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805

## THIRD SEMESTER EXAM

## OPERATIONS RESEARCH

(Part I, 5 hours; part II, 3 hours)

### I. Mathematical Programming

1. Consider the problem (P) of minimizing

(1) $$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

(2) $$x_{ij} + y_i - y_j \geq 0 \qquad i,j=1,\dots,n$$

(3) $$-y_1 + y_n \geq 1$$

(4) $$x_{ij} = 0 \text{ or } 1, \qquad i,j=1,\dots,n.$$

Let (P′) be the problem obtained from (P) by substituting = for ≥ in (2) and (3), and let (LP) and (LP′) be the problems obtained from (P) and (P′), respectively, by substituting $x_{ij} \geq 0$ for $x_{ij} = 0$ or 1 in (4).

(a) Discuss the relationship between (P), (P′), (LP) and (LP′), in the two cases when (a) $c_{ij} \geq 0$, $i,j=1,\dots,n$, and (b) $c_{ij}$ is of arbitrary sign, $i,j=1,\dots,n$.

(b) Give a sufficient condition for (P) to be solvable by a polynomial-time algorithm.

READ QUESTIONS WELL

2. Let (P) be a mixed integer program and (LP) its linear programming relaxation. Call a solution integer if it satisfies the integrality requirements, noninteger otherwise.

Consider the following procedure for solving (P):

Step 1. Apply the primal simplex method to (LP) for as long as you can pivot without making the solution noninteger. If this is not any longer possible, denote the current (integer) solution by $x^o$, and go to step 2.

Step 2. Perform a primal simplex pivot and denote the (noninteger) solution obtained by $x^1$. Generate a valid inequality that cuts off $x^1$, add it to the simplex tableau, and pivot in the cut row to obtain $x^o$ as a basic feasible solution to the amended linear program; then return to step 1.

(a) Show that step 2 can always be carried out. What is the difference between the old and the new basis associated with $x^o$? When does $x^o$ satisfy the cut with equality?

(b) The cut of step 2 can be generated directly (i.e., as a "primal" cut) from the first tableau associated with $x^o$, without actually carrying out the pivot that produces $x^1$. Derive a formula for this "primal" cut.

(c) Does the above procedure converge in a finite number of steps?

3. Suppose you are given a simple graph $G = (V,E)$ (finite, undirected, with no loops and no multiple edges). A matching in $G$ is any collection of edges, no two of which meet at a vertex. A cover of $G$ is any collection of vertices for which every edge of $E$ meets some vertex in the cover.

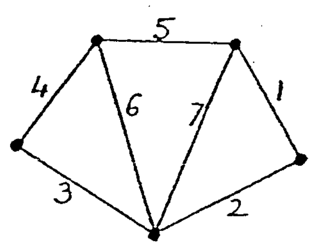
(a) State an integer program for finding a maximum cardinality matching in $G$.

(b) Use linear programming duality and fundamental properties of integer programs to show that when $G$ is bipartite the size of a maximum matching equals the size of a minimum cover.

(c) Conversely, can you say that if the size of a maximum matching equals the size of a minimum cover, then $G$ is bipartite? Justify your answer.

(d) To every matching in $G$ associate the vector $x = (x_e)_{e\in E}$ defined by $x_e = 1$ if the edge $e$ is in the matching, 0 if it is not. Let $P(G)$ be the convex hull of the 0-1 vectors associated with the matchings in $G$. What is the dimension of the polytope $P(G)$? Prove that for every $e\in E$ the inequality $x_e \geq 0$ is a facet of $P(G)$. For which vertices $v$ of $G$ is the inequality $\sum_{e\ni v} x_e \leq 1$ a facet of $P(G)$? Does $P(G)$ have other facets when $G$ is (i) bipartite, (ii) nonbipartite?

(e) Consider the graph $G$ with edges numbered 1 through 7:




Prove or disprove that

$$P(G) = \{x\geq 0 \mid x_2 + x_3 + x_6 + x_7 \leq 1,\ x_4 + x_5 + x_6 \leq 1,\ x_1 + x_5 + x_7 \leq 1,$$
$$x_1 + x_2 + x_7 \leq 1,\ x_5 + x_6 + x_7 \leq 1,\ x_3 + x_4 + x_6 \leq 1,\ x_1 + x_2 + x_3 + x_4 + x_5 \leq 2\}.$$

4. You are reminded of two results:

(A) If the linear program

$$\text{minimize} \quad c^T x \quad \text{subject to} \quad Ax = b \quad \text{and} \quad x \geqq 0$$

has an optimal solution, then it has one, say $\bar{x}$, such that if $\alpha$ denotes the index set of positive $\bar{x}$-variables, then the columns $A_{\cdot\alpha}$ are linearly independent.

(B) If the linear complementarity problem

$$q + Mz \geqq 0, \quad z \geqq 0, \quad z^T(q + Mz) = 0 \; .$$

has a solution, then it has one, say $\bar{z}$, such that if $\alpha$ denotes the index set of positive $\bar{z}$-variables, then the columns $M_{\cdot\alpha}$ are linearly independent.

Use these two results to give two proofs to the following assertion: If the convex quadratic program

$$\text{minimize} \; \tfrac{1}{2}y^T Dy + c^T x + d^T y \quad \text{subject to } y = Px \quad , \quad x \geqq 0$$

has an optimal solution, then it has one such that the number of positive x-variables will not exceed m which is the dimension of the vector y.

5. Let X and Y be two nonempty polyhedral sets with Y containing at least two points. Consider the problem: find a scalar s and a vector t to

$$\text{minimize} \quad s \quad \text{subject to} \quad Y \subseteqq sX + t \quad , \quad s \geqq 0$$

where

$$sX + t = \{ z : z = sx + t \quad \text{for some } x \in X \} .$$

In each of the following two cases, formulate the above problem as a linear program.

(i) $X = \{ x : Ax \leqq a\}$ and $Y = \{y : By \leqq b\}$

(ii) $X = \{ x : Ax \leqq a\}$ and $Y = \{y : y = P\lambda + Q\mu$ for some $\lambda, \mu \geqq 0$ and $e^T\lambda = 1\}$.

Here A, B, P and Q are arbitrary matrices of appropriate size; a and b are arbitrary vectors and e is the vector of ones.

Note: You need to give a computationally practical formulation for each case. In other words, you can not say that since (i) and (ii) are theoretically equivalent, therefore one formulation is sufficient.

6. Given a set X in $R^n$ and a mapping F from $R^n$ into $R^n$, consider the <u>stationary point problem:</u> find a vector $u \in X$ such that

$$(x - u)^T F(u) \geqq 0 \quad \text{for all } x \in X.$$

(i) Formulate this problem as one of finding a fixed point of a point - to - set mapping.

(ii) Use Kakutani's fixed point theorem to show that if X is compact and F is continuous then a stationary point exists.

80's

Third Semester Examination

Operations Research

Part II

Time 3 hours, 2:00-5:00 p.m. Open book.

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This examination contains one question on each of the following topics: integer programming, dynamic programming, convex analysis, and control theory.

1. Consider the integer program

$$(1)\quad z_I = \max \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$$

$$(2)\quad \sum_{j=1}^{n} x_{ij} \le 1 \qquad i = 1,\ldots,m$$

$$(3)\quad \sum_{i=1}^{m} x_{ij} \le b_j y_j \qquad j = 1,\ldots,n$$

$$(4)\quad x_{ij} \le y_j \qquad i = 1,\ldots,m;\ j = 1,\ldots,n$$

$$(5)\quad 0 \le x_{ij} \qquad i = 1,\ldots,m;\ j = 1,\ldots,n$$

$$(6)\quad 0 \le y_j \le 1 \qquad j = 1,\ldots,n$$

$$(7)\quad y_j \text{ integer} \qquad j = 1,\ldots,n$$

where $b_j$, $j = 1,\ldots,n$, are nonnegative integers and $c_{ij}$, $i = 1,\ldots,m$, $j = 1,\ldots,n$ are real numbers.

(a) Prove that the optimal value $z_I$ is not changed if the constraints (4) are dropped from the integer program (1)-(7).

(b) "The integer program (1)-(7) is NP-complete." Explain very briefly what this sentence means to you.

(c) Assume that one wants to find an optimal solution to (1)-(7). List the different methods that could be used. Also list the methods that you know in integer programming and which do not apply to this problem.

(d) Let $z_L = \max[\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} - \sum_{i=1}^{m} \lambda_i (\sum_{j=1}^{n} x_{ij}-1) - \sum_{j=1}^{n} \mu_j(\sum_{i=1}^{m} x_{ij}-b_jy_j)]$ subject to (4)-(7). How would you call this program? Prove that $z_L \ge z_I$ whenever $\lambda_i \ge 0$ for all $i = 1,\ldots,m$ and $\mu_j \ge 0$ for all $j = 1,\ldots,n$.

(e) Prove that the constraint matrix for (4)-(6) is totally unimodular.

1981

Third Semester Examination

Operations Research

Part I

January 8, 1981

Time 5 hours, 8:30-1:30. Open book.

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This examination contains two questions on linear programming, one on dynamic programming, two on graphs, one on nonlinear programming, and one on stochastic processes.

1. Consider the standard linear programming problem:

$$\text{Maximize} \quad cx$$

$$\text{Subject to} \quad Ax \leq b$$

$$x \geq 0$$

where $A$ is $m \times n$, and the dimensions of the other vectors are defined to correspond. Let $e = (1,\ldots,1)$ be an $n$-component vector of all 1's.

(a) Suppose the objective function is changed from $cx$ to $(c+\lambda e)x$, where $\lambda$ is a parameter. Write conditions that $\lambda$ must satisfy so that the optimal solution for the original problem also be optimal for the modified problem.

(b) Suppose ~~one of the constraints of~~ the original problem ~~is~~ has one additional constraint of the form $ex = 1/k$ where $k$ is a constant. How does this change your answer to (a)?

2. Consider the linear program

$$\begin{array}{ll} \text{Maximize} & x_2 \\ \text{Subject to} & \\ & -x_1 + x_2 \leq 1 \\ & x_1 + x_2 \leq 3 \\ & x_1 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

(a) Draw the bounding lines of the constraints and show that there are 9 points of intersection of these lines.

(b) Show that 5 of the 9 intersection points found in (a) are primal feasible, and the rest primal infeasible.

(c) Show that 3 of the 9 intersection points are dual feasible and the rest dual infeasible.

(d) State a simple geometric condition which characterizes such an intersection point as being dual feasible. [Hint. Use the outward pointing normals to the half spaces which define the feasible set, and the normal direction of the objective function.]

Questions 4 and 5 involve graphs. We state first some relevant definitions.

In a connected graph $G = (V,E)$ the distance $d(u,v)$ between vertices $u$ and $v$ is the length of the shortest path joining $u$ and $v$. (Here the length of a path means its number of edges.) The diameter $\delta(G)$ of $G$ is $\max_{u,v\in V} d(u,v)$. Finally you are reminded that a circuit is a closed path with positive length.

4. Show that the following statements are equivalent for a simple connected graph $G$.

   i) Every two vertices of $G$ are connected by at most one path of length $\delta(G)$ or smaller.

   ii) $G$ has no circuit of length $2\delta(G)$ or smaller.

   iii) Either $G$ is a cyclic or the length of the shortest circuit is $2\delta(G) + 1$.
   one word

5. Prove or disprove:

   i) If $\delta(G) = 2$, then $G$ has a spanning star.

   ii) If $G$ has a spanning star, then $\delta(G) = 2$.
   (A spanning star of $G = (V,E)$ is a subgraph of the form $(V,\{(v_0,w) \mid w\in V - \{v_0\}\})$ for some vertex $v_0\in V$.)

