IV Martingals

$(\Omega\ \mathcal{F}\ P)$, $\qquad \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \ldots \subseteq \ldots \subseteq \mathcal{F}$

$\nearrow$ seq. of σ-fields

Filtration: interpretation: $\mathcal{A}_n$ = σ field of ~~the~~ upto time n observable events.

$(X_n)_{n\geq 0}$ – stoch process – or seq. of RV-s

is called <u>adapted</u> (w.r.t the filtration $(\mathcal{A}_n)_{n\geq 0}$) if $\quad X_k \in \mathcal{A}_k \quad \forall k\geq 0$.

<u>previsible</u> if $\quad X_k \in \mathcal{A}_{k-1} \quad \forall k \geq 1$

<u>innovative</u> if $\quad X \in \mathcal{L}^1$

$$E[X_{n+1} - X_n \mid \mathcal{A}_n] = 0 \quad \text{a.s.}$$

$$(\Leftrightarrow E[X_{n+1} \mid \mathcal{A}_n] = X_n)$$

<u>Rem.</u> $\Rightarrow \forall\, n, k \geq 0$

$$E[X_{n+k} - X_n \mid \mathcal{A}_n] = \sum_{\ell=1}^{k} E[X_{n+\ell} - X_{n+\ell-1} \mid \mathcal{A}_n]$$

$$= \sum_{\ell=1}^{k} E\Big[\underbrace{E[X_{n+\ell} - X_{n+\ell-1} \mid \mathcal{A}_{n+\ell-1}]}_{=0 \text{ a.s}} \Big| \mathcal{A}_n\Big]$$

$= 0$, in particular $E[X_n] = E[X_0] + E[X_n - X_0]$

$= E[X_0] + E[E[X_n - X_0 \mid \mathcal{A}_0]]$

Def: The stochastic process $(X_n)_{\geq 0}$ is called a martingale wrt the filtration $(\mathcal{Q}_n)_{n\geq 0}$ if

1) adapted ie $X_n \in \mathcal{Q}_n \ \forall n$
2) innovative ie $X_n \in \mathcal{L}^1$
   and $E[X_{n+1} / \mathcal{Q}_n] = X_n$ as

EX 1) $Y_1, Y_2 \ldots$ iid RV. $\in \mathcal{L}^1$

$\mathcal{Q}_n = \sigma(Y_1 \ldots Y_n)$. then

$$X_n := \sum_{i=1}^{n} (Y_i - E[Y_i]), \quad X_0 = 0$$

is a martingal wrt $\mathcal{Q}_n$.

2) Let $X \in \mathcal{L}^1$ and $(\mathcal{Q}_n)$ be given

interpr.: X price of smth random. $E[X/\mathcal{Q}_n]$ = our best guess for that value.

then the successive prognoses

$$X_n := E[X / \mathcal{Q}_n] \quad \text{is a martingale}$$

$\ulcorner$ $E[X_{n+1} / \mathcal{Q}_n] = E[E[X/\mathcal{Q}_{n+1}] / \mathcal{Q}_n] =$

$= E[X/\mathcal{Q}_n]$ ✓ $\lrcorner$

Gambling systems and stopping times

$(X_n)_{n\geq 0}$ martingale wrt $\mathcal{A}_n$.

Let $(V_n)_{n\geq 1}$ a previsible sequence with

(*) $$V_n \cdot \Delta X_n = V_n \cdot (X_n - X_{n-1}) \in \mathcal{L}^1.$$

Set $$(V.X)_n := X_0 + \sum_{k=1..n} V_n \cdot \Delta X_n$$
$n \geq 1$ $\quad \Delta_n(V.X) = V_n \cdot \Delta_n X$

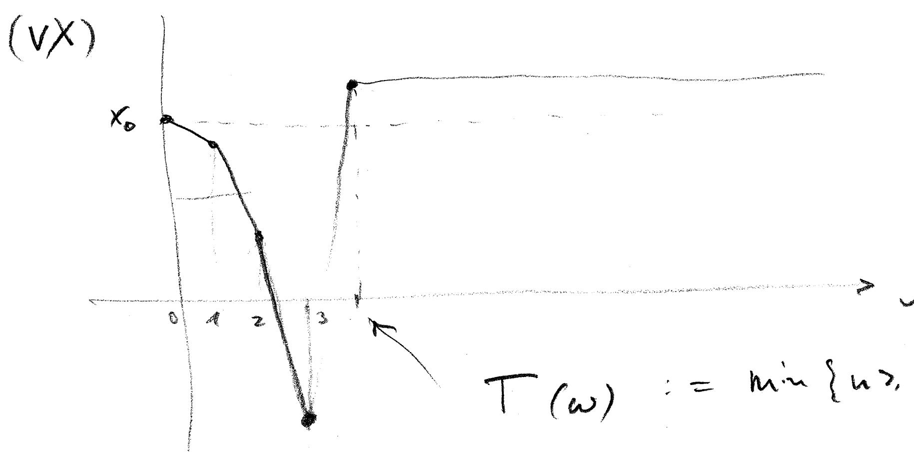
is called the <u>gambling system</u> assoc. with V.

