Dynamics and scaling in models of coarsening and coagulation

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Abstract

Clustering phenomena occur in numerous areas of science. This includes condensation at the nanoscale in condensed matter physics, smog and cloud formation in aerosol physics, processes of stellar and galactic clustering in astrophysics, random graph theory, and merging of lines of descent in ancestral trees. Some basic models of coagulation and aggregation take the form of rate equations for cluster size distributions, mathematically closely related to kinetic equations. In these lectures we describe some of the considerable progress made recently in this area, by workers in probability, PDE, and combinatorics. A topic of particular interest concerns the trend toward self-similar behavior. Methods and insights from probability in tandem with dynamical systems theory have been found very fruitful in developing a systematic approach to studying this topic and scaling dynamics more generally.

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Plan of the series of lectures:

1. The problem of universal behavior in complex systems. Some fundamental models of clustering and coagulation and their interrelations: Smoluchowski's coagulation equations, ballistic aggregation, Burgers' turbulence model (shock-wave clustering). Dynamic self-similarity in the simplest mean-field models.

2. A general framework for dynamic scaling analysis. Self-similar behavior and its lack in coagulation equations. Scaling-limit rigidity and power laws. Role of regular variation.

3. Scaling dynamics in general for a solvable coagulation equation. The scaling attractor and its Lévy-Khintchine representation. Conjugacy with dilational dynamics. Signatures of chaos. Analogy to stable laws of probability and infinite divisibility.

1 Lecture 1

1.1 General aims.

In these lectures we will survey some recent results involving dynamic scaling limits and scaling relations in systems that form patterns that develop by coarsening over time. The overarching aim is to understand something about systems that appear to behave *predictably*, but whose complexity precludes detailed analysis. One of the challenges in dealing with such systems, in fact, is to identify good *statistics*—some properties of the system —about which something can be said. Some of the main mathematical themes involve dynamic scaling limits and their connections to dynamical systems concepts and limit theorems in probability theory. These concepts can be compared to renormalization group methods, for example, but the fundamental ideas really originated with the pioneers of probability theory in the 1920s and 1930s, workers such as Lévy, Khintchine and Doeblin. The power of these ideas has not been fully appreciated in PDE theory, and I believe there to be considerable scope for extending their reach to address many problems not related to the ones we shall consider here.

1.2 A basic model for clustering.

Smoluchowski's coagulation equation is an oversimplified model for the aggregation or clustering of matter. It describes the evolution of a simple statistic, the *distribution of cluster size x at time t.* One describes this using a cumulative distribution function (CDF): Let

$$\nu_t(x) = \text{(expected) number of clusters of size} \le x \text{ at time } t$$

and let $n(x,t)$ denote a density for this distribution (presuming it exists), so that

$$\nu_t(x) = \int_0^x n(y,t)\,dy.$$

The mechanism of clustering can be indicated schematically as follows:

$$\boxed{x} \;+\; \boxed{y} \;\Longrightarrow\; \boxed{x+y}$$
$$\boxed{x-y} \;+\; \boxed{y} \;\Longrightarrow\; \boxed{x}$$

Clusters of size x and y join to form a cluster of size $x+y$ at rate presumed to be

$$K(x,y)n(x,t)n(y,t),$$

separately proportional to the populations of 'incoming' clusters. We assume the rate kernel is non-negative and symmetric: $K(x,y) = K(y,x) \ge 0$. Integration over y produces the rate of loss of size-x clusters. Size-x clusters are produced by joining clusters of size y and $x-y$. Integrating over y and avoiding double-counting yields the rate equation

$$\begin{aligned}\partial_t n(x,t) = &-\int_0^\infty K(x,y)n(x,t)n(y,t)\,dy && \text{(loss)} \\ &+\frac{1}{2}\int_0^\infty K(x-y,y)n(x-y,t)n(y,t)\,dy && \text{(gain)}\end{aligned} \tag{1.1}$$

This is Smoluchowski's coagulation equation (first written in this size-continuous form by Müller). Many forms of it appear in an *extensive* physical literature, in a wide range of fields: astrophysics, chemistry of colloids, polymers, aerosols (fog & smog), lines of descent in population biology, and also in probability theory (renewal processes), and random graph theory. About

1917 the great Polish physicist Smoluchowski derived the size-discrete form of this model corresponding to

$$K(x,y) = (x^{1/3} + y^{1/3})(x^{-1/3} + y^{-1/3}),$$

to model Brownian particles, and he solved an initial-value problem for an infinite set of ODEs corresponding to $K \equiv 2$.

1.3 Dynamic behavior.

In the pure coagulation process we are modeling, cluster sizes simply grow in time. Claimed in many physical papers and seen in numerics is *dynamic scaling behavior*: As time increases, the size distribution approaches a *universal self-similar form*,

$$n(x,t) \sim a(t) f(b(t)x)$$

with some scaling profile f. This raises compelling mathematical questions:

Is this true? When? And why?

Naturally one should expect such behavior only if the system is (asymptotically?) scale-free, meaning the kernel K is homogeneous; say

$$K(ax, ay) = a^{\lambda} K(x,y).$$

First remarks on dynamic scaling:

1. Conservation of mass imposes the constraint $a = b^2$ because

$$\int_0^\infty x n(x,t)\, dx = \int_0^\infty a f(bx)\, dx = \frac{a}{b^2} \int_0^\infty f(y) y\, dy$$

is a quantity that is constant in time.

2. Explicit self-similar solutions are known for special rate kernels (that we will consider further later):

$$K = 2: \quad n(x,t) = \frac{1}{t^2} \exp(-x/t) \qquad (t > 0) \tag{1.2}$$

$$K = x + y: \quad n(x,t) = \frac{1}{\sqrt{2\pi}} x^{-3/2} e^{-t} \exp(-x e^{-2t}/2) \qquad (-\infty < t < \infty)$$

$$K = xy: \quad n(x,t) = \frac{1}{\sqrt{2\pi}} x^{-5/2} \exp(-t^2 x/2) \qquad (-\infty < t < 0 = T_{\text{gel}})$$

The existence of self-similar solutions in general was a long-standing open problem; the first results were achieved in 2004 by Fournier and Laurençot and Escobedo, Mischler and Rodriguez-Ricard. Some kernels important in applications are not covered by these results, however.

1.4 Some goals of these lectures

1. Derive Smoluchowski's equation with $K = x + y$ formally by modeling *random shock clustering* in the inviscid Burgers equation

$$\partial_t u + u \partial_x u = 0.$$

This formal model actually turns out to be a *rigorous* description of time evolution for a class of random initial data that includes (one-sided) Brownian motion: This result is due to Carraro & Duchon (1994) and Bertoin (1998), and is explicitly described in:

[MP0] G. Menon and R. L. Pego, Universality classes in Burgers turbulence, Comm. Math. Phys. 273 (2007) 177–202.