6. Let $A$ be an $n \times n$ symmetric positive semi-definite matrix and $F$ an $m \times n$ matrix. Let $C$ denote the set of all vectors $y \in R^m$ satisfying

$$AF^T y = 0, \quad f^T y = 0 \quad \text{and} \quad y \geqq 0$$

where $f$ is a m-vector.

(i) Show that $C$ is the characteristic cone of all nonempty level sets

$$Y(\lambda) = \{y \geqq 0: \quad \varphi(y) \leqq \lambda\}$$

where

$$\varphi(y) = (f + FAa)^T y + \frac{1}{2} y^T FAF^T y.$$

(ii) Deduce from (i) that if the LCP$(f + FAa, \ FAF^T)$ is feasible, then it has a bounded solution set if and only if $C = \{0\}$.

1981

Third Semester Examination

Operations Research

Part II

Time 3 hours, 2:00-5:00 p.m. Open book.

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This examination contains one question on each of the following topics: integer programming, dynamic programming, convex analysis, and control theory.

1. Consider the integer program

(1) $$z_I = \max \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$$

(2) $$\sum_{j=1}^{n} x_{ij} \le 1 \qquad i = 1,\ldots,m$$

(3) $$\sum_{i=1}^{m} x_{ij} \le b_j y_j \qquad j = 1,\ldots,n$$

(4) $$x_{ij} \le y_j \qquad i = 1,\ldots,m;\; j = 1,\ldots,n$$

(5) $$0 \le x_{ij} \qquad i = 1,\ldots,m;\; j = 1,\ldots,n$$

(6) $$0 \le y_j \le 1 \qquad j = 1,\ldots,n$$

(7) $$y_j \quad \text{integer} \qquad j = 1,\ldots,n$$

where $b_j$, $j = 1,\ldots,n$, are nonnegative integers and $c_{ij}$, $i = 1,\ldots,m$, $j = 1,\ldots,n$ are real numbers.

(a) Prove that the optimal value $z_I$ is not changed if the constraints (4) are dropped from the integer program (1)-(7).

(b) "The integer program (1)-(7) is NP-complete." Explain very briefly what this sentence means to you.

(c) Assume that one wants to find an optimal solution to (1)-(7). List the different methods that could be used. Also list the methods that you know in integer programming and which do not apply to this problem.

(d) Let $z_L = \max[\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} - \sum_{i=1}^{m} \lambda_i (\sum_{j=1}^{n} x_{ij} - 1) - \sum_{j=1}^{n} \mu_j (\sum_{i=1}^{m} x_{ij} - b_j y_j)]$

subject to (4)-(7). How would you call this program? Prove that $z_L \ge z_I$ whenever $\lambda_i \ge 0$ for all $i = 1,\ldots,m$ and $\mu_j \ge 0$ for all $j = 1,\ldots,n$.

(e) Prove that the constraint matrix for (4)-(6) is totally unimodular.

1983

Ph.D. Qualifying Exam 83-84

Operations Research

Part I

Mathematical Programming

and

Combinatorics

1. Consider the following linear program having two objective functions:

$$\text{Maximize} \quad c^{(1)}x$$

$$\text{Maximize} \quad c^{(2)}x$$

Subject to

$$Ax \leq b$$
$$x \geq 0$$

A <u>Pareto optimal solution</u> to this problem is a feasible vector $x^*$ such that there is no other feasible vector satisfying

$$c^{(1)}x \geq c^{(1)}x^*$$
$$c^{(2)}x \geq c^{(2)}x^*$$

with at least one of the inequalities being <u>proper</u>.

Discuss an algorithm for finding the <u>frontier set</u>, i.e. the set of all Pareto optimal solutions by using the ordinary linear program

$$\text{Maximize} \quad c^{(1)}x$$

Subject to

$$c^{(2)}x = d$$
$$Ax \leq b$$
$$x \geq 0 .$$

and performing parametric variation on the number $d$.

2. Let $G = (V,E)$ be an undirected graph. Define a cut in $G$ as an edge set of the form

$$(S,V\setminus S) = \{(i,j) \in E \mid i \in S,\ j \in V \setminus S\}$$

for some $S \subseteq V$.

(a) How many cuts are there in G?

(b) Which problem is easier: finding a minimum cardinality cut, or a maximum cardinality cut in G? Outline a method for solving the easier of these two problems. Is it polynomial in (V)?

3. Consider the problem

$$
\begin{aligned}
&\min cx \\
(P) \qquad & Ax \geq b \\
& x \geq 0 \\
& \prod_{j\in S} x_j = 0, \ \forall S \subseteq N: |S| = k,
\end{aligned}
$$

where A is $m \times n$, c and b are conformable with A, $N \doteq \{1, \ldots, n\}$, $\pi$ means product, and k is a positive integer, $1 \leq k \leq n$.

(a) Find $k_0$ such that for every $k \geq k_0$, (P) is polynomially solvable.

(b) Outline an algorithm for solving (P) for some $k < k_0$.

(c) Characterize the convex hull of the feasible solutions of (P).

4. Consider an undirected graph $G = (V,E)$. Define a tour of $G$ as a cycle (closed walk) going <u>at least</u> once through each node of $G$. (Note: nodes as well as edges can be traversed twice or more.)

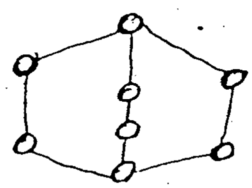
(a) Relate tours to the vectors $x = (x_e : e \in E)$ which satisfy

(i) $x_e \geq 0$ and integer for all $e \in E$,

(ii) $\sum_{e \in C(S,V-S)} x_e \geq 2$ for all $S \subset V$, $S \neq \emptyset$ where $C(S,V-S) = \{(i,j) \in E: i \in S,\ j \in V-S\}$, and

(iii) $\Sigma(x_e:$ e incident with node $i)$ is even for all $i \in V$.

(b) Let $G$ be the following graph:



For this graph, are there any vectors $x$ which satisfy conditions (i) and (ii) but not condition (iii)? Is the inequality $\sum_{e \in E} x_e \geq 10$ valid for all vectors which satisfy conditions (i) and (ii)? Is it valid for vectors which satisfy (i), (ii) and (iii)?

(c) Consider again the general graph $G$. Let $P$ be the convex hull of the vectors satisfying (i), (ii) and (iii). Show that $P$ is full dimensional if $G$ is connected and empty otherwise. Is $P$ bounded?

(d) Show that $x_e \geq 0$ is a facet of $P$ if and only if the graph $(V, E-\{e\})$ is connected.

(e) How would you modify conditions (i) - (iii) in order to obtain the incidence vectors of the Hamilton cycles of $G$? Let $Q$ be the convex hull of the incidence vectors of Hamilton cycles of $G$. Is it correct to say that the polytope $Q$ is a face of $P$?

5. (a) Let $C \subseteq R^p$ be a convex set. Show that $X = \{x \in R^n : x = Ap, p \in C\}$, where $A$ is a given $n \times p$ real matrix, is a convex set in $R^n$.

(b) Find minima of the function $f(x,y) = (x^2-y)^2$ among all the points satisfying necessary conditions for an extremum.

(c) Let $x^*$ be an optimal solution of the problem

$$\min \sum_{j=1}^{n} f_j(x_j)$$

subject to $x_j \geq 0 \quad j = 1, \ldots, n$

$$\sum_{j=1}^{n} x_j = 1,$$

where $f_j$ are differentiable functions. Show that there exists a number $\mu$ such that

$$f_j'(x_j^*) = \mu \text{ if } x_j^* > 0$$
$$f_j'(x_j^*) \geq \mu \text{ if } x_j^* = 0$$

where the prime indicates differentiation.

1985 1985

Ph.D. Qualifying Exam

Operations Research

Part I

Mathematical Programming

and

Combinatorics

1. If a system of m linear equations (which may arise from m linear inequalities to which slack variables have been added) has a nonnegative solution, then it has a solution with at most $m$ variables positive. [Hint. Use pivoting techniques.]

2. A certain firm has a vector x which measures both its production and sales efforts and which must satisfy the following constraints:

$$Ax \leq b, \quad x \geq 0 \tag{1}$$

The firm has two goals for next year, a profit goal of $p$ and a sales goal of $s$. For activity $x$ it can achieve a profit of $cx$ and sales of $dx$. In order to measure its deviation from the above goals it defines nonnegative quantities $p^+$, $p^-$, $s^+$ and $s^-$ as follows:

$$cx - p = p^+ - p^-, \quad p^+ \geq 0, \quad p^- \geq 0 \tag{2}$$

$$dx - s = s^+ - s^-, \quad s^+ \geq 0, \quad s^- \geq 0. \tag{3}$$

(a) Show that in any basic solution of (2) and (3) the following hold

$$p^+p^- = 0,$$

$$s^+s^- = 0$$

(b) Show that

$$|cx - p| = p^+ + p^-$$

$$|dx - s| = s^+ + s^-$$

measure the deviations from each goal.

(c) Show that to find a plan to achieve its goals the firm should solve the following linear program:

Minimize $p^+ + p^- + s^+ + s^-$

Subject to (1), (2), and (3).

3. Consider the polytope

$$X = \text{conv}\{x \in \{0,1\}^4 | x_1 + 3x_2 + 3x_3 + 4x_4 \geq 4\}.$$

(a) Show that $X \equiv P$, where

$$P = \text{conv} \bigcup_{i=1} P_i,$$

with

$$P_1 = \{x \in K_4 | x_1 + x_2 \geq 2\}$$

$$P_2 = \{x \in K_4 | x_1 + x_3 \geq 2\}$$

$$P_3 = \{x \in K_4 | x_2 + x_3 \geq 2\}$$

$$P_4 = \{x \in K_4 | x_4 \geq 1\}$$

and

$$K_4 = \{x \in \mathbb{R}^n | 0 \leq x_j \leq 1, \quad j = 1,\ldots,4\}.$$

(b) Find a minimal system of linear inequalities defining X.

[Hint: use polarity.]

(c) Let Y be the 0-1 knapsack polytope obtained from X by setting $y_i = 1 - x_i$, $i = 1,\ldots,4$. Show that not all the facets of Y can be obtained by sequential lifting.

4. Associate with the undirected graph $G = (V,E)$ the symmetric bipartite graph $G' = (V_1 \cup V_2, E')$, where $V = \{1,\ldots,n\}$, $V_1 = \{11,\ldots,n1\}$, $V_2 = \{12,\ldots,n2\}$, and $(i,j) \in E$ implies $(i1, j2) \in E'$, $(j1, i2) \in E'$, while $(i,j) \notin E$ implies $(i1, j2) \notin E$, $(j1, i2) \notin E$. Call a matching $M \in G'$ antisymmetric if for all $i,j$, $(i1, j2) \in M$ implies $(j1, i2) \notin M$. Show that there is a 1-1 correspondence between 2-matchings in $G$ and antisymmetric 1-matchings in $G'$. Formulate the antisymmetric matching problem in a symmetric bipartite graph as a linear program.

[Hint: the linear programming formulation of the 2-matching problem in $G$ is

$$\max \sum_{i \in V} \sum_{j > i} c_{ij} x_{ij}$$

$$\sum_{j \mid (j,i) \in E} x_{ji} + \sum_{j \mid (i,j) \in E} x_{ij} \leq 2 \qquad i \in V$$

$$0 \leq x_{ij} \leq 1 \qquad (i,j) \in E$$

$$\sum_{(i,j) \in E(S)} x_{ij} + \sum_{(i,j) \in F} x_{ij} \leq |S| + \left\lfloor \frac{|F|}{2} \right\rfloor, \quad S \subset V,\ F \subset (S, V \setminus S),\ |F| \text{ odd},$$

where $E(S)$ is the set of edges with both ends in $S$.]

