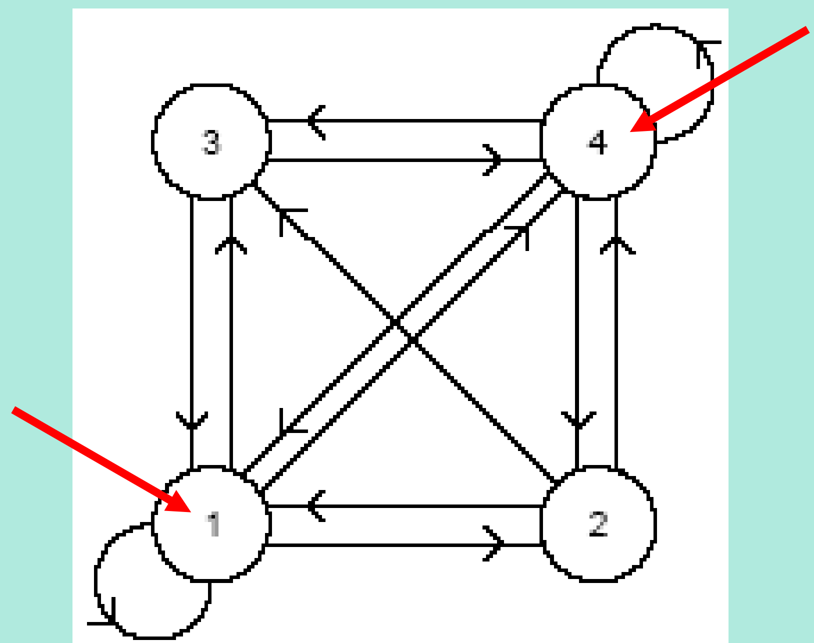
- When do we get a fractal that can be produced by a standard IFS without forbidden transitions?
  - We call such an attractor *IFS-able*
- Might there be an infinite # of transformations?
  - We call such an attractor *Infinitely IFS-able*
- When do we get a fractal that *cannot* be produced by a standard IFS?
  - We call such an attractor *Non-IFS-able*
- How can we determine the answer by looking at the transition graph and/or probability matrix?

Full States

- We say that the state k is *full* iff k can immediately follow any other transition:
  - All of $1 \to k$, $2 \to k$, $3 \to k$, $4 \to k$ are allowed

1 4

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/2 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$

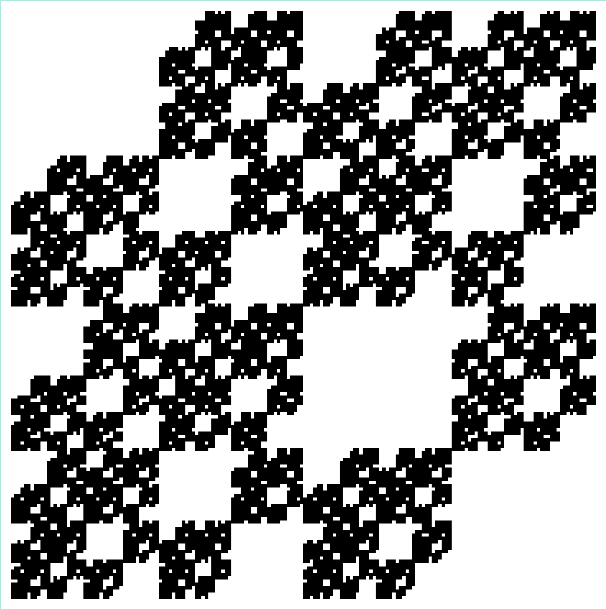


Criteria for Classification

- IFS-able
  - There exists a full state
- Infinitely IFS-able
  - There exists a full state, and an infinite sequence of non-full states
- Non-IFS-able
  - There are no full states

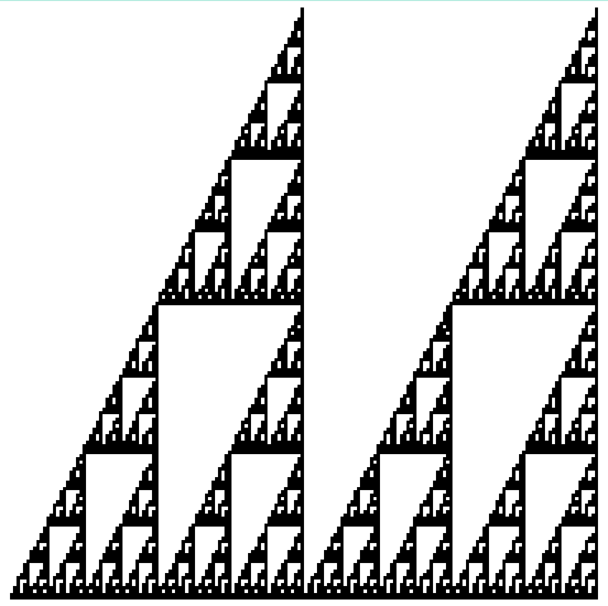
“When is a recurrent IFS attractor a standard IFS attractor?”
M. Frame, J. Lanski, *Fractals*, 7 (1999), 257-266.