For recent results concerning time evolution for a wide class of Markov-process initial data, see Menon and Srinivasan (preprint, arXiv:0909.4036).

2. Address questions of *dynamic scaling*, including well-posedness, classification of all self-similar solutions, characterization of their domains of attraction, classification of cluster points (the *scaling attractor*) and description of the "ultimate dynamics" on the scaling attractor. This will be done for the particular case $K = x + y$, which is 'solvable' by using the Laplace transform and characteristics. Analyses in this solvable case have suggested results later proved to hold for other kernels and in related models.

Most of the main results are adapted (and improved) from two papers:

[MP1] G. Menon and R. L. Pego, Approach to self-similarity in Smoluchowski's coagulation equations, Comm. Pure Appl. Math. 57 (2004) 1197-1232.

[MP2] G. Menon and R. L. Pego, The scaling attractor and ultimate dynamics for Smoluchowski's coagulation equations, J. Nonl. Sci. 18 (2008) 143-190.

For a treatment of the simpler case $K = 2$ and discussion of a variety of other models of coarsening (e.g., 1D bubble bath, Mullins-Sekerka, LSW) and methods of analysis, see the following book chapter (available online):

[P] R. L. Pego, Lectures on dynamics in models of coarsening and coagulation, in *Dynamics in Models of Coarsening, Coagulation, Condensation and Quantization* (Lec. Notes Ser. Inst. Math. Sci. Nat. Univ. Singapore), Eds. Weizhu Bao and Jian-Guo Liu, World Scientific (Singapore), 2007.

1.5 Ballistic motion and clustering

Shock clustering in the inviscid Burgers equation is closely related to a more general problem of ballistic aggregation that we now describe.

In an influential 1989 paper in large-scale cosmology, Shandarin and Zeldovich discuss how the distribution of matter in the universe might become inhomogeneous, starting from a toy model for mass density $\rho(x,t)$ evolving by *free streaming*: Particles follow straight-line paths

$$x = x_0 + tv_0(x_0),$$

where $v_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is smooth, say. This naturally leads to the advection equations

$$\partial_t v + v \cdot \nabla v = 0, \quad \partial_t \rho + \nabla \cdot (\rho v) = 0.$$

"Pancake" singularities typcially form, when some eigenvalue of the deformation gradient

$$\frac{\partial x}{\partial x_0} = I + t\nabla v_0$$

first vanishes. (This is a nice thing to discuss in teaching a basic PDE course!)

The problem arises, how should one continue in time past these singularities? Some options are:

- Continue free streaming, with multiple velocities allowed at each "point", via Vlasov-type kinetic equations

- Preserve single-valuedness via ad hoc means such as introducing "viscosity"

$$\partial_t v + v \cdot \nabla v = \epsilon \Delta v.$$

  (This appears in the literature, but whether it is a faithful model of physical mechanisms is unclear.)

- Forbid interpenetration of matter by a ballistic aggregation mechanism, e.g., forming and transporting singular "mass sheets", say. There is work on such 'sticky particle' models in 1D: Brenier and Grenier (∼1998), Gangbo and Tudorescu (∼2008). The problem is completely open in higher dimensions.

1.6 Random shock clustering for the inviscid Burgers equation

Here we give a heuristic description of the surprising connection between the inviscid Burgers equation and the Smoluchowski coagulation equation with $K = x + y$, which actually appeared first as a rigorous result in the theory of 'Burgers turbulence', the study of how nonlinear dynamics propagates random initial data.

We start considering (entropy) solutions of

$$\partial_t u + u\partial_x u = 0, \quad x \in \mathbb{R},\ t \geq 0, \tag{1.3}$$

consisting of a (random) staircase of shocks (downward jumps):

$$u(x,t) = \sum_k -s_k H(x - x_k(t)) \tag{1.4}$$

We suppose this looks roughly linear on a (very) large scale.

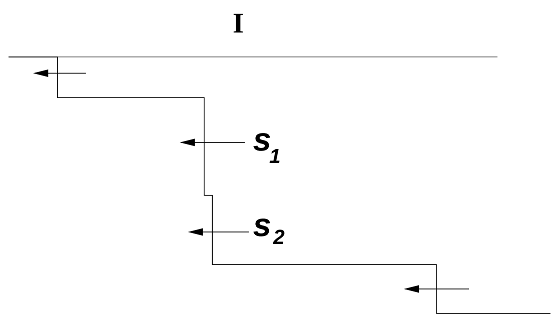

Figure 1.1: Binary clustering of shocks

Each shock at position $x_j(t)$ has constant size s_j and constant speed

$$\dot{x}_j = \frac{1}{2}(u_- + u_+)$$

between collisions. The shocks aggregate upon collision like ballistic particles, conserving total 'mass' (size) and 'momentum,' due to the standard jump conditions.

We formally describe a mean-field statistical model for the shock-size distribution as follows: Let $n(s,t)\,ds$ be the expected number of shocks per unit length I (on the large scale) with size in $[s, s+ds]$, assuming this distribution is stationary in space. There are two mechanisms of evolution of the density n:

1. Net influx of shocks into I: The velocity difference across I is

$$|u_R - u_L| \approx \int_0^\infty s\, n(s,t)\, ds =: m_1(t).$$

The net influx of shocks at size in $[s, s+ds]$ then should be the product of the rate they are swept into I and the density:

$$m_1(t) n(s,t)\, ds$$

2. *Shock coalescence* in I occurs at relative velocity

$$\dot{x}_2 - \dot{x}_1 = \frac{1}{2}(s_2 + s_1)$$

(The labels should be taken to indicate any two consecutive shocks.) The expected number of pairs with sizes in $[s_1, s_1 + ds_1]$, $[s_2, s_2 + ds_2]$ is

$$n(s_1, t) n(s_2, t)\, ds_1\, ds_2$$

and the probability that these are near enough to collide in time dt is proportional to the distance swept by the relative speed in time dt, namely $\frac{1}{2}(s_1 + s_2) dt$. So the number of such collisions in time dt is expected to be

$$\frac{1}{2}(s_1 + s_2) n(s_1, t) n(s_2, t)\, ds_1\, ds_2\, dt.$$

Considering all cases with $s = s_1 + s_2$ fixed and adding up over gain and loss terms, we obtain the density rate equation

$$\partial_t n(s,t) = m_1(t) n(s,t) + Q(n,n), \tag{1.5}$$

$$Q(n,n) = -\int_0^\infty n(s,t) n(s_2,t)(s + s_2)\, ds_2 + \frac{1}{2}\int_0^s n(s - s_1, t) n(s_1, t) s\, ds_1. \tag{1.6}$$

Integration over s yields (since $\int_0^\infty sQ(n,n)\, ds = 0$, see below),

$$\dot{m}_1 = \partial_t \int_0^\infty s n(s,t)\, ds = m_1^2,$$

whence one easily derives

$$\frac{1}{m_1}\partial_t \left(\frac{n}{m_1}\right) = Q\left(\frac{n}{m_1}, \frac{n}{m_1}\right) \tag{1.7}$$

Up to a change in time scale ($d\hat{t} = m_1 dt$) and size distribution ($\hat{n} = n/m_1$), this is exactly Smoluchowski's coagulation equation with additive kernel $K(s_1, s_2) = s_1 + s_2$.