5. Let $f$ be a continuously differentiable concave function, let $c$ be an r-vector and let $B$ be an $r \times n$ matrix, $r \leq n$, with linearly independent rows. Consider the problem

$$(1) \qquad \begin{array}{l} \max f(x) \\ \quad Bx = c \end{array}$$

(a) Use classical optimality conditions to show that $x^*$ is optimal for this problem if and only if

$$Bx^* = c$$
$$\nabla f(x^*) + B^T u = 0$$

where $u$ is an unconstrained r-vector.

(b) Let $x$ be a feasible solution for (1) and suppose that $[I - B^T(BB^T)^{-1}B]\nabla f(x) = 0$.

Prove that $x$ is optimal for (1).

(c) Prove $[I - B^T(BB^T)^{-1}B]^T[I - B^T(BB^T)^{-1}B] = I - B^T(BB^T)^{-1}B$.

Now define $d = [I - B^T(BB^T)^{-1}B]\nabla f(x)$.

(d) Prove $\nabla f(x)^T d = d^T d$.

(e) Show that, if $x$ is not optimal for (1), then

$$\nabla f(x)^T d > 0.$$

(f) Let $x$ be feasible for (1). Prove that $x + \theta d$ is also feasible for any real number $\theta$.

6. (a) Let $M = \{x_1,\ldots,x_n\}$ be a set of $n$ affinely dependent points in $\mathbb{R}^d$. Let $\lambda_1,\ldots,\lambda_n$ be real numbers, not all zero, such that $\sum_{i=1}^{n} \lambda_i x_i = 0$, $\sum_{i=1}^{n} \lambda_i = 0$. Let $M_1 = \{x_i \in M: \lambda_i > 0\}$ and $M_2 = \{x_i \in M: \lambda_i \leq 0\}$. Show that $\operatorname{conv} M_1 \cap \operatorname{conv} M_2 \neq \emptyset$, where conv K denotes the convex hull of K.

(b) Let $M = \{x_1,\ldots,x_n\}$ be a set of $n$ points in $\mathbb{R}^d$, where $n \geq d + 2$. Show that there exist nonempty subsets $M_1$ and $M_2$ of $M$ with $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = M$ such that $\operatorname{conv} M_1 \cap \operatorname{conv} M_2 \neq \emptyset$.

(c) Let $\{C_1,\ldots,C_n\}$ be a family of convex sets in $\mathbb{R}^d$, where $n \geq d + 2$. Show that, if every subfamily of $n-1$ sets $C_i$ has a nonempty intersection, then all $n$ sets $C_i$ have a point in common. (Hint: Use the result proved in (b)).

(d) Let $\{C_1,\ldots,C_n\}$ be a family of convex sets in $\mathbb{R}^d$, where $n \geq d + 2$. Show that, if every subfamily of $d + 2$ sets $C_i$ has a nonempty intersection, then all $n$ sets have a point in common.

1985/86

Ph.D. Qualifying Exam

Operations Research

Part I

Mathematical Programming

and

Combinatorics

Six questions, four hours

1. Let $x^o$ and $v^o$ be optimal primal and dual solutions to the linear program

$$\begin{aligned} &\text{Maximize} \quad cx \\ &\text{Subject to} \quad Ax \le b^o \\ &\qquad\qquad\qquad x \ge 0 \end{aligned}$$

where $A$ is an $m \times n$ matrix and the other dimensions are compatible. If $x^*$ is an optimal solution to the same linear program but with $b^o$ replaced by $b^*$, show that

$$c(x^o - x^*) \ge v^o(b^o - b^*).$$

2. Consider the following linear program:

$$\text{Max} \quad cx + ps$$

Subject to

$$Ax + s \leq b$$

$$s \leq u$$

$$x, s \geq 0$$

where $A$ is $m \times n$ and the rest of the vectors are dimensioned accordingly. The variable $s_i$ of the vector $s$ represents sales at price $p_i$ per unit of the ith resource which is available in amount $b_i$ and the constant $u_i$ represents the upper bound on the sales of that resource.

Assume that a revised simplex code has been used to optimize the x-solution vector <u>before</u> considering the $s$ variables. Under what conditions on the dual variable $v_i$ and selling price $p_i$ will the revised simplex solution for the whole problem make the variable $s_i$ be positive?

3. Let $X_I$ be the set of those $x \varepsilon \{0,1\}$ satisfying

$$(1) \quad \begin{aligned} &\sum_{j=1}^{n} a_{1j}x_j \geq b_1 \\ &\sum_{j=2}^{n} a_{2j}x_j \geq b_2 , \end{aligned}$$

and assume that $a_{11} > 0$, $a_{21} < 0$, $b_1 > 0$ and $b_2 - a_{21} > 0$.

(a) Use the condition $x_{11} \varepsilon \{0,1\}$ to derive a valid disjunctive cut for $X_I$ in which the coefficient of $x_1$ vanishes, and which is violated by at least one $x \varepsilon \mathbb{R}^n$ satisfying (1).
(Hint: restate (1) as a disjunction between two sets defined by pairs of inequalities, one obtained by setting $x_1 = 0$, the other by setting $x_1 = 1$; then consider the family of cuts implied by by the disjunction and choose an appropriate member.)

(b) Strengthen the cut derived under (a) by using the integrality conditions on $x_j$, $j = 2, \ldots, n$.

4. Let $G = (N,A)$ be a digraph, with $N = \{1, \ldots, n\}$. A cycle-decomposition of G is a partition of the arc set into node-disjoint directed cycles. Let $G^* = (V,E)$ be an undirected bipartite graph that has two nodes, $i1 \in V_1$ and $i2 \in V_2$ for every node i of G, with $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \phi$, $|V_1| = |V_2|$, and an undirected edge $(i1,j2) \in E \subset V_1 \times V_2$ for every directed arc $(i,j) \in A$ of G.

(a) What, if any, is the relationship between cycle decompositions of G and perfect matchings of $G^*$?

(b) Give a necessary and sufficient condition for G to have a cycle-decomposition.

(c) Give an algorithm for finding a maximal (with respect to set inclusion) induced subgraph of G that has a cycle-decomposition. State the complexity of your algorithm.

5. A wheel W is a graph with n + 1 nodes $v_0$, $v_1$, ..., $v_n$ and 2n edges defined as follows. The nodes $v_1$, ..., $v_n$ induce a cycle and $v_0$ is joined by an edge to each of the nodes $v_1$, ..., $v_n$. (See Fig. 1.) In the remainder we assume that $n \geq 4$.

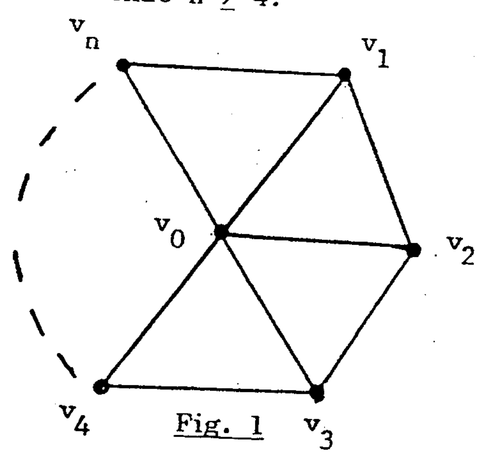



Fig. 1

(a) What is the number of Hamilton cycles in W?

(b) Show that the traveling salesman polytope on W has dimension n − 1. (The traveling salesman polytope is the convex hull of the incidence vectors of the Hamilton cycles.)

(c) The edges of the cycle induced by $v_1$, ..., $v_n$ are called the rim edges of the wheel. Show that, for any rim edge j, the constraint $x_j \leq 1$ defines a facet of the traveling salesman polytope.

(d) Conclude that the traveling salesman polytope is defined by the linear system

$\Sigma(x_j$ : edge j is incident with node $v_i) = 2$ for i = 0, 1, ..., n

$x_j \leq 1$ for every rim edge j.

6. A common strategy for unconstrained nonlinear optimization is to design a method that works well on quadratic problems and apply it to general problems. Newton's method has such a strategy, but it requires the expensive recalculation of the inverse Hessian $H^{-1}$ at each iteration. Quasi-Newton procedures save time by using an approximation $D$ of $H^{-1}$ at each iteration. One such method is the Davidon-Fletcher-Powell method, which begins with a guess $D_1$ of the inverse Hessian at the starting point $y_1$. In step $j$ the search direction is $d_j = -D_j\nabla f(y_j)$, and the new point is $y_{j+1} = y_j + \lambda_j d_j$, where $\lambda_j$ minimizes $f(y_j + \lambda d_j)$. The new estimate of $H^{-1}$ is got from a rank 2 update of $D_j$:

$$D_{j+1} = D_j + \frac{p_j p_j^T}{p_j^T q_j} - \frac{D_j q_j q_j^T D_j}{q_j^T D_j q_j},$$

where $p_j = \lambda_j d_j$ and $q_j = \nabla f(y_{j+1}) - \nabla f(y_j)$. Let's apply this method to a quadratic problem in which $f(x) = c^T x + (1/2)x^T Hx$, where $H$ is symmetric and positive definite.

a) What is $\nabla f(x)$?

b) What is $d_1$?

c) Show that $\lambda_1 = 1$. (*Hint.* $(d/d\lambda)\ f(y_1 + \lambda d_1) = d_1^T\nabla f(y_1 + \lambda d_1)$.)

d) Suppose that by pure luck we guess the inverse Hessian correctly, so that $D_1 = H^{-1}$. Show that the rank 2 update preserves this guess. That is, show $D_2 = H^{-1}$. (*Hint.* Express $q_1$ in terms of $p_1$.)

e) Given the assumption of part (d), how many iterations are required to reach the minimum?

1985/86

Ph.D. Qualifying Exam

Operations Research

Part II

Dynamic Programming,

Control Theory

and

Stochastic Processes

1. Let $I = \{i_1, i_2, \ldots, i_k\}$ where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, and let

$$W(I) = \sum_{t=1}^{k+1} w(i_{t-1} + 1, i_t),$$

where $i_o = 1$, $i_{k+1} = n$ and $w$ is an arbitrary function satisfying $w(i,j) = 0$ if $i > j$.

(a) Construct a functional equation which would enable you to solve

(P1) minimize $W(I)$ s.t. $I \subseteq \{1,2,\ldots,n\}$
in $O(n^2)$ time.

(Hint: let $f(m) = \min.\ W(I)$ subject to $I \subseteq \{1,2,\ldots,m\}$)

(b) Construct a functional equation which would enable you to solve

(P2) minimize $W(I)$ s.t. $I \subseteq \{1,2,\ldots,n\}$, $|I| = p$
in $O(n^2 p)$ time.

(c) Suppose that in (b), $p = 2^k$ and $w$ depends only on $j-i$. Show that P2 is solvable in $O(n^2 k)$ time.

(d) Show how the following location problem can be formulated as a special case of P2: there are n points placed at $x_1, x_2, \ldots, x_n$ on the real axis. We want to find p points $y_1, y_2, \ldots, y_p$ so as to

$$\text{minimize} \sum_{i=1}^{n} \underbrace{\min_{j=1,\ldots,p} \{|x_i - y_j|\}}_{\text{this is the distance from } x_i \text{ to its nearest } y_i}$$

1987/88 Burj C...

1987

Ph.D. Qualifying Exam

Operations Research

Part I

Mathematical Programming

and

Combinatorics

Six questions, four hours

1. Consider the following linear system:

$$(P) \quad \begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

where A is an mxn matrix and the dimensions of x and b are consistently determined. Let S be the set of all solutions to (P); assume $S \neq \phi$.