EX $\quad X_0 = x_0$
$X_n = \sum_{i=1..n} Y_i$ $\qquad V_k = \begin{cases} 0 & \text{if otherwise} = 1 \\ 2^{k-1} & Y_1 = Y_2 = \dots = Y_{k-1} = -1 \end{cases}$

Y_i iid $\left\langle \begin{matrix} +1 & 1/2 \\ -1 & 1/2 \end{matrix} \right.$ $\qquad$ V_k previsible!

$\Delta X_k = Y_k$, $\quad \mathcal{A}_k = \sigma(Y_1, \dots, Y_k)$

$\Rightarrow$ $(V.X)_n$ = your balance at time n



$T(\omega) := \min\{n \geq 1 \mid Y_n = 1\}$

Note that $(V.X)_{T(\omega)} = x_0 + 1$ so you end up ~~with~~ winning.

$$T(\omega) = \min \{ n \geq 1 \mid Y_n(\omega) = 1 \}.$$

<u>Thm</u>. if $V.X$ is a gambling system

$\Rightarrow$ $(V.X)_n$ is a martingale

$\ulcorner$

$(V.X)$ is adapted $(V.X)_n = \underbrace{V_n}_{\mathfrak{A}_{n-1}} \cdot (\underbrace{X_n - X_{n-1}}_{\in \mathfrak{A}_n})$

and $\in \mathcal{L}^1$ (by assumption)

$$E[\underbrace{(V.X)_{n+1} - (V.X)_n}_{\Delta_{n+1}(V.X)} \mid \mathfrak{A}_n] = E[\overbrace{V_{n+1} X_{n+1} - V_{n+1} X_n}^{V_{n+1} \cdot \Delta_{n+1} X} \mid \mathfrak{A}_n]$$

$$= E[V_{n+1} X_{n+1} \mid \mathfrak{A}_n] - V_{n+1} \cdot X_n$$

$$= V_{n+1} \big(\underbrace{E[X_{n+1} \mid \mathfrak{A}_n] - X_n}_{\overline{\overline{a.s.}}\ 0} \big) = 0 \text{ a.s.} \quad \checkmark \lrcorner$$

$$\Rightarrow \quad E[(V.X)_n] = E[(V.X)_0] = X_0$$

$\lrcorner$

Stopping times.

$$T : \Omega \longrightarrow \overline{\mathbb{N}} \quad \text{st}$$

$$\{T = n\} \in \mathcal{A}_n \quad \forall n = 1, 2, \ldots$$

Interpretation: at time n I know whether $\{T = n\}$ or $\{T \leq n\}$ &, $\{T > n\}$ happened or not.

⚠ T is not (necessarily) $\mathcal{A}_n$-measurable!

<u>Ex</u> 1) $A \in \mathbb{R}$, (X_n) adapted on $\mathcal{A}_n$.

$\Rightarrow$ (first) entrance time into A

$$T_A(\omega) := \min \{ n \geq 0 \mid X_n(\omega) \in A \}$$

<u>is</u> a stopping time.

$$\{T_A \leq n\} = \bigcup_{k=0}^{n} \{\underbrace{X_k \in A}_{\in \mathcal{A}_k \subseteq \mathcal{A}_n}\} \in \mathcal{A}_n.$$

$$\Rightarrow \{T_A = n\} = \underbrace{\{T_A \leq n\} \cap (\{T_A \leq n-1\}^c)}_{\in \mathcal{A}_n}.$$

$\triangle$ $$L_A(\omega) := \sup\{n \geq 0 \mid X_n(\omega) \in A\}.$$

"last visit"

is not a stopping time

$\boxed{\underline{\text{Def}}: \quad X^T_\cdot = \text{"stopped process"} \quad X^T_n(\omega) := X_{T(\omega)\wedge n}(\omega)}$

Let T be a stopp. time.