Exercise: There is a very nice symmetry of the inviscid Burgers equation which allows one to add a constant slope to initial data, and express the solution with initial data $\tilde{u}_0(x) = u_0(x) + ax$ in terms of the solution with initial data $u_0(x)$. Can you find it?

1.7 Weak form of Smoluchowski's equation

To model cluster size distributions that may either be discrete or have continuous densities in a unified framework, it is desirable to have a theory of measure solutions, which satisfy a weak form of the equation. This is based on a *generalized moment identity* for Smoluchowski's coagulation equation. Multiplying (1.1) by a test function $a(x)$ and integrating we formally get

$$\partial_t \int_0^\infty a(x) n(x,t)\, dx = \frac{1}{2}\int_0^\infty \int_0^\infty \big(a(x+y) - a(x) - a(y)\big) K(x,y) n(x,t) n(y,t)\, dx\, dy.$$

This is most easily verified backwards: Using $x + y = \hat{x}$, $y = \hat{y}$,

$$\frac{1}{2}\int_0^\infty \int_0^\infty a(x+y) K(x,y) n(x) n(y)\, dx\, dy = \frac{1}{2}\int_0^\infty \int_0^{\hat{x}} a(\hat{x}) K(\hat{x}-\hat{y}, \hat{y}) n(\hat{x}-\hat{y}) n(\hat{y})\, d\hat{y}\, d\hat{x}.$$

Integrating in t yields the desired weak form for measure solutions $t \mapsto \nu_t(dx)$:

$$\int_0^\infty a(x)\nu_t(dx) = \int_0^\infty a(x)\nu_{t_0}(dx) + \frac{1}{2}\int_{t_0}^t \int_0^\infty \int_0^\infty \big(a(x+y) - a(x) - a(y)\big) K(x,y) \nu_s(dx)\nu_s(dy)\, ds$$

for a suitable class of test functions $a(x)$ (that depends on K and we will not specify here).

Formally, total mass is always conserved: $a(x) = x \Rightarrow \frac{d}{dt}\int_0^\infty x n(x,t)\, dx = 0$. This can *fail* however for fast-growing rate kernels K homogenous of degree > 1:

$$K(bx, by) = K(x,y) b^\lambda \quad \text{with} \quad \lambda > 1.$$

Solutions in this case can start to lose mass in finite time. This phenomenon is called *gelation* and has attracted much interest. There are basic math papers on it by Jeon (1998) and by Escobedo, Mischler and Perthame (2002), and many physical papers, for example by Leyvraz (2003) and Lushnikov. Much remains to be understood, however. The term 'gelation' suggests the loss of mass to an 'infinite cluster,' but the model as presented does not contain such a cluster.

1.8 Solutions for $K = x + y$

For definiteness we restrict attention to this kernel. Consider moments $m_p = \int_0^\infty x^p n(x,t)\,dx$. By conservation of mass, $m_1(t)$ is constant — often we normalize so $m_1 = 1$. For $p = 0$, using $a(x) = 1$ in the weak form yields

$$\begin{aligned} \dot{m}_0 &= -\frac{1}{2}\int_0^\infty \int_0^\infty (x+y)n(x)n(y)\,dx\,dy = -m_1 m_0, \\ m_0(t) &= m_0(0)\exp(-tm_1). \end{aligned}$$

Then the expected cluster size grows exponentially: $m_1/m_0 = Ce^{m_1 t}$.

Weak solutions generally for $K = x + y$ can be characterized using a variant of the Laplace transform: Note

$$a(x) = 1 - e^{-qx} \quad \Rightarrow \quad a(x+y) - a(x) - a(y) = -a(x)a(y).$$

Then

$$\varphi(t,q) := \int_0^\infty (1 - e^{-qx})\nu_t(dx)$$

satisfies $\partial_q \varphi = \int_0^\infty e^{-qx} x \nu_t(dx)$ and the evolution equation

$$\begin{aligned} \partial_t \varphi &= -\frac{1}{2}\int_0^\infty \int_0^\infty (1-e^{-qx})(1-e^{-qy})(x+y)n(x)n(y)\,dx\,dy \\ &= -(m_1 - \partial_q\varphi)\varphi, \end{aligned}$$

or, normalizing $m_1 = 1$,

$$\partial_t \varphi - \varphi \partial_q \varphi = -\varphi.$$

This is a damped inviscid Burgers equation, of course, and this time, solutions should be analytic. Solutions can be found from a implicit equation determined by the method of characteristics as follows.

Solution via characteristics. Along a characteristic curve $q = q(t, \alpha)$ with $q(0, \alpha) = \alpha$ we have

$$\frac{dq}{dt} = -\varphi, \quad \frac{d}{dt}\varphi(t, q(t,\alpha)) = \partial_t\varphi - \partial_q\varphi\dot{q} \quad = -\varphi,$$

Hence

$$\varphi = e^{-t}\varphi_0(\alpha), \quad e^t\varphi = \varphi_0(\alpha), \quad \frac{d}{dt}(q-\varphi) = 0,$$

so

$$q - \varphi(t,q) = \alpha - \varphi_0(\alpha) = \int_0^\infty (e^{-\alpha x} - 1 + \alpha x)\, \nu_0(dx) =: \psi_0(\alpha).$$

Now

$$\alpha = q + (e^t - 1)\varphi$$

so to find φ from given (t,q) we can solve the implicit equation

$$q = \varphi + \psi_0(q + (e^t-1)\varphi). \tag{1.8}$$

Or, to find $\alpha = \alpha(t,q)$ from given (t,q), note

$$(e^t-1)q = (e^t-1)\varphi + (e^t-1)\psi_0(\alpha)$$

so

$$e^t q = \alpha + (e^t-1)\psi_0(\alpha) \tag{1.9}$$

Observe:

$$\psi_0'(\alpha) = \int_0^\infty (1-e^{-\alpha x})x\,\nu_0(dx) \ \in (0,1) \quad \forall \alpha > 0, \tag{1.10}$$

$$\alpha \mapsto \psi_0''(\alpha) = \int_0^\infty e^{-\alpha x} x^2 \nu_0(dx) \tag{1.11}$$

$$\frac{\partial\alpha}{\partial q}(t,q) = 1 + (e^t-1)\frac{\partial\varphi}{\partial q} \tag{1.12}$$

1.9 Well-posedness of the initial-value problem

Here we describe a well-posedness theorem from [MP1], for measure solutions to Smoluchowski's equation with $K = x + y$, which requires the initial data to have only finite mass. A corresponding result for a general class of homogeneous kernels was obtained by Fournier and Laurençot (2006), using a Gronwall inequality for a Wasserstein-type distance. (The required finite moment corresponds to the degree of homogeneity of the kernel.)