A variable $x_j$ is said to be a <u>null variable</u> if $x_j = 0$ for all $x \varepsilon S$.

(a) For the system

$$6x_1 + 5x_2 + 3x_3 + 4x_4 = 8$$

$$3x_1 + 2x_2 + x_3 + 2x_4 = 4$$

$$x_1, x_2, x_3, x_4 \geq 0$$

show that $x_2$ and $x_3$ are null variables.

(b) Show that if there exists a vector $v \neq 0$ such that $vA \geq 0$, $vb = 0$ and $(vA)_j > 0$ (where $(vA)_j$ is the jth component of vA), then $x_j$ is a null variable.

(c) Prove the converse of the statement in (b) by considering the linear program

$$\begin{aligned} \text{Maximize} \quad & e^j x \\ \text{Subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

where $e^j$ is the unit vector with 1 in the jth position and 0's elsewhere.

2. Consider the following problem in n variables:

Maximize $cx$

Subject to

$$Ax \leq b \tag{1}$$

$$x \geq 0 \tag{2}$$

$$(\alpha x \leq \alpha_0) \text{ or } (\beta x \leq \beta_0) \tag{3}$$

where A is mxn, and $\alpha$ and $\beta$ are n component row vectors and $\alpha_0$ and $\beta_0$ constants. A vector x will be said to be <u>feasible</u> if it satisfies (1), (2) and either one or both of the constraints in (3).

(a) Use parametric programming to describe a practical solution procedure.

(b) Describe how you could tell when the problem has a solution. In how many ways can it have or not have a solution?

(c) To what extent could your method generalize to more complicated constraints of the form of (3)?

3. An <u>edge cover</u> of (the vertices of) a graph $G = (V,E)$ is a set $C \subseteq E$ such that every vertex is incident with some edge in C. There is a close parallel between the theory of minimum edge covers and maximum matchings in a graph.

(a) Give a necessary and sufficient condition for an edge cover to be minimum, in terms of alternating paths. Prove the validity of your condition.

[Hint: Define analogs of the concepts of exposed node and augmenting path relative to a matching. Note that, unlike in matching, the alternating paths used may be closed.]

(b) Give a linear characterization of the edge covering polytope (the convex hull of incidence vectors of edge covers).

[Hint: There is a one to one correspondence between the inequalities of the systems defining the edge covering polytope and the matching polytope.]

(c) Let G be an edge-weighted graph. Give an efficient procedure for finding an edge cover that minimizes the weight of the heaviest edge in the cover.

[Hint: No connection to matching here, but some connection to spanning trees. Use the obvious fact that a graph has an edge cover if and only if it has no isolated vertex.]

## Solution

(a) Call a vertex covered (overcovered, twice overcovered) if it is incident with at least one (at least two, at least three) edges in C.

Call an alternating trail a C-reducing trail if its extreme edges are in C and its endpoints are either distinct and overcovered or identical and twice overcovered.

An edge cover C is minimum if and only if there exists no C-reducing trail. The "only if" part is immediate; for the "if" part, if C and C' are edge covers such that $|C'| < |C|$, then the symmetric difference of C and C' can be shown to contain a C-reducing trail.

(b) The edge covering polytope is defined by the system

$$\sum_{j \in I(i)} x_j \geq 1, \qquad i \in V$$

$$\sum_{j \in I(S)} x_j \geq (|S| + 1)/2, \qquad S \subseteq V,\ |S| \text{ odd and } \geq 3$$

$$x_j \geq 0, \qquad j \in E$$

where $G = (V,E)$ and

$$I(i) := \{ j \in E \mid j \text{ is incident with } i \},\ i \in V$$

(c) Delete edges of G in order of decreasing edge weight, until no further edge can be deleted without isolating a vertex. At that point the remaining subgraph G' has an edge cover, and any edge cover of G' is an edge cover of G that minimizes the weight of the heaviest arc in the cover. For suppose G has an edge cover whose heaviest edge, $j_*$, is of lower weight than the heaviest edge of G'. Then all edges of G' heavier than $j_*$ can be deleted without isolating any vertex, a contradiction.

4. Consider the integer program (IP)

$$Z_{IP} = \text{Max} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij}$$

$$(1) \quad \sum_{i=1}^{n} x_{ij} = 1 \qquad \text{for } j = 1, \ldots, n$$

$$(2) \quad \sum_{j=1}^{n} x_{ij} = 1 \qquad \text{for } i = 1, \ldots, n$$

$$(3) \quad \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_{ij} \leq b$$

$$(4) \quad x_{ij} = 0 \text{ or } 1 \qquad \text{for } i = 1, \ldots, n \text{ and } j = 1, \ldots, n.$$

Denote by $Z^{(i)}$ the value of the Lagrangian dual obtained by relaxing contraint (i) in a Lagrangian fashion. Let $Z_{LP}$ be the value of the linear programming relaxation.

(a) Show that $Z_{IP} \leq Z^{(1)} \leq Z^{(1)(2)} \leq Z_{LP} = Z^{(3)} = Z^{(1)(3)}$

(b) Give an example where $Z^{(1)(2)} < Z_{LP}$.

5. Let $A = (a_{ij})$ be a 0,1 matrix with m rows and n columns. Assume that A has no dominated row (row i <u>dominates</u> row k if $a_{ij} \geq a_{kj}$ for all $j = 1, \ldots, n$). Denote by $Q_A$ the polyhedron defined by the inequalities $Ax \leq 1$, $x \geq 0$, and let $P_A = \text{Conv}(\{0,1\}^n \cap Q_A)$ be the corresponding set packing polytope.

(a) Show that $x_j \geq 0$ defines a facet of $P_A$ for every $j = 1, \ldots, n$.

(b) Show that $\sum_{j=1}^{n} a_{ij}x_j \leq 1$ defines a facet of $Q_A$ for every $i = 1, \ldots, m$.

(c) Does $\sum_{j=1}^{n} a_{ij}x_j \leq 1$ always define a facet of $P_A$?

[Hint: consider $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ ]

(d) Let $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. Assume that $P_{A_1} = Q_{A_1}$ and $P_{A_2} = Q_{A_2}$. Show that $P_A = Q_A$.

(e) Given 0,1 matrices $A_1$ and $A_2$ such that $P_{A_1} = Q_{A_1}$ and $P_{A_2} = Q_{A_2}$, can you give other ways of composing $A_1$ and $A_2$ into a matrix A so that $P_A = Q_A$? (No proof is required for question (e).)

6. Consider the problem

$$\begin{aligned} \min \quad & x^3 + 2y^3 + 2z^3 \\ \text{s.t.} \quad & x^2 + y^2 + z^2 = 6 \end{aligned}$$

Give a bound on the asymptotic convergence ratio of the projected gradient method at the local minimum point $(x, y, z) = (2, 1, 1)$.

1827

PHD QUALIFYING EXAM IN OR

PART I

6 questions

4 hours

Tuesday, January 10, 1989 (morning)

1. Suppose the simplex method has performed $k$ pivot steps in the solution of a linear program so that the current tableau is $T^{(k)}$ with entries $t_{ij}^{(k)}$. Suppose also that there are exactly two incoming pivot columns, $j_1$ and $j_2$, which have negative reduced costs (indicators), and that corresponding pivot rows, $i_1$ and $i_2$, respectively (with $i_1 \neq i_2$) have been selected by the ratio rule.

(a) If $t_{i_1 j_2}^{(k)} \leq 0$ and $t_{i_2 j_1}^{(k)} \leq 0$, show that at least two more pivots will be needed to complete the solution.

(b) If $t_{i_1 j_2}^{(k)} \geq 0$ and $t_{i_2 j_1}^{(k)} \geq 0$, show that at least one more pivot will be needed to complete the solution. Explain why only one pivot might be enough.

(c) If $t_{i_1 j_2}^{(k)} \geq 0$ and $t_{i_2 j_1}^{(k)} \leq 0$, show that in order to (possibly) reduce the number of pivots, it is better to first perform the first pivot.

2. A k-goal programming problem is of the form (for k = 3):

Minimize $\quad P_1 s_1^- + P_2 s_2^- + P_3 s_3^-$

Subject to

$$
\begin{aligned}
c^{(1)}x + s_1^- - s_1^+ &= g_1 \\
c^{(2)}x + s_2^- - s_2^+ &= g_2 \\
c^{(3)}x + s_3^- - s_3^+ &= g_3 \\
Ax \qquad\qquad &\leq b \\
x, s_i^-, s_i^+ \geq 0 \quad &\text{for } i = 1,2,3
\end{aligned}
$$

Here $c^{(i)}x$ is an objective function and $g_i$ is the desired goal value for that function. Also $P_i$ is the penalty for not achieving the goal $g_i$. Assume the constraints $Ax \leq b$, $x \geq 0$ have feasible solutions.

(a) Explain the meaning of the optimal solution to the goal programming problem.

(b) Assume $P_1 \gg P_2 \gg P_3$, that is, goal $g_1$ is infinitely more important than goal $g_2$, and goal $g_2$ is infinitely more important than goal $g_3$. Show how to solve this problem by first minimizing $s_1^-$; then holding $s_1^-$ fixed and minimizing $s_2^-$; etc.

3. Generate all the facet defining inequalities for the vertex packing polytope defined on the graph G of Figure 1. What is the facet defining inequality of highest (Chvatal) rank that you can get?

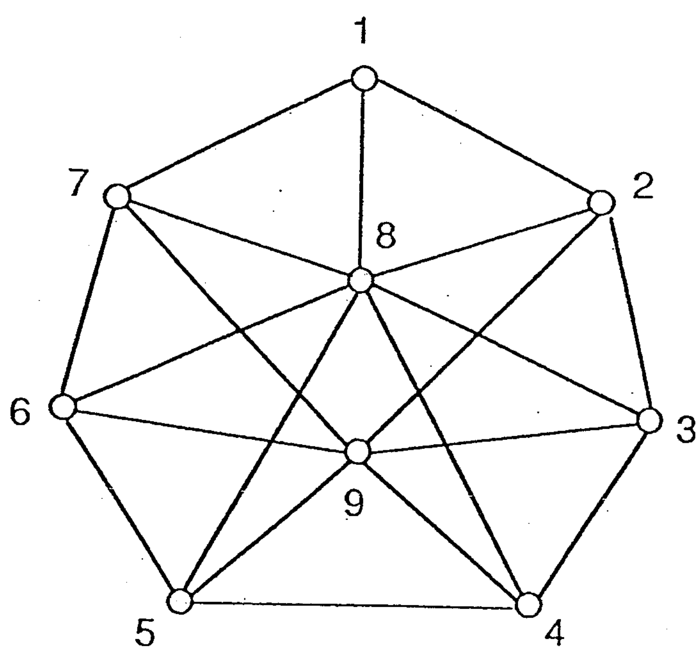



Figure 1

4. Let S be the set of $x \in \mathbb{R}^2$ satisfying

$$x_1 - x_2 \geq a_{12} \qquad \vee \qquad x_2 - x_1 \geq a_{21}$$

$$b_1 \leq x_1 \leq c_1$$

$$b_2 \leq x_2 \leq c_2$$

a. Give a set of linear inequalities in $\mathbb{R}^2$ that define the convex hull of S.

b. Introduce additional variables and give a set of linear inequalities in a higher dimensional space that define a polytope whose projection on $\mathbb{R}^2$ is the convex hull of S.