Repr. as a gambling system

$$V_n := 1_{\{T \geq n\}} \quad \text{is previsible}$$

$$(V \cdot X)_n(\omega) = X_0 + \sum_{k=1}^{T\wedge n} (X_k - X_{k-1})(\omega) \quad (*)$$

$$= X_{T\wedge n(\omega)}(\omega)$$

$\underline{\text{is}}$ a gambling system

since $V_n(X_n - X_{n-1}) = 1_{\{T\geq n\}} \cdot (X_n - X_{n-1})$

$\in \mathcal{L}^1$!

$\Rightarrow$ $(X_{T\wedge n})_{n\geq 1}$ is a martingale.

$$(*) = X_0 + \sum_{k=1}^{n} 1_{\{T \geq k\}} (X_k - X_{k-1}) =$$

$$= X_0 + \left\{ \begin{array}{ll} \sum_{k=1}^{n} (X_k - X_{k-1}) \; ; & \text{if } T \geq n \\ \sum_{k=1}^{T} (X_k - X_{k-1}) \; ; & \text{if } T < n \end{array} \right\} = X_0 + \sum_{k=1}^{T\wedge n} \cdots$$

<u>Uniform integrability</u>. $(\Omega, \mathcal{F}, P)$.

$\mathcal{H} \subset \mathcal{L}^1$ is unif. integrable if

$$\lim_{c \to \infty} \sup_{\mathcal{H}} \int_{\{|X| > c\}} |X| \, dP = 0$$

<u>Rem.</u> if $\mathcal{H}$ is finite $\Rightarrow$ unif. int.

$$|X| = \lim_{c \to \infty} \nearrow 1_{\{|X| \le c\}} |X|$$

$$\Rightarrow E[|X|] = \lim_{c \to \infty} \int_{\{|X| \le c\}} |X| \, dP$$

but

$$E[|X|] = \underbrace{\int_{\{|X| \le c\}} |X| dP}_{\xrightarrow[c\to\infty]{} E[|X|]} + \underbrace{\int_{\{|X| > c\}} |X| dP}_{\Rightarrow \xrightarrow[c\to\infty]{} 0}$$

<u>Lemma</u>. $\exists\, g \ge 0$, with $\frac{g(x)}{x} \nearrow \infty \quad (x \to \infty)$

s.t. $\sup_{\mathcal{H}} \int g(|X|) \, dP < \infty$

$\Longleftrightarrow$ $\mathcal{H}$ is unif. int.

<u>Thm</u>: $f_n \to f$ in $\mathcal{L}^1$ $\Longleftrightarrow$ 1) $f_n \to f$ in measure
2) $\{f_n\}$ unif. int.

<u>Remark</u>: Let $(X_n)_n$ be unif int and assume $X_n \to: X_\infty$ a.s.

$\Rightarrow \{X, X_0 X_1 X_2 \dots X_n \dots\}$ is also unif. int

$\Rightarrow \{|X - X_n|\}$ also u.i.

($\Rightarrow$ $|X - X_n| \to 0$ as and $E[|X_n - X_\infty|] \to 0$ meaning $X_n \to X_\infty$ in $\mathcal{L}^1$.)

$\ulcorner$ $X_n \underset{a.s}{\xrightarrow{n\to\infty}}: X_\infty \Rightarrow |X_n| \to |X_\infty|$ as and by Fatou for $\underline{\lim}$ $E[|X_\infty|] \leq \underline{\lim}\, |X_n| \leq K < \infty$
$\mathcal{L}^1$-bdd.

$\Rightarrow X_\infty \in \mathcal{L}^1$.

Any u.i. family $\mathcal{H}$ + an $\mathcal{L}^1$-variable is u.i:

$\ulcorner$ $$\sup_{Y \in \mathcal{H} \cup \{X\}} E[|Y| ; |Y| \geq c] \leq \underbrace{\sup_{Y \in \mathcal{H}} E[|Y| ; |Y| \geq c] + E[|X| ; |X| \geq c]}_{\to 0 \text{ as } c \to \infty.} \lrcorner$$

Finally $|X - X_n| \leq |X| + |X_n|$ and $\{|X| + |X_n|\}_{n \geq 0}$ is also u.i. ...

$\lrcorner$

make a sheet about

Convergence:

a.s. $\overset{?}{\longleftrightarrow}$ in probab $\overset{?}{\longleftrightarrow}$ in L^1

in distribution (?)

i

Thm (X_n) is $(\mathcal{G}_n)$ martingale, T stopping time. Then

$\boxed{\text{OPTIONAL STOPPING}}$

1) $(X_{T\wedge n})$ is a martingale and $E[X_{T\wedge n}] = E[X_0]$.