Theorem 1.1. *Let ν_0 be any measure on $(0,\infty)$ such that $m_1 = \int_0^\infty x\,\nu_0(dx) < \infty$. Then there exists a unique weakly continuous map $t \mapsto x\nu_t(dx)$ such that $\int_0^\infty x\,\nu_t(dx) = m_1$ for all $t \geq 0$ and ν_t is a weak solution to Smoluchowski's coagulation equation.*

1.10 Self-similar limits with finite 2nd moment

To study self-similar limits it is convenient to consider the normalized mass distribution, which corresponds to a probability distribution function:

$$F_t(x) = \int_{[0,x]} y\,\nu_t(dy) \Big/ \int_0^\infty y\,\nu_t(dy)$$

The following result is a dynamic version of the *central limit theorem* for this system. Assuming finiteness of one more moment, there is a universal scaling behavior:

Theorem 1.2. *(Leyvraz (2003), [MP1]) Suppose $\int_0^\infty x^2\nu_0(dx) < \infty$. (We assume $= 1$ without loss of generality.) Let $\lambda(t) = e^{2t}$. Then as $t \to \infty$, in the weak sense of probability measures*

$$F_t(\lambda(t)\,dx) \to F_*(dx) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x}\,dx.$$

Proof. 1. Due to the *classical continuity theorem* that makes weak convergence of probability measures equivalent to pointwise convergence of the Laplace transform, to establish the conclusion it is equivalent to show

$$\int_0^\infty e^{-qx} F_t(\lambda dx) \to \mathcal{L}F_*(q) := \int_0^\infty e^{-qx} F_*(dx) \qquad \forall q > 0.$$

We have

$$\int_0^\infty e^{-qx} F_t(\lambda dx) = \mathcal{L}F_t(q/\lambda) = \frac{\partial\varphi}{\partial q}(t, q/\lambda)$$

2. Recall

$$\frac{q}{\lambda} - \varphi\left(t, \frac{q}{\lambda}\right) = \psi_0(\alpha) \quad \text{with} \quad \alpha = \alpha(t, q/\lambda) = \frac{q}{\lambda} + (e^t - 1)\varphi\left(t, \frac{q}{\lambda}\right)$$

As $t \to \infty$ we have for any fixed $q > 0$ that

$$\alpha + (e^t - 1)\psi_0(\alpha) = \frac{e^t q}{\lambda} \to 0, \quad \text{hence} \qquad \alpha\left(t, \frac{q}{\lambda}\right) \to 0.$$

3. Since $\int_0^\infty x^2\nu_0(dx) = \psi_0''(0) = 1$, as $\alpha \to 0$ we have

$$\psi_0'(\alpha) = \alpha(1 + o(1)), \qquad \psi_0(\alpha) = \frac{1}{2}\alpha^2(1 + o(1)).$$

Since $\lambda = e^{2t}$ now it follows that for any fixed $q > 0$,

$$q = e^t\alpha + \frac{1}{2}e^{2t}\alpha^2(1 + o(1))$$

as $t \to \infty$. From this we can infer $e^t \alpha \to \varphi_* = \varphi_*(q)$ where

$$q = \varphi_* + \frac{1}{2}\varphi_*^2, \qquad \varphi_*(q) = -1 + \sqrt{1+2q}.$$

4. Then, by differentiating the implicit characteristic relation (1.8), we find

$$\frac{\partial \varphi}{\partial q}\left(t, \frac{q}{\lambda}\right) = \frac{1 - \psi_0'(\alpha)}{1 + (e^t - 1)\psi_0'(\alpha)} \to \frac{1}{1 + \varphi_*(q)} = \frac{1}{\sqrt{1+2q}}.$$

But this yields the result, since

$$\int_0^\infty e^{-(q+1/2)x} \frac{dx}{\sqrt{x}} = \frac{1}{\sqrt{q+1/2}} \int_0^\infty e^{-y} \frac{dy}{\sqrt{y}} = \frac{\sqrt{2\pi}}{\sqrt{2q+1}} = \sqrt{2\pi}\mathcal{L}F_*(q).$$

We remark that if one assumes existence of further moments, rates of convergence have been obtained by Cañizo, Mischler and Mouhot (to appear) for $K = 2$ and Srinivasan for $K = x + y$ and xy (to appear).

2 Lecture 2

2.1 A framework for scaling analysis

In the theorem of the last lecture, for convergence to self-similarity we required finiteness of the 2nd moment. This is analogous to the condition of finite variance that one sees in the standard central limit theorem in probability. Without this condition, however, one can achieve a zoo of other possible limits, involving the Lévy stable laws and infinitely divisible laws.

The question then arises, what kind of scaling limits can arise in coagulation without a second finite moment? It turns out that there is an extensive set of analogies with classical limit theorems in probability. The pioneers of that subject deeply investigated conditions under which scaling limits are obtained, and the results are beautifully laid out in W. Feller's superb book, Introduction to Probability Theory and its Applications, vol. 2 (2nd ed. 1971).

Using the language of dynamical systems theory, one can list a number of questions about scaling limits for Smoluchowski's equation with solvable kernel, that one can rather completely address using this analogy:

1. What are all the scaling solutions that exist?
2. What are their domains of attraction? (Can one characterize which initial data converge to self-similar form as $t \to \infty$?)
3. What other scaling limit points can exist, along subsequences $t_n \to \infty$? (We will call the set such points the *scaling attractor* for the system.)
4. Can one describe precisely the "ultimate dynamics" on the scaling attractor?

In probability theory one considers scaled sums of iid random variables, $S_n = (X_1 + \ldots + X_n)/c_n$. The scaling limits as $n \to \infty$ are the Lévy stable laws, and there is a complete classification of their domains of attraction in terms of the 'tails' of the distribution of the X_j. Limits along subsequences $n_j \to \infty$ are the infinitely divisible laws, which are represented by the famous Lévy-Khintchine formula in terms of a class of measures.

In this lecture, for $K = x + y$ we will address the question of domains of attraction. In the next lecture we discuss the characterization of the scaling attractor by a Lévy-Khintchine-like representation, and indicate how, in terms of this representation, the ultimate dynamics becomes conjugate to *pure dilation.*

The results indicate a kind of sensitive dependence of long-time dynamics on the tails of the initial distribution—a form of 'chaos.' In probability theory it was Wolfgang Doeblin who first saw this kind of behavior, among the many results he achieved before he died at a young age in World War II.

2.2 Scaling limits and regular variation

The following basic results on scaling limits and the Laplace transform are taken from Feller's book.