5. Consider a finite set $E$ and a partition of $E$ into nonempty subsets $E_1$, $E_2, \ldots, E_m$. To each $E_i$, assign two integers $\ell_i$ and $u_i$ ($\ell_i \le u_i$). Let $k$ be a positive integer.

Consider $B = \{X \subseteq E : \ell_i \le |X \cap E_i| \le u_i \text{ for } i = 1,2,\ldots,m; \; |X| = k\}$. Assume $B \ne \emptyset$.

(a) Let $X, X' \in B$ and $x \in X \setminus X'$. Show that there exists $x' \in X' \setminus X$ such that both $(X \setminus \{x\}) \cup \{x'\}$ and $(X' \setminus \{x'\}) \cup \{x\}$ belong to $B$.

(b) Show that $B$ is the family of bases of a matroid.

(c) Describe the circuits of this matroid.

(d) Assume that each $e_j \in E$ has a weight $w_j$. Describe an algorithm for the following problem.

Problem (P) : Find a set $X \in B$ having the smallest weight $\sum_{j \in X} w_j$.

(e) Assume $E$ is the edge set of a connected graph with $k + 1$ nodes. Is the following problem NP-hard?

Problem (Q) : Find a minimum weight spanning tree $X$ with the additional condition that $X \in B$.

(f) Remove the condition that $E_1, E_2, \ldots, E_m$ is a partition. Can problem (Q) be solved in polynomial time? What about problem (P)?

6. Consider the nonlinear program,

$$\min \quad x^T H x$$

$$\text{s.t.}\, e^T x = 1$$

where H is an n×n symmetric matrix and e a vector of n ones.

(a) Write first-order necessary conditions for an optimal point $\bar{x}$.

(b) Write second-order sufficient conditions for an optimal point $\bar{x}$.

(c) Suppose we solve the problem with the projected gradient method. Write an expression for the search direction vector (i.e., the negative projected gradient) at the current point $x^k$. Simplify as much as possible.

(d) Suppose we solve with the reduced gradient method. Write an expression for the reduced gradient vector at the current point $x^k$. Simplify as much as possible. Let $x_1$ be the basic variable, and define

$$x = (x_1, \bar{x})$$

$$H = \begin{bmatrix} h_{11} & h_1^T \\ h_1 & \bar{H} \end{bmatrix}$$

Hint: if $x = g(y)$,

$$\nabla_y f(g(y), y) = \nabla_y f(x,y) + \nabla g(y)^T \nabla_x f(x,y).$$

(e) Let us write the problem in the form

$$\min x^T H x + \mu\phi(x)$$

where $\phi$ is an exact penalty function. Define a suitable function $\phi$ and state a lower bound for the parameter $\mu$ if $x^*$ is the optimal point.

1993

# 8

Ph.D. Qualifying Exam
Algorithms, Combinatorics & Optimization
GSIA portion

5 questions

Time: 4 hours

12:15 - 4:15

Friday, Jan 8, 1993

1. Let $g(x) = cx+d$, where c is a 1xn vector, x is an nx1 vector, and d is a scalar. Given p such functions it is possible to define a piecewise linear function f(x) as:

$$f(x) = \text{Minimum } (c^{(1)}x + d^{(1)},\ c^{(2)}x + d^{(2)}, \ldots, c^{(p)}x + d^{(p)}).$$

For such a function, consider the problem

$$\begin{array}{ll} \text{Maximize} & f(x) \\ \text{Subject to} & Ax = b \\ & x \geq 0 \end{array}$$

where A is an mxn matrix and b is an mx1 vector. Show how to convert this problem into a linear programming problem.

2. A graph G is *chordal* if every cycle of length greater than three possesses a chord, i.e. an edge joining two nonconsecutive vertices of the cycle. Given two vertices x,y, denote by d(x,y) the smallest number of edges in a path from x to y. The *eccentricity* of a vertex v is the maximum distance from v to any vertex in G. The *radius* r(G) of G is the smallest eccentricity of a vertex in G and the *diameter* of G is the maximum eccentricity. The *center* C(G) of G is the subgraph induced by the set of all vertices with eccentricity r(G).

(a) Let x and y be nonadjacent vertices of a chordal graph and let k be an integer such that $1 \le k \le d(x,y)-1$. Show that the set of vertices z such that $d(x,z) = k$ and $d(z,y) = d(x,y)-k$ induces a clique.

(b) Let G be a chordal graph with diameter equal to 2r(G). Show that C(G) is a clique.

(c) Give an example of a chordal graph such that C(G) is not a clique.

3. Consider a bipartite graph, with bipartition $(A,B)$. Each $i \in A$ has an integral supply $a_i$, while each $j \in B$ has an integral demand $d_j$. There exist edges $(i,j)$ for some pairs $(i,j)$ with $i \in A$ and $j \in B$. An edge allows an arbitrary flow from $i$ to $j$. The objective is to determine if there is a feasible flow in this graph (one that meets all supplies and demands exactly).

(a) Give an algorithm for this problem (possibly by reducing it to an already studied problem) and prove its correctness. Your algorithm need not be the fastest possible, but should be "appropriate" (don't solve a multicommodity flow problem when a shortest path problem is sufficient).

(b) Based on this algorithm, give necessary and sufficient conditions for a feasible flow to exist. These conditions should go beyond "If the algorithm works, a flow exists; otherwise, not."

(c) How would your algorithm and conditions change if each edge had a capacity $u_{ij}$?

4. Consider the following two problems:

Set covering:

$$\text{(SC)}\qquad \begin{aligned} &\min \Sigma\,(c_j x_j : j \in N) \\ &\quad \Sigma\,(x_j : j \in N_i) \geq 1, i \in M \\ &\qquad\quad x_j \in \{0,1\}, j \in N \end{aligned}$$

Facility location:

$$\text{(FL)}\qquad \begin{array}{lll} \min \Sigma\,(c_j x_j : j \in N) & & \\ \Sigma\,(y_{ij} : j \in N_i) & \geq 1, & i \in M \\ -\Sigma\,(y_{ij} : i \in M_j) + |M_j| x_j & \geq 0, & j \in N \\ y_{ij} \geq 0,\ \forall\ i,j;\ x_j \in \{0,1\}, & & j \in N \end{array}$$

(a) Show that under a certain definition of $M_j$, $j \in N$, (SC) and (FL) are equivalent: $\bar{x} \in \mathbb{R}^N$ is an optimal solution to (SC) if and only if there exists $\bar{y} \in \mathbb{R}^{M\times N}$ such that $(\bar{y},\bar{x})$ is an optimal solution to (FL).

(b) Compare the strength of the linear programming relaxation of (FL) to that of (SC) by projecting the feasible set of (FL) onto the subspace of the x variables.

(c) Give a different formulation of (FL), say (FL′), using the same set of variables, but such that the linear programming relaxation of (FL′) is stronger than that of (FL).

(d) Compare the strength of the linear programming relaxation of (FL′) to that of (SC) by the same method as under (b).

5. Let $P := P_1 \cup P_2$, $Q := Q_1 \cup Q_2$, where for $i = 1,2$, $P_i$ and $Q_i$ are nonempty polyhedra. For a set S, let conv(S) denote the closed convex hull of S.

(a) Which, if any, of the following two relations is always true (in each case give a proof or a counterexample):

a.1. $\mathrm{conv}(P \cap Q) \subseteq \mathrm{conv}(P) \cap \mathrm{conv}(Q)$

a.2. $\mathrm{conv}(P \cap Q) \supseteq \mathrm{conv}(P) \cap \mathrm{conv}(Q)$.

(b) Give necessary and sufficient conditions for both relations to be true simultaneously.

1993

# Ph.D. Qualifying Exam

## Algorithms, Combinatorics & Optimization

## GSIA portion

5 questions

Answer 4 questions out of 5

Time: 4 hours

1. Let $P = \{x \varepsilon \mathbb{R}^n : Ax = b, x \geq 0\}$. Prove that a point $x \varepsilon P$ is a vertex of $P$ if and only if the columns of $A$ corresponding to positive components of $x$ are linearly independent. (No assumption is made on the rank of $A$ or the dimension of $P$.)

1993

① Sse LD. Then $\exists v \neq 0$ s.t. $A^+ v = 0$.

$\Rightarrow$ So $A^+ \lambda v = 0$. $A\lambda v' = 0$.

$A(x + \lambda v') = 0$

$A(x - \lambda v') = 0$

---

Sse $x$ not a vtx. $\exists\ y, z \in P$ s.t. $x = \lambda y + (1-\lambda) z$ some $0 < \lambda <$

$A^+(y-z)^+ = 0$

---

(2) Dual

③ (b) Projection, Lifting ... Balas.

(a) $\Delta$, $\cap$

(c)

2. (a) Develop an efficient (not simplex, ellipsoid or interior point) algorithm for the linear program

$$\text{Max} \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad x_1 \le b_1$$

$$x_i - \sum_{j=1}^{i-1} a_{ij} x_j \le b_i \qquad i = 2,3,\ldots,n$$

$$x_j \ge 0 \quad j = 1,2,\ldots,n.$$

where all $b_i$'s and $a_{ij}$'s are nonnegative (the $c_j$'s may have arbitrary signs.)

(b) Apply your algorithm to the problem

$$\begin{array}{llllll} \text{Max} & 4x_1 - 3x_2 + 2x_3 - 5x_4 & & & & \\ & x_1 & & & & \le 4 \\ & -2x_1 + x_2 & & & & \le 3 \\ & -x_1 - 3x_2 + x_3 & & & & \le 1 \\ & -3x_1 & - & x_3 & + \; x_4 & \le 4 \\ & x_1,\; x_2,\; x_3,\; x_4 \ge 0. & & & & \end{array}$$

(c) Use your algorithm from (a) to show that the same basis $B$ is optimal for all nonnegative right hand sides.

(d) Prove $B^{-1} \ge 0$.

3. Consider the set partitioning problem

(SP) $$\min\{cx : Ax = 1,\ x\varepsilon\{0,1\}^n\},$$

where A is an mxn 0-1 matrix, and let $M := \{1,\ldots,m\}$, $N := \{1,\ldots,n\}$, $N_i := \{j\varepsilon N : a_{ij} = 1\}$, $i\varepsilon M$.

Let $i,k\varepsilon M$ be such that

$$N_i \cap N_k \neq \emptyset,\ N_i \backslash N_k \neq \emptyset,\ N_k \backslash N_i \neq \emptyset.$$

Then there exists a unique 2-partition of $N_{ik} := N_i \cup N_k$ into subsets $N^1_{ik}$ and $N^2_{ik}$ such that the dichotomy

(1) $(x_j = 0, j\varepsilon N^1_{ik}) \vee (x_j = 0, j\varepsilon N^2_{ik})$

is a valid branching rule, and $N^1_{ik}$ is (inclusion-) maximal subject to this condition.

(a) Prove the above statement and find the unique 2-partition in question.

(b) Let

$$C = \text{conv}\{x\varepsilon\mathbb{R}^n : Ax = 1,\ x \geq 0,\ x \text{ satisfies (1)}\}.$$

Introduce 2(n+1) additional variables to describe C by a system of roughly 3n+2m linear equations and inequalities. Projection, lifting ....

(c) Use projection to describe C by a system of linear inequalities in the original variables. How many inequalities do you need? Which inequalities are facet defining?

4. Consider the problem

$$\min\{cx : x \varepsilon X\},$$

where X is the solution set of the system

(1) $\Sigma(x_{ij} : j \varepsilon N) = 1 \qquad i \varepsilon N$

(2) $\Sigma(x_{ij} : i \varepsilon N) = 1 \qquad j \varepsilon N$

(3) $x_{ij} + x_{ji} \leq 1 \qquad i, j \varepsilon N$

(4) $x_{ij} \varepsilon \{0, 1\} \qquad i, j \varepsilon N.$

(a) Interpret the above problem on the complete directed graph D with node set N.