2. If T is bdd, i.e. $T \le N$ a.s.

$$\Rightarrow E[X_T] = E[X_{T\wedge N}] = E[X_0].$$

3.) if $T < \infty$ a.s. and $X_{T\wedge n}$ is uniformly integrable $\Rightarrow$

$$E[X_T] = E[X_0].$$

Pf: 1), 2) ok. 3)

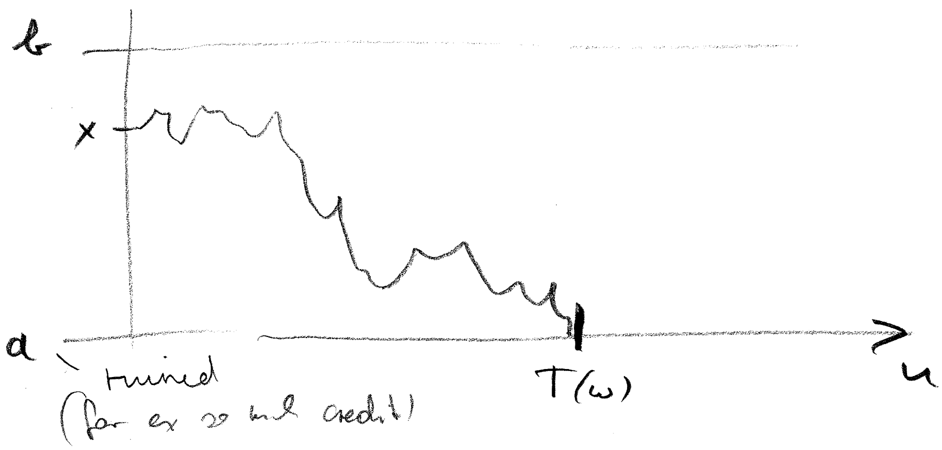
$$E[X_T] = E[\lim_{n\to\infty} X_{T\wedge n}] \overset{\text{unif. int + a.s convergence}}{=} \lim_{n\to\infty} \underbrace{E[X_{T\wedge n}]}_{=E[X_0]}$$

$$= E[X_0].$$

Appl. classical ruin problem.
(Gambling fairly to make $(b-x_0)$ \$ with credit level a)

$$X_n = x + S_n \quad , \quad S_n = \sum_{i\geq 1}^{n} Y_i$$

$$Y_i < \begin{matrix} 1; & p \\ -1; & 1-p \end{matrix}$$



$$T(\omega) = \min \{ n \geq 0 \mid X_n(\omega) \notin (a,b) \}$$

is a stopping time.

By Borel-Cantelli: $T < \infty$ a.s. $\ulcorner * \lrcorner$ HW?

$$r(x) = P[X_T = a].$$

1) $p = 1/2$. then X_n is a martingale

and $(X_{n\wedge T})$ is bdd (uniformly) $\Rightarrow$ glu int.

$$\Rightarrow \underset{x}{\underbrace{E[X_0]}} = E[X_T] = b \cdot \overbrace{P[X_T = b]}^{1-P[X_T=a]} + a P[X_T = a]$$

$$x = b(1 - r(x)) + a\, r(x)$$

$$\boxed{r(x) = \frac{b-x}{b-a}}$$

2) $p \neq 1/2$ $\qquad h(x) = \left(\frac{1-p}{p}\right)^x$

$h(X_n)$ is a martingale

(⚠ T is still the same! ie X has to hit $(a,b)^c$ NOT $h(X)$!!!)

$\Rightarrow$

$$\underset{E[h(X_0)]}{h(x)} = E[h(X_T)] = h(b)(1-r(x)) + h(a)\,r(x)$$

$$\Rightarrow \quad r(x) = \frac{h(b)-h(x)}{h(b)-h(a)} = \frac{1-\frac{h(x)}{h(b)}}{1-\frac{h(a)}{h(b)}}$$

$$= \frac{1-\left(\frac{p}{1-p}\right)^{b-x}}{1-\left(\frac{p}{1-p}\right)^{b-a}} = (*)$$

<u>$p < 1/2$</u>:

$$r(x) \geq 1-\left(\frac{p}{1-p}\right)^{b-x}$$

and this bound doesn't depend on a!

$p = \frac{18}{37}$ $\qquad b-x = 128$ is sufficient for

$r(x) \geq 0.999$!