Lemma 2.1. *Suppose $U > 0$ is increasing on $[0, \infty)$ and*

$$\frac{U(tx)}{U(x)} \underset{t\to\infty}{\longrightarrow} \psi(x) \leq \infty \quad \textit{for all } x \in A, \textit{ dense in } [0, \infty).$$

Then $\psi(x) = x^p$ for some $p \in [0, \infty]$.

Proof.

$$\frac{U(tx_1x_2)}{U(t)} = \frac{U(tx_1x_2)}{U(tx_2)}\frac{U(tx_2)}{U(t)}.$$

If $x_j \in A$, $\psi(x_j) < \infty$ then

$$\psi(x_1x_2) = \psi(x_1)\psi(x_2). \tag{2.1}$$

ψ is increasing and $\psi(1) = 1$. If $0 < \psi(x) < 1$ for *some* $x_0 < 1$ then this is true for all $x_0 \in A \cap (0, 1)$. Extend ψ_0 as right continuous to $\mathbb{R}^+$. Then (2.1) holds for all $x > 0$; Hence $\psi(e^r) = \psi(e)^r$ for all r (at first in $\mathbb{N}$, then in $\mathbb{Q}$, then in $\mathbb{R}$). Now with $x = e^r$, $r = \log x$ so $\psi(x) = (e^p)^{\log x} = x^p$ with $p = \log \psi(e)$.

Corollary 2.2. *Let $L(t) = U(t)/t^p$ if $p < \infty$. Then for all $x > 0$,*

$$\frac{L(tx)}{L(t)} \underset{t\to\infty}{\longrightarrow} 1. \tag{2.2}$$

Definition 2.1. A function $L : (0, \infty) \to (0, \infty)$ is *slowly varying at ∞* if (2.2) holds for all $x > 0$. Any function of the form $U(t) = t^p L(t)$ is *regularly varying at ∞ with exponent p.*

Lemma 2.3. *(Rigidity of scaling limits) Suppose $U > 0$ is increasing on $(0, \infty)$ and*

$$a_n U(b_n x) \underset{n\to\infty}{\longrightarrow} g(x) \leq \infty \quad \textit{for all } x \in A, \textit{ dense in } [0, \infty),$$

where $b_n \to \infty$, $a_{n+1}/a_n \to 1$ *and* $0 < g(x) < \infty$ *in some interval.*

Then necessarily $U(x) = x^p L(x)$ *where* L *is slowly varying at* ∞, *and* $g(x) = cx^p$ *for some* $c > 0$, $p \in (0, \infty)$.

Proof. Without loss (rescale otherwise) we assume $1 \in A$. For large $t > 0$ we map $t \mapsto n = \min\{m : b_{m+1} > t\}$. Then

$$\frac{a_{n+1}}{a_n}\frac{a_n}{a_{n+1}}\frac{U(b_n x)}{U(b_{n+1})} \le \frac{U(tx)}{U(t)} \le \frac{U(b_{n+1}x)}{U(b_n)}\frac{a_{n+1}}{a_n}\frac{a_n}{a_{n+1}}$$

Taking $t \to \infty$ we infer that

$$1 \cdot \frac{g(x)}{g(1)} = \lim_{t\to\infty} \frac{U(tx)}{U(t)} = \frac{g(x)}{g(1)} \cdot 1$$

for all $x \in A$. Apply the previous Lemma to conclude.

2.3 Karamata's Tauberian theorem

Let U be a measure on $[0, \infty)$ with $U(0) = 0$, and

$$\omega(q) = \int_0^\infty e^{-qx} U(dx)$$

We use the notation $U(x) = \int_{[0,x]} U(dx)$, $U(0) = 0$. For $t > 0$ let

$$\tau = \frac{1}{t}, \quad \text{then} \quad \omega(\tau q) = \int_0^\infty e^{-q\hat{x}} U(t\, d\hat{x}).$$

Suppose that for some $p \in [0, \infty)$,

$$\frac{U(tx)}{U(t)} \underset{t\to\infty}{\longrightarrow} x^p \quad \forall x > 0. \tag{2.3}$$

Then (with $q = 1$)

$$\frac{\omega(\tau)}{U(t)} = \int_0^\infty e^{-x} \frac{U(t\,dx)}{U(t)} \underset{t\to\infty}{\longrightarrow} \int_0^\infty e^{-x} d(x^p) = p\Gamma(p) = \Gamma(1+p).$$

This means

$$\omega(\tau) \sim U(t)\Gamma(1+p) \tag{2.4}$$

Then

$$\frac{\omega(\tau q)}{\omega(\tau)} \sim \frac{U(t/q)}{U(t)} \underset{\tau\to 0}{\longrightarrow} \frac{1}{q^p}. \tag{2.5}$$

So (2.3) $\implies$ (2.5). We claim (2.5) $\implies$ (2.3). Note $\int_0^\infty e^{-qx} d(x^p) = q^{-p}\Gamma(1+p)$.

$$\frac{\omega(\tau q)}{\omega(\tau)} = \int_0^\infty e^{-qx} \frac{U(t\,dx)}{\omega(\tau)} \to \frac{1}{q^p}$$

implies, by the usual (2nd) continuity theorem for Laplace transforms, that

$$\frac{U(tx)}{\omega(\tau)} \to \frac{x^p}{\Gamma(1+p)} \quad \forall x > 0.$$

Taking $x = 1$ gives (2.4), then

$$\frac{U(tx)}{U(t)} \sim \frac{\omega(\tau/x)}{\omega(\tau)} \to x^p.$$

This gives the following classic theorem.

Theorem 2.4. *If L is slowly varying at ∞ and $0 \le p < \infty$, then the following are equivalent:*
(i) $U(t) \sim t^p L(t)$ as $t \to \infty$.
(ii) $\omega(\tau) \sim \tau^{-p} L(1/\tau)\Gamma(1+p)$ as $\tau \to 0^+$.

2.4 Necessary & sufficient for self-similarity with $K = x + y$

We shall see that *necessary and sufficient* conditions for convergence to self-similar form involve regular variation. Here we emphasize necessity:

Theorem 2.5. *Let $F_0(dx) = x\,\nu_0(dx)$ be a probability measure on $(0,\infty)$ and let $F_t(dx) = x\,\nu_t(dx)$ $(0 \le t < \infty)$ correspond to the weak solution of Smoluchowski's equation with $K = x + y$.*
Suppose (1):

$$F_t(\lambda(t)x) \to F_*(x) \quad \text{as } t \to \infty \text{ for a.e. } x > 0,$$

where $\lambda(t) \to \infty$ and $F_*(dx)$ is a (proper) probability measure on $(0,\infty)$.
Then (2):

$$\int_{[0,x]} y^2\,\nu_0(dy) = x^\rho L(x),$$

where $0 \le \rho < 1$ and L is slowly varying at ∞.

(The converse (2) $\implies$ (1) is also true, with F_* depending on ρ, and λ depending on ρ and L.)