Let $\tilde{X}$ be the solution set of the system (1), (2), (3) and

(4′) $x_{ij} \geq 0, \qquad i, j \varepsilon N.$

Does the polytope $\tilde{X}$ have fractional vertices? Prove your answer.

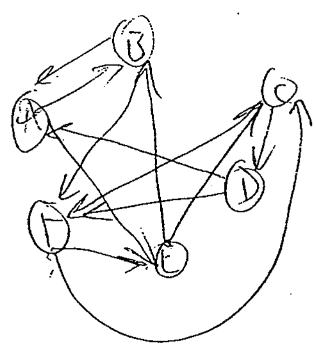
(b) Let (1′) and (2′) be the constraints obtained from (1) and (2), respectively, by replacing "=" with "≤", and let X′ be the solution set of (1′), (2′), (3), (4). Further, let G be the intersection graph of the coefficient matrix of the system (1′), (2′), (3). Show that the vertex packing polytope P(G) defined on G is isomorphic to conv X′. Use this fact to derive a class of valid inequalities for conv X′ associated with the odd holes of G. Do these inequalities define facets of conv X′? Can lifting be applied to them?

5. Six philatelists have been to the British Museum on the day the unique old stamp collection disappeared. Each had entered once, stayed for some time, and then left. If two were present at the same time, at least one of them saw the other.

Scotland Yard detectives questioned the philatelists and collected the following data: Abe said he saw Burt and Eddie at the Museum. Burt said he saw Abe and Frank. Charlie said he saw Dennis and Frank. Dennis said he saw Abe and Frank. Eddie said he saw Burt and Charlie. Frank said he saw Charlie and Eddie.

The detectives didn't know what to make of these testimonies; they did not seem to yield any clues. So they called in a consultant, Sherlock Holmes. Upon a cursory examination of the material, Sherlock exclaimed: at least one of these guys is lying! What led him to this observation?

[Hint: Blessed by nature with curiosity, Sherlock has devoted one of his vacations to the study of graph theory.]



## Solutions

1. W.l.o.g., assume the first p components of x are positive and the last n-p components are 0. Let $x = \begin{pmatrix} y \\ 0 \end{pmatrix}$, $y > 0$. Denote by B the first p columns of A. Then $Ax = By = b$.

Suppose the columns of B are not linearly independent. Then there exists $w \neq 0$ such that $Bw = 0$. Therefore, $B(y \pm \varepsilon w) = By = b$. For small enough $\varepsilon$, $y \pm \varepsilon w > 0$. Consequently $x' = \begin{pmatrix} y+\varepsilon w \\ 0 \end{pmatrix}$ and $x'' = \begin{pmatrix} y-\varepsilon w \\ 0 \end{pmatrix}$ are both in P. Since $x = \frac{1}{2}x' + \frac{1}{2}x''$, x is not a vertex of P.

Suppose x is not a vertex of P. Then $x = \lambda x' + (1-\lambda)x''$ where $x', x'' \in P$, $x' \neq x''$ and $0 < \lambda < 1$. Since $x, x' \in P$, $A(x-x') = Ax - Ax' = 0$. Furthermore, the last n-p components of $x'$, and hence of $x-x'$, must be 0. There fore the columns of B are linearly dependent.

2. (a) If $c_n \leq 0$, set $x_n = 0$.

Otherwise, set $x_n = b_n + \sum_{j=1}^{n-1} a_{nj}x_j$

Replace $x_n$ in the objective function and remove the last constraint. Now the linear program has the same form but one fewer variable. When only one variable is left, set $x_1 = \begin{cases} 0 \text{ if } c_1 \leq 0 \\ b_1 \text{ if } c_1 > 0 \end{cases}$ and plug back to get $x_2, \ldots, x_n$.

(b) $x_4 = 0$

$x_3 = 1 + x_1 + 3x_2$ Updated $c_1 = 4+2$ Updated $c_2 = -3+6$

$x_2 = 3 + 2x_1$

$x_1 = 4.$

Now we get $x_2 = 11$, $x_3 = 38$.

(c) The algorithm depends on the signs of the terms in the objective function which, after updating, are expressed in terms of the $c_j$'s and $a_{ij}$'s only. So the optimal basis does not depend on the $b_i$'s.

(d) $x = B^{-1}b$. Assume some element of $B^{-1}$, say $b_{ij}$, is strictly negative. Then, let $b = e_j$ be the $j^{th}$ unit vector. This implies $x_i < 0$, a contradiction to (c).

3. (a) Define $S_1 := \{j \varepsilon N_{ik} : a_{ij} = a_{kj} = 1\}$, $S_2 := \{j \varepsilon N_{ik} : a_{ij} + a_{kj} = 1\}$, and let $N^1_{ik} \subseteq S_1$, $N^2_{ik} = N_{ik} \setminus N^1_{ik}$. Then any $x \varepsilon \{0,1\}^n$ that satisfies $\sum_{h=1}^{n} a_{hj} x_i = 1$ for $h = i,k$, satisfies the disjunction (1). Hence (1) is a valid branching rule for any such 2-partition. The one for which $N^1_{ik}$ is maximal is $N^1_{ik} = S_1$, $N^2_{ik} = S_2$. For suppose $N^1_{ik} \supseteq S_1$, $N^1_{ik} \cap S_2 \neq \emptyset$. Then there exists $x \varepsilon \{0,1\}^n$ such that $x_j = 1$ for some $j \varepsilon N^1_{ik} \cap S_2$ and $x_\ell = 1$ for some $\ell \varepsilon N^2_{ik} \cap S_2$, and obviously there exists some (SC) for which this x is feasible; hence the disjunction (1) is not valid.

(b) Let
$$P_1 := \{x \varepsilon R^n : Ax = 1,\ x \geq 0,\ x_j = 0,\ j \varepsilon N^1_{ik}\}$$
$$P_2 := \{x \varepsilon R^n : Ax = 1,\ x \geq 0,\ x_j = 0,\ j \varepsilon N^2_{ik}\}$$
Then $C = \mathrm{conv}(P_1 \cup P_2)$. Further,

$$\mathrm{conv}(P_1 \cup P_2) := \{x : \begin{array}{lllll} x - x^1 & & - x^2 & & = 0 \\ & Ax^1 - 1x^1_0 & & & = 0 \\ & x^1_j & & & = 0,\ j \varepsilon N^1_{ik} \\ & & Ax^2 - 1x^2_0 & & = 0 \\ & & x^2_j & & = 0,\ j \varepsilon N^2_{ik} \\ & x^1_0 & & + x^2_0 & = 1 \\ & x^1, x^1_0 \geq 0, & x^2,\ x^2_0 \geq 0 & & \} \end{array}$$

(c) The projection of the above system on the subspace of the x variables is the set of inequalities $\alpha x \geq \alpha_0$ for all $(\alpha, \alpha_0)$ in the cone W defined by

$$\begin{array}{llllll} -\alpha_j & + ua_j & & & \geq 0 & j \varepsilon N \setminus N^1_{ik} \\ -\alpha_j & & + vA_j & & \geq 0 & j \varepsilon N \setminus N^2_{ik} \\ & - u1 & & + \alpha_0 & \geq 0 & \\ & & - v1 & + \alpha_0 & \geq 0. & \end{array}$$

Further, an inequality $\alpha x \geq \alpha_0$ defines a facet of C if and only if $(\alpha, \alpha_0)$ is an extreme ray of W. The number of extreme rays may be exponential in n.

If we can find $\bar{u}$, $\bar{v}$ s.t. $\bar{u}1 = \bar{\alpha}_0 = 1$, $\bar{v}1 = \bar{\alpha}_0 = 1$, $\bar{u}a_j = 0$, $j \varepsilon N^2_{ik}$, $\bar{v}a_j = 0$, $j \varepsilon N^1_{ik}$, then $\alpha x \geq 1$ is valid with

$$\alpha_j \doteq \begin{cases} 0 & \text{for } j \varepsilon N^1_{ik} \cup N^2_{ik} \\ \max\{\bar{u}a_j, \bar{v}a_j\}, & j \varepsilon N \setminus N^1_{ik} \cup N^2_{ik} \ \} \end{cases}$$

4. (a) The problem is the assignment problem without cycles of length 2 and without loops. On the complete digraph on n nodes D, x is the incidence vector of arcs and every solution to (1), (2), (3), (4) is a union of dicycles of length at least 3.

$\tilde{X}$ can have fractional vertices. An example for $n = 3$ is $x_{11} = x_{22} = x_{12} = x_{21} = \frac{1}{2}$, $x_{33} = 1$, $x_{13} = x_{23} = x_{31} = x_{32} = 0$.

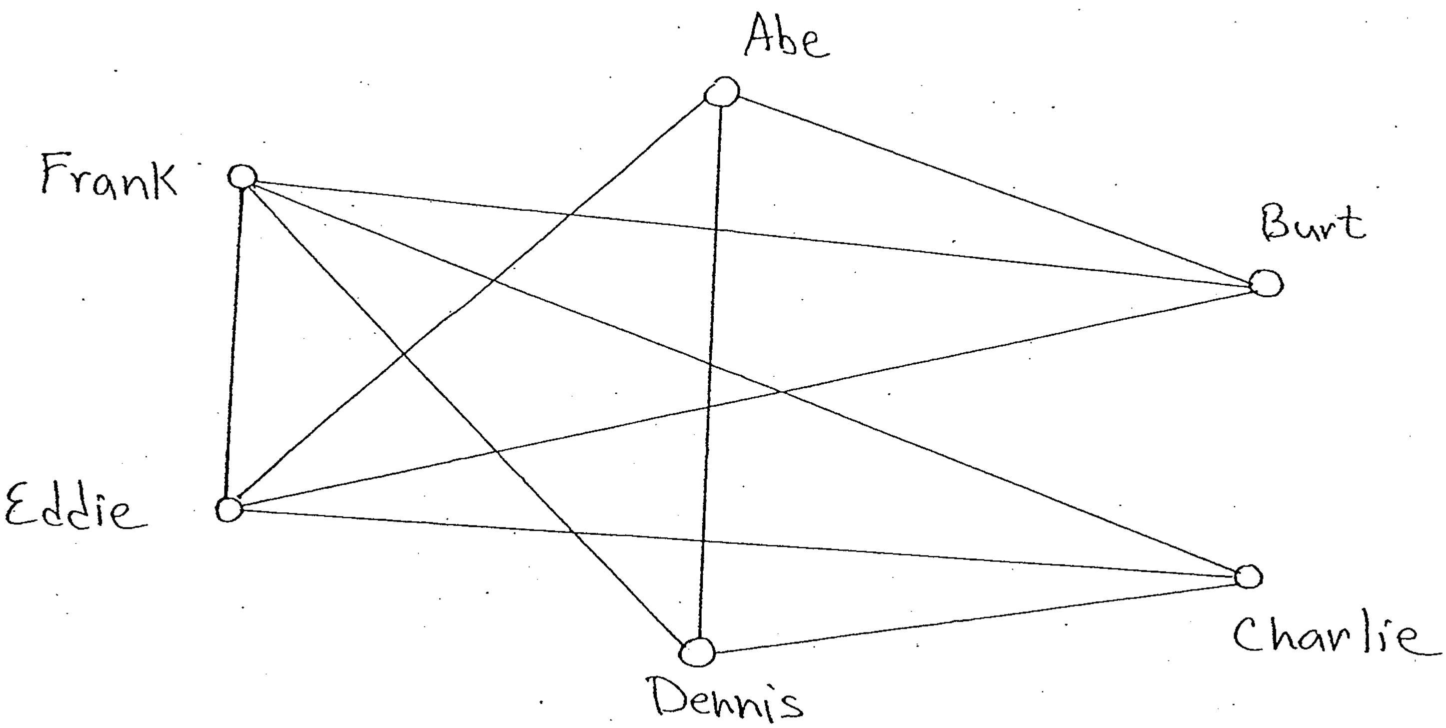
(b) P(G) is the convex hull of 0-1 vectors satisfying (1′), (2′), (3), which is conv X′.