Some Classified Attractors



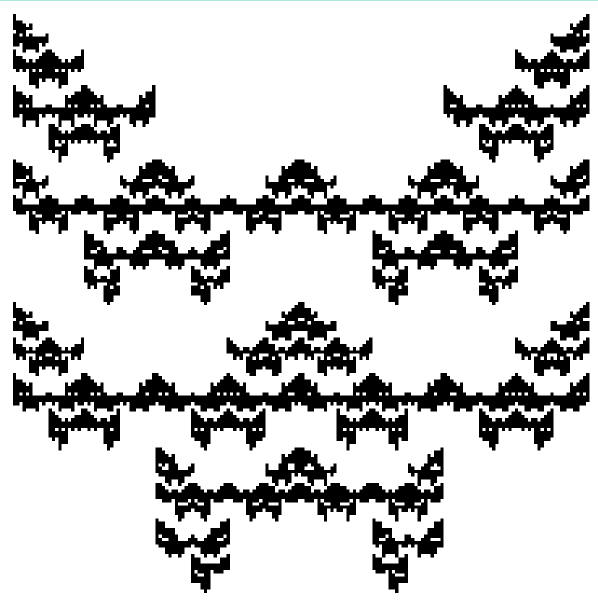
Finitely IFS-able

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/2 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$



Infinitely IFS-able

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$



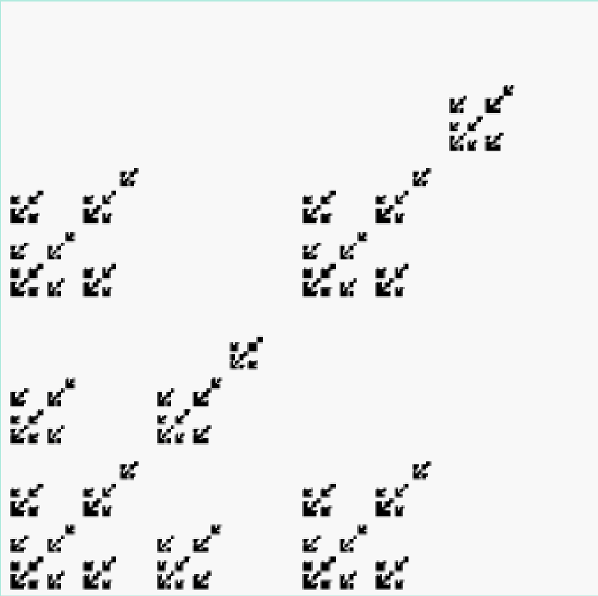
Non-IFS-able

$$P = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{pmatrix}$$

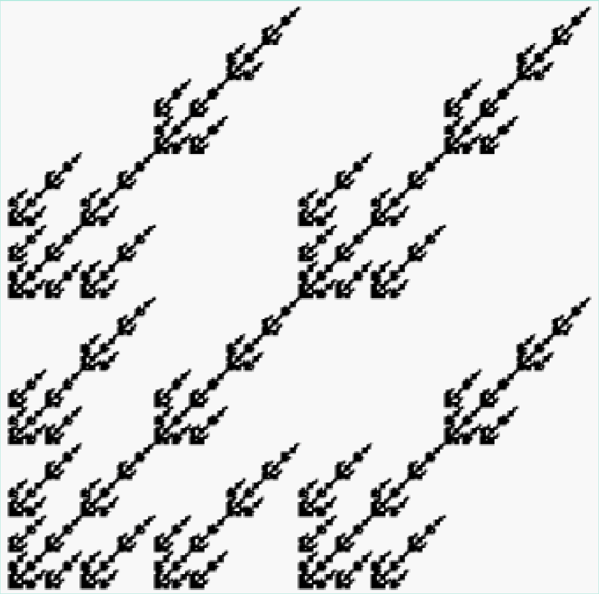
Forbidding Triples

- For example, $1 \rightarrow 1 \rightarrow 3$ is forbidden
- We ask the same question:
  - When do we get a fractal that is IFS-able, Infinitely IFS-able, or Non-IFS-able?
  - Also, when do we get a fractal that cannot also be generated by a 1-memory system?
- We believe we have established criteria to classify an attractor produced by a 2-memory system as IFS-able, Infinitely IFS-able, or Non-IFS-able
  - Must look at probability matrix, not transition graph

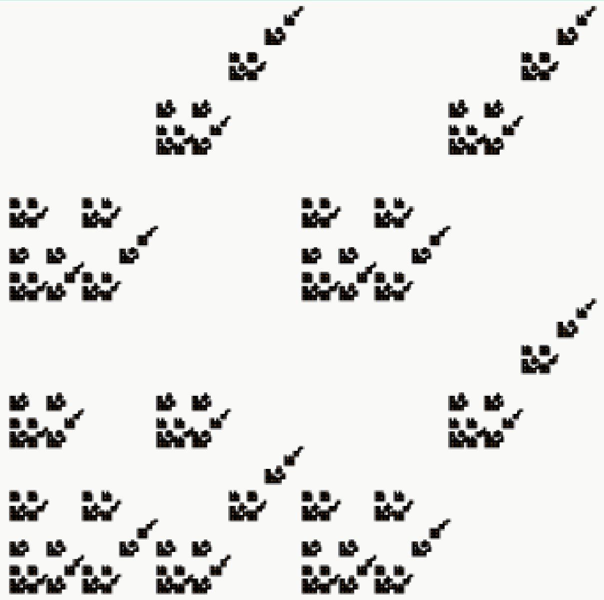
Examples of 2-Memory Fractals



Finitely IFS-able



Infinitely IFS-able



Non-IFS-able

We also believe none of these 3 attractors are produce-able by a 1-memory system.

Generalizing The Problem

- Open questions:
  - When is an attractor produced by an *n*-level memory system also produce-able by an *m*-level memory system ($m \leq n$)?
  - Given a fractal, what is the least integer *n* such that the attractor can be generated by an *n*-level memory IFS system?

References

- "When is a recurrent IFS attractor a standard IFS attractor?" M. Frame, J. Lanski, *Fractals*, 7 (1999), 257-266.
- "Fractal Geometry," M. Frame, B. Mandelbrot, N. Neger, http://classes.yale.edu/fractals/
- *Fractal and Multifractal Geometry: A Gateway to Advanced Mathematics*, M. Frame, N. Neger, Yale University.

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