Proof. 1. Assume (1). Then

$$\int_0^\infty e^{-qx} F_t(\lambda(t)dx) = \mathcal{L}F_t(q/\lambda) = \partial_q \varphi(t, q/\lambda) \to \mathcal{L}F_*(q) \quad \forall q > 0.$$

The limit is decreasing in q and lies in $(0,1)$, and we infer

$$\lambda\varphi\left(t, \frac{q}{\lambda}\right) = \int_0^q \partial_q \varphi\left(t, \frac{\hat{q}}{\lambda}\right) d\hat{q} \to \varphi_*(q) := \int_0^q \mathcal{L}F_*(\hat{q}) d\hat{q}.$$

2. Recall

$$\frac{q}{\lambda} - \varphi\left(t, \frac{q}{\lambda}\right) = \psi_0(\alpha), \qquad \alpha = \alpha(t, q/\lambda) = \frac{q}{\lambda} + (e^t - 1)\varphi\left(t, \frac{a}{\lambda}\right)$$

As $t \to \infty$,

$$q - \lambda\varphi\left(t, \frac{q}{\lambda}\right) = \lambda\psi_0(\alpha) \to q - \varphi_*(q).$$

Hence $\alpha \to 0$ and hence $e^t \ll \lambda$ (since $e^t\varphi(t, q/\lambda) = (e^t/\lambda)(\varphi_*(q) + o(1)) \to 0$. It follows

$$\alpha \sim \beta(t)\varphi_*(q) \qquad \text{where} \quad \beta(t) = e^t/\lambda \to 0.$$

3. Recall

$$\frac{\partial\varphi}{\partial q} = \frac{1 - \psi_0'(\alpha)}{1 + (e^t - 1)\psi_0'(\alpha)} \to \frac{1}{1 + \varphi_*(q)} \to \mathcal{L}F_*(q) \qquad \forall q > 0.$$

With a little work (using comparisons $\alpha - \varepsilon < \beta(t)\varphi_*(q) < \alpha + \varepsilon$, not shown) we can infer that there is a limit $g(q)$ depending on q such that

$$e^t \psi_0'(\beta(t)\varphi_*(q)) \sim e^t \psi_0'(\alpha) \to g(q) = \hat{g}(\varphi_*) \qquad (0 < \varphi_* < \infty).$$

4. By the scaling rigidiy argument (inverted) we have that *necessarily*

$$\psi_0'(\alpha) = \alpha^p L(1/\alpha) \quad \text{as } \alpha \to 0,$$

where L is slowly varying at ∞. Since

$$\psi_0'(\alpha) = \int_0^\infty (1 - e^{-\alpha x}) x \, \nu_0(dx),$$

one can show (it's not trivial but not hard) that

$$\psi_0''(\alpha) = \int_0^\infty e^{-\alpha x} x^2 \, \nu_0(dx) \sim p\alpha^{p-1} L(1/\alpha) \quad \text{as } \alpha \to 0.$$

By the Tauberian theorem it follows

$$\int_{[0,x]} y^2\, \nu_0(dy) \sim px^{1-p} \frac{L(x)}{\Gamma(2-p)} \quad \text{as } x \to \infty.$$

This finishes the proof of the Theorem.

Self-similar profiles. Let us go a little further to describe the limit F^*. By the scaling rigidity argument, we also infer

$$\hat{g}(\psi_*) = c\psi_*^p \qquad \text{for some } c > 0,\ p \in \mathbb{R}.$$

Then

$$\mathcal{L}F_*(q) = \partial_q \varphi_*(q) = \frac{1}{1 + c\varphi^p}.$$

This is decreasing in p so $p > 0$, and we can *scale* so that $c = p + 1$, whence we find

$$q = \varphi_* + \varphi_*^{p+1}.$$

Also $p \le 1$ follows by computing $-(\mathcal{L}F_*)'(q)$ and observing that this must be positive and decreasing in q, so φ_*^{p-1} can't vanish at $q = 0$.

By use of the Lagrange inversion formula, one can derive an explicit series representation for $F_*(x)$, and it turns out quite remarkably that

$$F_*(dx) = x\nu_*(dx) = n(x^\beta)x^{\beta-1}\,dx$$

where $n(x)$ is the density of one of the *Lévy stable laws* of probability theory! See [MP1] for details.

3 Lecture 3

In this lecture our aim is to describe the scaling attractor, its measure representation (based on a result of Bertoin 2002) analogous to the Lévy-Khintchine formula for infinitely divisible laws, and how the dynamics on the attractor is made purely dilational in terms of this representation. Although the proofs available now certainly rely on the Laplace transform, it seems plausible that only the scaling properties of Smoluchowski's equation should be important. Exactly how this should work for a general class of kernels is a mystery at present.

Scaling symmetries. Let ν_t be a measure solution of Smoluchowski's equation with $K = x + y$ (for $t \geq 0$, $m_1 = 1$). Then for all $T \in \mathbb{R}$ and $\lambda > 0$ the quantity

$$\tilde{\nu}_t(dx) = \lambda \nu_{t+T}(\lambda dx)$$

is a measure solution also (for $t \geq -T$, $\tilde{m}_1 = 1$). Correspondingly,

$$\tilde{\varphi}(t,q) = \int_0^\infty (1 - e^{-qx}) \tilde{\nu}_t(dx) = \lambda \varphi(t+T, q/\lambda), \qquad \tilde{F}_t(dx) = F_{t+T}(\lambda dx).$$

3.1 The scaling attractor, and eternal solutions

Consider any sequence of solutions $\nu_t^{(n)}(dx)$ (defined for $t \geq 0$, with $m_1 = 1$) and suppose sequences T_n, $\lambda_n \to \infty$ such that

$$F_{T_n}^{(n)}(\lambda_n dx) \underset{n\to\infty}{\longrightarrow} \hat{F}_0(dx) \tag{3.1}$$

where $\hat{F}_0(dx)$ is a probability distribution on $[0, \infty]$. Note that one can always pass to a subsequence to ensure that such a $\hat{F}_0$ exists—with the usual weak topology, the space of probability distributions on $[0, \infty]$ is compact. Of course, defective limits in $(0, \infty)$ are possible; probability can concentrate at 0 or leak off to ∞.

The set of $\hat{F}_0$ that arise in this way comprise all the cluster points of the set of solutions up to an arbitrary rescaling of cluster size.

Definition 3.1. The set of probability measures $\hat{F}_0$ on $[0, \infty]$ such that sequences exist yielding (3.1) is the *scaling attractor* $\mathcal{A}$.