A set S of vertices of G induces a hole in G if it can be ordered into a sequence $v_1, \ldots, v_s$, $s = |S|$, such that for $i = 1, \ldots, s$, $v_i$ is adjacent to $v_{i-1}$ and $v_{i+1}$ and to no other vertex in S. Two arcs $(i,j), (k,\ell)$ of D correspond to adjacent vertices of G if $i = k$ or $j = \ell$ or $i = \ell$ and $j = k$. Thus a closed sequence of arcs $(i_1, j_1), \ldots, (i_q, j_q)$ of odd length in D corresponds to an odd hole in G if and only if for $k = 1, \ldots, q$, $i_k = i_{k+1}$ or $j_k = j_{k+1}$ or $i_k = j_{k+1}$ and $j_k = i_{k+1}$, and for all $\ell \neq k+1$, $i_k \neq i_\ell$ and $j_k \neq j_\ell$, and either $i_k \neq j_\ell$ or $j_k \neq i_\ell$. If H is the arc set of D corresponding to an odd hole in G, the inequality

$$x(H) \leq \lfloor |H|/2 \rfloor$$

is valid for conv X′. Such an inequality is facet defining if and only if the corresponding odd hole inequality for P(G) is facet defining. If the inequality does not define a facet of conv X′, a facet defining inequality can be obtained from it by lifting.

5. Assigning a vertex to each of the six philatelists and an edge to each pair of philatelists who testified to have seen each other, we obtain the graph G shown below.




Since each philatelist spent a single time interval at the British Museum, G by construction must be an interval graph. But an interval graph is triangulated, and G has a 4-hole: Abe-Dennis-Frank-Eddie. This is a contradiction.

1994

# Ph.D. Qualifying Exam

## Part I (5 hours):

Answer 6 questions out of 7.

## Linear Programming 1

1. The well known Klee-Minty (KM) example of size $n$ can be stated as:

$$\begin{array}{lrcll} \text{Maximize} & \displaystyle\sum_{j=1}^{n} 10^{n-j} x_j & & & \\ \text{Subject to} & \displaystyle\{2\sum_{j=1}^{i-1} 10^{i-j} x_j\} + x_i & \leq & 100^{i-1} & (i = 1, 2, \ldots, n) \\ & x_j & \geq & 0 & (j = 1, 2, \ldots, n). \end{array}$$

   (a) Write the KM problem for $n = 3$.

   (b) Show that the maximum $z$-change (maximum objective function change) pivoting rule will find the optimal solution in one simplex pivot.

   (c) Write the dual problem to the KM problem for $n = 3$. Find the optimal solution to the dual problem by inspection. Show that knowledge of this solution essentially reduces the KM example to a one-by-one problem

   (d) Indicate why the results in (b) and (c) hold for the KM problem for any $n$.

## Linear Programming 2

2. The aim is to solve the linear programming problem

$$
\begin{aligned}
\min \quad & cx \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0.
\end{aligned}
\tag{1}
$$

It will be solved by first solving the more restricted problem,

$$
\begin{aligned}
\min \quad & cx \\
\text{s.t.} \quad & Ax = b \\
& x \geq d
\end{aligned}
\tag{2}
$$

for some nonnegative $d \neq 0$. The bounds $d$ will then be gradually reduced, and (2) repeatedly reoptimized, until an optimal solution of the original problem is obtained. The motivation for this strategy is that the more restricted problem (2) excludes many extreme points that might have been visited if (1) were solved first. The solution of (2) may be close to the optimum and therefore allow one to avoid visiting these extreme points when the bound $d$ is relaxed. Assume for the sake of argument that this motivation is valid.

(a) It may be possible to terminate the algorithm even before $d = 0$. State a condition that is sufficient for the algorithm to be terminated. *Hint.* Solve (2) after the change of variable $y = x - d$, and let $B$ be the optimal basis.

(b) Write an expression for a value of $x$ that is optimal in (1) at termination.

(c) What type of simplex algorithm would you recommend for reoptimization? Why?

(d) If the termination condition in part (a) is not satisfied, the algorithm replaces $d$ with $d'$ (where $0 \leq d' \leq d$), and (2) is reoptimized. Give a sufficient condition on $d'$ for which $B$ is still optimal. The new bound $d'$ is chosen so as to violate this condition.

(e) Let $d'$ be chosen as above. Show that $Ad' < Ad$. *Hint.* Does $cy$ increase or decrease?

(f) Assume that (2) is solved and reoptimized with a simplex algorithm that is guaranteed to terminate. Prove that the algorithm described above terminates with an optimal solution after finitely many iterations, if an optimal solution exists.

Convex Analysis [10 points]

3. Identify which of the following sets are convex and which are not. Explain why.

   (a) [2 points] A point $x$ in $\mathbb{R}^{\{1,2,\ldots,n\}\times\{1,2,\ldots,n\}}$ is said to be *metric* if it obeys the following conditions.

      i. $x_{ij} \geq 0$ for all $i, j \in \{1, 2, \ldots, n\}$.

      ii. $x_{ii} = 0$ for all $i \in \{1, 2, \ldots, n\}$.

      iii. $\forall i, j \in \{1, 2, \ldots, n\}, \quad x_{ij} = x_{ji}$.

      iv. $\forall i, j, k \in \{1, 2, \ldots, n\}$ the triangle inequality $x_{ij} \leq x_{ik} + x_{kj}$ holds.

   Is the set of all metric points convex?

   (b) [4 points] Define a point $x$ in $\mathbb{R}^{\{1,2,\ldots,n\}\times\{1,2,\ldots,n\}}$ to be a *cycle metric* if it is metric and in addition, we have

      v. for any pair $i, j$ not adjacent in the cyclic order $1, 2, \ldots, n$, the value of $x_{ij}$ equals the shorter of the two alternative distances (using the $x$-values as lengths) between $i$ and $j$ obtained by using the paths in the cycle numbered $1, 2, \ldots, n$ in the clockwise and counter-clockwise directions respectively.

   Note that a cycle metric is uniquely defined by the positive values assigned to the $n$ entries $x_{12}, x_{23}, \ldots, x_{n1}$.

   Is the set of all cycle metric points convex?

   (c) [4 points] Define a point $x$ in $\mathbb{R}^{\{1,2,\ldots,n\}\times\{1,2,\ldots,n\}}$ to be a *tree metric* if it is metric and in addition,

      vi. there is a spanning tree $T$ on the node set $\{1, 2, \ldots, n\}$ such that for any pair $i, j$, the value of $x_{ij}$ is the sum of the $x$-values of all the edges in the unique path in $T$ between $i$ and $j$.

   Is the set of all tree metric points convex?

## Integer Programming

4. Consider the set covering problem

$$\min\{cx : Ax \geq 1,\ x \in \{0,1\}^n\},$$

where $A$ is $m \times n$ with $\sum_j a_{ij} = 3$, and the system $Ax \geq 1$ is of the form

$$x_{j_1(i)} + x_{j_2(i)} + x_{j_3(i)} \geq 1, \quad i = 1, \dots m.$$

Describe a lift and project procedure for generating strong cuts from the disjunctions

$$x_{j_1(i)} = 1 \ \vee \ x_{j_2(i)} = 1 \ \vee \ x_{j_3(i)} = 1. \tag{1}$$

In particular:

(a) Describe by a linear system in $\mathbb{R}^{4n+3}$ the convex hull of the union of the three polyhedra obtained from the LP relaxation by imposing (1);

(b) Show how to project this linear system onto the subspace $\mathbb{R}^n$ of the $x$ variables;

(c) Discuss briefly the properties of the cuts obtained.

5. The precedence-constrained ATSP asks for a minimum-cost tour which satisfies the following constraints: given the home city, 1, where the tour starts and ends, for a specified list of ordered pairs $[i, j]$, $i \neq 1 \neq j$, each city $i$ is visited by the tour before the corresponding city $j$.

   (a) Which arcs are unusable in any tour (which variables are forced to 0) because of the precedence constraints?

   (b) Define the precedence graph $G^*$ as having a node for every city other than 1, and an arc $(i, j)$ for every pair such that $i$ has to precede $j$. Use $G^*$ to give a necessary and sufficient condition for the problem to be feasible.

   (c) Formulate the precedence-constrained ATSP as a 0-1 program in the arc variables only. Can the inequalities that you are using be strengthened?

## Graph Theory

6. Prove or disprove.

   (a) Any undirected graph that is the union of two edge-disjoint spanning trees is biconnected (2-vertex connected).

   True (b) Recall that a dominating set of an undirected graph $G$ is a subset of nodes s.t. that every node of $G$ is adjacent to at least one node from this subset. We assume that every node is adjacent to itself.

   indep set
   vertex cover

   There exist a pair of node-disjoint dominating sets in any connected undirected graph.

   (c) Let $G$ be a cubic (3-regular) graph with a proper 3-edge coloring using colors 1,2 and 3. For any node subset $S$, let $\delta_i(S)$ denote the number of edges colored $i$ that have exactly one endpoint in $S$. For any node subset $S$, all three numbers $\delta_1(S), \delta_2(S)$ and $\delta_3(S)$ have the same parity.

   True

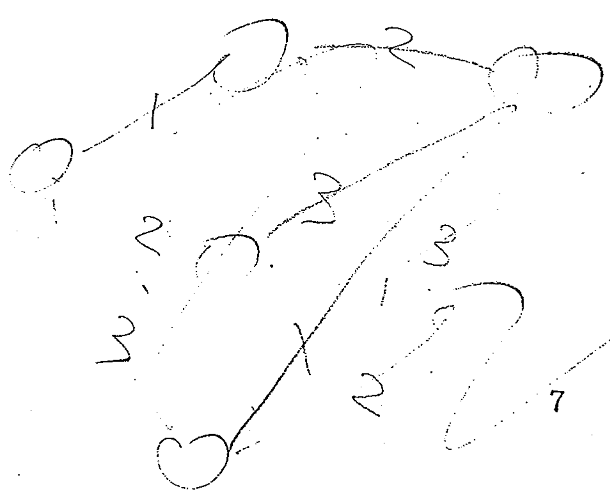
key induction

IS

VC

is a dominating set

c)



## Networks and Matchings [10 points]

7. (a) Let $G = (V, A)$ be a directed acyclic graph (DAG) with positive weights $w : A \rightarrow \mathbb{R}^+$ on the arcs. Furthermore, let $r \in V$ be a node such that there is a directed path from $r$ to every node $v \in V$. A *directed spanning tree* of $G$ rooted at $r$ is defined as an (arc-) minimal subgraph of $G$ in which there is a directed path from $r$ to every node $v \in V$. Note that this tree has $|V| - 1$ arcs and every node in this tree has indegree exactly one, except for $r$ which has zero indegree.

   Give a linear time ($O(|A|)$) algorithm to find a directed spanning tree of $G$ of minimum total weight. (Partial credit for algorithms with higher running time.)