We aim to characterize elements of this scaling attractor and their dynamics. In terms of corresponding 'Laplace exponents,'

$$\begin{aligned}\tilde{\varphi}^{(n)}(0,q) = \lambda_n \varphi^{(n)}\left(T_n, \frac{q}{\lambda_n}\right) &= \int_0^\infty \frac{1-e^{-qx}}{x} F_{T_n}^{(n)}(\lambda_n dx) \\ &\underset{n\to\infty}{\longrightarrow} \int_0^\infty \frac{1-e^{-qx}}{x} \hat{F}_0(dx) =: \hat{\varphi}_0(q).\end{aligned}$$

Since $\tilde{\varphi}^{(n)}(t,q)$ is defined for $t > -T_n$, one can show without much difficultly that if we allow a finite advance or delay in time, convergence still holds. This means that *for each fixed* $t \in (-\infty, \infty)$ and all $q > 0$,

$$\tilde{\varphi}^{(n)}(t,q) \underset{n\to\infty}{\longrightarrow} \hat{\varphi}(t,q),$$

and then one can deduce (details omitted) that the corresponding probability measures converge weakly:

$$F_{t+T_n}^{(n)}(\lambda_n dx) \underset{n\to\infty}{\longrightarrow} \hat{F}_t(dx), \qquad -\infty < t < \infty \tag{3.2}$$

where $\hat{F}_t$ is a probability measure on $[0,\infty]$, with

$$\hat{\varphi}(t,q) = \int_{[0,\infty]} \frac{1-e^{-qx}}{x} \hat{F}_t(dx), \qquad q > 0.$$

(This has to be interpreted properly to account for possible atoms at 0 and ∞.)

Naturally, by forward-backward continuity of the solution along characteristics for the PDE, the limit $\hat{\varphi}(t,q)$ satisfies the same PDE:

$$\partial_t \hat{\varphi} - \varphi \partial_q \varphi = -\hat{\varphi}, \qquad \varphi(0,q) = \hat{\varphi}_0(q), \quad q > 0. \tag{3.3}$$

It is striking that rescaled limits of solutions that concentrate probability at 0 or ∞ have Laplace exponents $\hat{\varphi}$ that solve the same damped Burgers equation. We take this to mean that F_t corresponds to a solution of Smoluchowski's equation in an *extended* sense that allows for the presence of 'dust' and 'gel' — here, a positive probability that mass distribution includes clusters of size 0 or ∞. (Practically, this means sizes that are either negligible or too huge to account for precisely.)

It turns out that 'dust' and 'gel' cannot appear or disappear in finite time: For all t,

- $\hat{F}_0(0) > 0 \quad \Leftrightarrow \quad \mathcal{L}F_t(0) = \partial_q \hat{\varphi}(t,\infty) > 0 \quad$ (dust)
- $\hat{F}_0(\infty^-) < 1 \quad \Leftrightarrow \quad \hat{F}_t(\infty^-) < 1 \quad \Leftrightarrow \quad \partial_q \varphi(t, 0^+) < 1 \quad$ (gel)

Definition 3.2. The *proper scaling attractor*

$$\mathcal{A}_{\mathrm{p}} := \{\hat{F}_0 \in \mathcal{A} : \hat{F}_0 \text{ is a proper probability distribution on } (0,\infty)\}.$$

Theorem 3.1. $\hat{F}_0 \in \mathcal{A}_{\mathrm{p}} \quad \Leftrightarrow \quad \hat{F}_0 = \hat{F}_t|_{t=0}$*, where* $\hat{F}_t$ *is a (proper) eternal solution, defined for* $-\infty < t < \infty$*.*

We remark that the (proper) scaling attractor is *invariant*: $\hat{F}_0 \in \mathcal{A}_{\mathrm{p}}$ if and only if $\hat{F}_t \in \mathcal{A}_{\mathrm{p}}$ for all t.

3.2 Characterizing eternal solutions via $t \to -\infty$

The key result here is due to Bertoin (2002). My main aim here is to give some idea how one might discover such a thing, and hint at the consequences for dynamics that were described in [MP2]. We first scale to simplify the dynamics of the damped inviscid Burgers equation:

1. Let $\nu_t(dx)$ be a (proper) eternal solution (for $K = x + y$ as usual) with $\int_0^\infty x\, \nu_t(dx) \equiv 1$. Recall

$$\varphi(t,q) = \int_0^\infty (1 - e^{-qx}) \nu_t(dx), \qquad \partial_t \varphi - \varphi \partial_q \varphi = -\varphi \quad (q > 0,\ -\infty < t < \infty).$$

Along characteristics $\varphi(t, q(\alpha,t)) = e^{-t}\varphi(\alpha)$,

$$q - \varphi(t,q) = \alpha - \varphi_0(\alpha) = \psi_0(\alpha),$$

$$\alpha = q + (e^t - 1)\varphi = e^t q - (e^t - 1)(q - \varphi).$$

2. We rescale via $x = s(t)\hat{x}$ with $s(t) = e^t$. (An important point is that *it is not well-understood why this choice of rescaling works!*) Then

$$\int_0^\infty (1 - e^{-q\hat{x}}) \nu_t(s\, d\hat{x}) = \varphi\left(t, \frac{q}{s}\right),$$

and replacing q by q/s yields

$$\psi(s,q) := \frac{q}{s} - \varphi\left(t, \frac{q}{s}\right) = \psi_0(\alpha)$$

with

$$\alpha = e^t \frac{q}{s} - (e^t - 1)\left(\frac{q}{s} - \varphi\right) = q - (s-1)\psi.$$

Thus

$$\psi(s,q) = \psi_0(q - (s-1)\psi), \quad \forall s = e^t > 0,\ q > 0. \tag{3.4}$$

This is just the implicit formula for the *usual inviscid Burgers equation*:

$$\partial_s \psi + \psi \partial_q \psi = 0, \qquad \psi(1,q) = \psi_0(q). \tag{3.5}$$

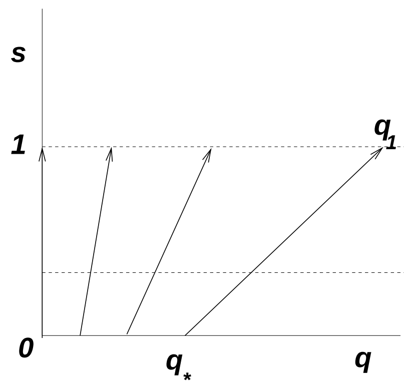

Figure 3.1: Geometry of characteristics

3. As usual, ψ is *constant along straight-line characteristics*, whose speed increases with increasing q:

$$\frac{dq}{ds} = \psi = \psi_0(\alpha), \qquad q(1,\alpha) = \alpha, \qquad q = \alpha + (s-1)\psi.$$

At this point it seems natural to wonder what happens as $s \downarrow 0$. It is not hard to see there is a limit:

Proposition 3.2. *For each $q > 0$, $\Psi(q) := \lim_{s\to 0^+} \psi(s,q)$ exists and determines $\psi(s,q)$ via $\psi = \Psi(q - s\psi)$ for all $s, q > 0$.*

Proof. For all $s, q > 0$ we have $\psi > 0$, $\partial_q \psi > 0$ hence $\partial_s \psi = -\psi \partial_q \psi < 0$. Thus we only need to prove the bound

$$\Psi(q) = \sup_{s>0} \psi(s,q) < \infty \qquad \forall q > 0. \tag{3.6}$$

Along characteristics: $q = \hat{q}(s,\alpha) = \alpha + (s-1)\psi_0(\alpha)$, so

$$\hat{q}(0^+,\alpha) = \alpha - \psi_0(\alpha) = \varphi_0(\alpha) = \int_0^\infty (1 - e^{-\alpha x})\nu_0(dx) > 0 \quad (\alpha > 0).$$

1. For $s > 0$, $0 < q < q_* := \varphi_0(1)$, say, by monotonicity in q we have the bound

$$\psi(s,q) \le \psi(s, \varphi_0(1) + s\psi_0(1)) = \psi_0(1).$$

2. The map $q \mapsto \psi(s,q)/q^2$ is *decreasing*, since

$$\begin{aligned} \frac{\psi(s,q)}{q^2} &= \int_0^\infty \left(\frac{e^{-qx} - 1 + qx}{q^2x^2}\right) x^2 \nu_t(s\,dx) \\ &= \int_0^\infty \left(\int_0^1 (1-y)e^{-qxy}dy\right) x^2\nu_t(s\,dx) \end{aligned} \tag{3.7}$$

and the inner integral is decreasing in q. Hence $\forall s > 0$ and $\forall q \ge q_*$, by monotonicity again,

$$\psi(s,q) \le q^2 \frac{\psi(s,q_*)}{q_*^2} \le Cq^2.$$

This proves the Proposition.

Once one sees this, there arises the question:

What does this proposition mean in terms of the measure solution $\nu_t(dx)$?

An answer is suggested by the formula in (3.7). Define the measure

$$G_t(dx) = x^2 \nu_t(e^t\,dx). \tag{3.8}$$

Then the convergence $\psi(s,q) \to \Psi(q)$ suggests (and it is true) that

$$G_t(dx) \to H(dx) \quad \text{weakly as } t \to -\infty,$$

where H is some measure. (One should note $\int_0^\infty x^{-1}G_t(dx) = 1/s = e^{-t} \to \infty$, and possibly $\int_0^\infty G_t(dx) = \infty$.)

Now one can ask:

What measures H are possible limits?

The answer formulated in [MP2] goes as follows. We write $a \wedge b = \min(a,b)$.

Definition 3.3. A measure G on $[0,\infty)$ is a *g-measure* if

$$\int_{[0,\infty)} (1 \wedge y^{-1}) G(dy) < \infty.$$

We say a sequence $G^{(n)} \xrightarrow{g} G$ as $n \to \infty$ if $x(G^{(n)}) \to x(G)$ for a.e. $x > 0$, where

$$x(G) := \int_{[0,\infty)} (x^{-1} \wedge y^{-1}) G(dy) = \frac{1}{x} G(x) + \int_x^\infty y^{-1} G(dy).$$

We say G is a *divergent* g-measure if $x(G) \to \infty$ as $x \to 0^+$.

Theorem 3.3. *(Bertoin 2002, [MP2]) 1. If $t \mapsto \nu_t$ is a (proper) eternal solution for $K = x + y$, then as $t \to -\infty$ we have $G_t \xrightarrow{g} H$ for some divergent g-measure H.*

2. For each divergent g-measure H, there is a unique proper eternal solution ν_t such that $G_t \xrightarrow{g} H$ as $t \to -\infty$. The solution ν_t is determined from

$$\Psi(q) := \int_{[0,\infty)} \frac{e^{-qx} - 1 + qx}{x^2} H(dx)$$

through the implicit relation $\psi(s, q) = \Psi(q - s\psi)$ and (3.7).

3. The correspondence

$$\hat{F}_0(dx) \leftrightarrow H(dx),$$

from probability measures on $(0, \infty)$ to divergent g-measures, is a bicontinuous bijection.

This result is the analog of the *Lévy-Khintchine representation formula* for infinitely divisible laws in probability theory. The proof in [MP2] relies on a (rather easy) extended continuity theorem for '2nd order Laplace exponents' of g-measures. Roughly, a sequence $G^{(n)} \xrightarrow{g} G$ if and only if $\psi^{(n)}(q) \to \psi(q)$ for all $q > 0$, where

$$\psi^{(n)}(q) = \int_{[0,\infty)} \frac{e^{-qx} - 1 + qx}{x^2} G^{(n)}(dx).$$

See [MP2] for details.

3.3 Scaling dynamics on the scaling attractor

The limit theorem above gives a remarkable simple representation of non-linear clustering dynamics on the scaling attractor, as follows.

Let $T \in \mathbb{R}$, $\lambda > 0$. Then we map eternal solutions to eternal solutions by the scaling

$$\nu_t(dx) \quad \to \quad \tilde{\nu}_t(dx) = \lambda \nu_{t+T}(\lambda\, dx).$$

This is the time-T map, scaling size by λ. Correspondingly,

$$F_0(dx) = x\,\nu_0(dx) \quad \to \quad F_T(\lambda\,dx),$$

$$G_t(dx) = x^2\nu_t(e^t dx) \quad \to \quad \tilde{G}_t(dx) = \frac{e^{2T}}{\lambda}G_{t+T}(\lambda e^{-T}dx) \tag{3.9}$$

because

$$x^2\tilde{\nu}_t(e^t dx) = x^2\lambda\nu_{t+T}(e^t\lambda\,dx) = \nu_{t+T}(e^{t+T}\lambda e^{-T}dx)(\lambda e^{-T}x)^2\frac{e^{2T}}{\lambda}.$$

Now, just using the previous theorem to take the limit $t \to -\infty$ in the scaling relation (3.9) yields the following result!!

Theorem 3.4. *To the map $\nu_t \mapsto \tilde{\nu}_t$ on the scaling attractor $\mathcal{A}_\mathrm{p}$ corresponds the map $H \mapsto \tilde{H} = H_{T,\lambda}$ of divergent g-measures given by*

$$H_{T,\lambda}(dx) = \frac{e^{2T}}{\lambda}H(\lambda e^{-T}dx).$$

To summarize: In the Lévy-Khintchine representation of an eternal solution ν_t by divergent g-measure H, the time-T map corresponding to nonlinear coagulation dynamics is represented by a pure dilational scaling as in the theorem. This is analogous to Bernoulli shift dynamics on sequences (after a log change of variables), and suggests a form of sensitive dependence on initial conditions, the shape of the tail in the distribution of large clusters in particular. For further developments of this idea see [MP2].

So, we have seen on the one hand an analog of the central limit theorem, for well-localized initial data, and on the other hand, how for heavy-tailed data one can classify domains of attraction and cluster points mod scaling on the other hand. These results are strongly analogous to the treatment of the stable laws and infinite divisibility in probability, as in Feller's book for example. A few results, such as asymptotic shadowing of solutions with initially similar tails, for example, go a little farther, however.