   (b) Let $G = (V, E)$ be a connected undirected graph with positive lengths $w : E \rightarrow \mathbb{R}^+$ on the edges. Recall that a *shortest path tree* of $T$ rooted at a node $r \in V$ is a spanning tree $T_r$ of $G$ such that for any node $v \in V$, the path in $T_r$ from $v$ to $r$ is a shortest length path from $v$ to $r$ in $G$.

   i. Give an example of a graph and a root node for which the shortest path tree rooted at this node is not unique.

   ii. Design a polynomial-time algorithm to find a shortest path tree of an undirected graph rooted at a given node with the minimum *total* length. (The total length is the sum of lengths of all edges in the spanning tree).

1994

# Ph.D. Qualifying Exam

## Part II (3 hours):

Answer all 4 questions.

## Nonlinear Programming

1. (a) Let $A$ be an $n \times m$ matrix with full row rank. The orthogonal projection of a vector $c \in I\!R^n$ onto $\{x \mid Ax = 0\}$ is

$$d = c - A^T(AA^T)^{-1}Ac.$$

Use first and second-order optimality conditions to show that $d$ is the closest point to $c$ in $\{x \mid Ax = 0\}$. That is, show $x = d$ is the unique solution of

$$\begin{aligned} \min \quad & \|x - c\|^2 \\ \text{s.t.} \quad & Ax = 0. \end{aligned}$$

(b) We wish to solve the linear programming problem

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

by solving the problem

$$\begin{aligned} \min \quad & cx - \alpha \sum_j \ln x_j \\ \text{s.t.} \quad & Ax = b \end{aligned} \tag{1}$$

We are given a starting solution $x^0 > 0$ that is feasible in (1). In iteration $k$ of the algorithm, the current iterate is $x^k$. One step of Newton's method is applied to the unconstrained problem,

$$\min \ cx - \alpha \sum_j \ln x_j,$$

using $x^k$ as the starting point, to obtain the point $\bar{x}$. Then the orthogonal projection $d$ of $\bar{x} - x^k$ onto $\{x \mid Ax = 0\}$ is computed. The next iterate is

$$x^{k+1} = x^k + d.$$

The parameter $\alpha$ is gradually reduced to zero as the algorithm proceeds.

i. Given that the current iterate is $x^k$, write an expression for $\bar{x}$. It is convenient to let $D = \mathrm{diag}(x_1^k, \ldots, x_n^k)$, so that $x^k = De$, where $e$ is a vector of ones, and the gradient[1] $\nabla(\sum_j \ln x_j)$ evaluated at $x = x^k$ is $D^{-1}e$.

ii. Let $d_p$ be the orthogonal projection of $-D^2c$ onto $\{x \mid Ax = 0\}$, and let $d_c$ be the orthogonal projection of $x^k$ onto $\{x \mid Ax = 0\}$. Show that the direction $d$ is a linear combination of $d_p$ and $d_c$.

Notes: The above method is a *projected Newton barrier method* for solving an LP. The directions $d_c$, $d_p$ are *predictor* and *corrector* steps. Note that the corrector is given less and less weight as the *barrier parameter* $\alpha$ goes to zero. For appropriate $\alpha$'s, the method is identical to Karmarkar's famous projective scaling method for solving LP's.

[1] The gradient $\nabla f(x)$ is $(\frac{\partial}{\partial x_1} f(x), \ldots, \frac{\partial}{\partial x_n} f(x))$.

**Jerry Thompson's LP1 Solution**

1. (a)

$$
\begin{array}{llcccccl}
\text{Maximize} & 100x_1 & + & 10x_2 & + & x_3 & & \\
\text{Subject to} & x_1 & & & & & \le & 1 \\
 & 20x_1 & + & x_2 & & & \le & 100 \\
 & 200x_1 & + & 20x_2 & + & x_3 & \le & 10,000 \\
 & & & & & x_1, x_2, x_3 & \ge & 6
\end{array}
$$

Bringing $x_3$ into the basis gives $z = 10,000$ and the reduced costs of $x_1$ and $x_2$ become negative, $-100, -10$ so no improvements are possible. Hence solution is $x_1 = x_2 = 0$, $x_3 = 10,000$, $z = 10,000$. If we bring in any other variable on the first step a lesser change in the objective function occurs.

(b)

$$
\begin{array}{llcccccl}
\text{Minimize} & v_1 & + & 100v_2 & + & 10,000v_3 & & \\
\text{subject to} & v_1 & + & 20v_2 & + & 200v_3 & \ge & 100 \\
 & & & v_2 & + & 20v_3 & \ge & 10 \\
 & & & & & v_3 & \ge & 1
\end{array}
$$

Clearly $v_1 = 0$, $v_2 = 0$, $v_3 = 1$ satisfies all of the inequalities and $z = 10,000$. Since the first two constraints are oversatisfied they can be ignored.

(c) For (a) the reduced costs of variable $j$ after the pivot are $-10^{n-i}$ and $z = 100^{n-1}$. for (b) the surplus variable for the $j$-th constraint when $v_n = 1$ and $v_1 = v_2 = \cdots = v_{n-1} = 0$ is $100^{n-j}$.

**John Hooker's LP2 Solution**

(a) After the change of variable $y = x - d$, (2) becomes

$$\begin{array}{ll} \min & cy + cd \\ \text{s.t.} & Ay = b - Ad \\ & y \geq 0. \end{array} \tag{2}$$

Let $B$ be the optimal basis, so that the optimal solution is $(y_B, y_N) = (B^{-1}(b - Ad), 0)$. (2) is identical to (1) when $d = 0$. So the algorithm can be terminated if $B$ is optimal when $d = 0$. Because $d$ does not affect the reduced costs, it is enough for $B$ to be feasible, i.e.,

$$B^{-1}b \geq 0.$$

(b) $(x_B, x_N) = (B^{-1}b, 0)$ is optimal in (1).

(c) It suffices that $B^{-1}(b - Ad') \geq 0$, or

$$Ad' \leq B^{-1}b.$$

(d) The dual simplex method, because the dual solution $c_B B^{-1}$ remains feasible in the dual.

(e) Let $y, y'$ be the solutions of (2) corresponding to $d, d'$. Because $B$ becomes infeasible in (2) when $d'$ replaces $d$, the corresponding dual solution becomes suboptimal. The new optimal dual value is therefore strictly greater than the previous one, which implies by strong duality that $cy' > cy$. Meanwhile the optimal value of (2) cannot increase because $d' \leq d$. So $cy' + Ad' \leq cy + Ad$, which implies $Ad' < Ad$.

(f) Because $cy$ strictly increases in each iteration, an optimal basis $B$ that becomes infeasible in (2) cannot become feasible again. Because there are a finite number of bases, the algorithm terminates with an optimal solution.

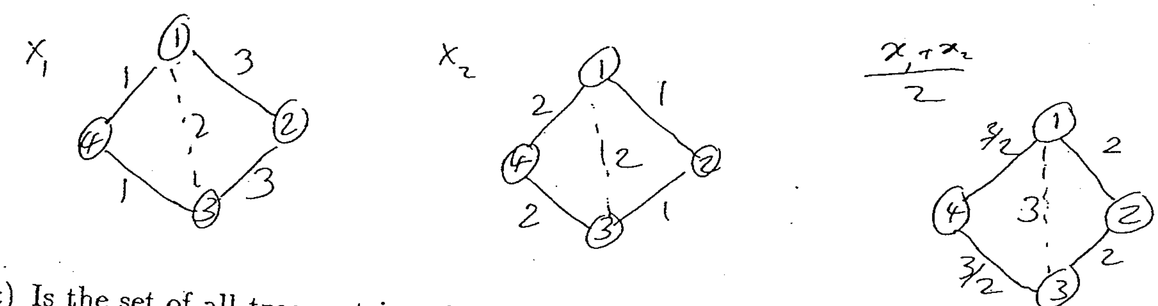
**Ravi's Convex Analysis Solution**

(a) Is the set of all metric points convex?

YES. There are a couple of obvious ways to show this. The most direct is to observe that this set is described as the intersection of the finite set of closed halfspaces given by (*i*) through (*iv*). The more roundabout way is to verify convexity directly by showing that all points in the line between any pair of metric points are metric.

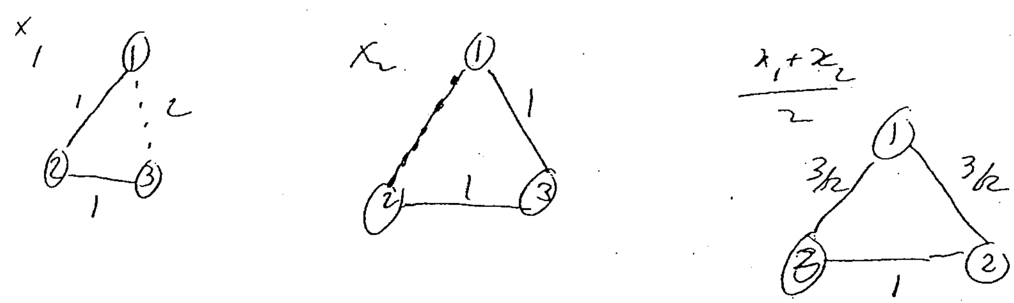
(b) Is the set of all cycle metric points convex?

NO. Here is an example, where $x_1$ and $x_2$ are cycle metrics, but $x = \frac{x_1+x_2}{2}$ is not one.




(c) Is the set of all tree metric points convex?

NO. Here is an example of two points $x_1$ and $x_2$ both tree metrics but their midpoint is not.



4. (a) Let the system $Ax \geq 1, x \geq 0, -x \geq -1$, be written as $\tilde{A}x \geq \tilde{1}$. Then the convex hull is the set of $x \in \mathbb{R}^n$ for which there exist vectors $(y^k, y_0^k) \in \mathbb{R}^{n+1}$, $k = 1, 2, 3$, satisfying

$$\begin{array}{rcll}
x - \sum_{k=1,2,3} y^k & = & 0 & \\
\tilde{A}y^k - \tilde{1}y_0^k & \geq & 0 & \\
y^k_{j(ik)} - y_0^k & = & 0 & k = 1,2,3 \\
\sum_{k=1,2,3} y_0^k & = & 1 & \\
y_0^k & \geq & 0, & k = 1,2,3
\end{array}$$

(b) The projection cone $W$ is the set of $(\alpha, \{u^k, u_0^k\}_{k=1,2,3}, \beta) \in \mathbb{R}^{n+3(m+2n+1)+1}$ satisfying

$$\begin{array}{rcll}
\alpha - u^k\tilde{a}_j & = & 0, & j \in \{1,\dots,n\} - j(ik) \\
\alpha - u^k\tilde{a}_{j(ik)} - u_0^k & = & 0 & \\
u^k\tilde{1} + u_0^k - \beta & \geq & 0, & k = 1,2,3,
\end{array}$$

where $\tilde{a}_j$ is the $j$-th column of $\tilde{A}$.

(c) The extreme rays of $W$ give rise to cuts $\alpha x \geq \beta$ that are facets of the convex hull of the union of the three polyhedra.

## Balas' Linear Programming 2 Solution

5. (a) Unusable arcs:

   $(1, j)$ for any $j$ that has a predecessor

   $(i, 1)$ for any $i$ that has a successor

   $(j, i)$ for any pair $[i, j]$ such that $i$ has to precede $j$

   $(i, j)$ for any pair $[i, j]$ for which there exists $k \neq 1$ such that $i$ has to precede $k$ and $k$ has to precede $j$.

   For any other arc, one can exhibit a feasible tour containing it.

   (b) There exists a feasible tour if and only if the precedence graph $G^*$ is acyclic.

   $\Rightarrow$ obvious $\Leftarrow$ if $G^*$ is acyclic, any tour visiting the cities in the topological order defined on $G^*$ is feasible.

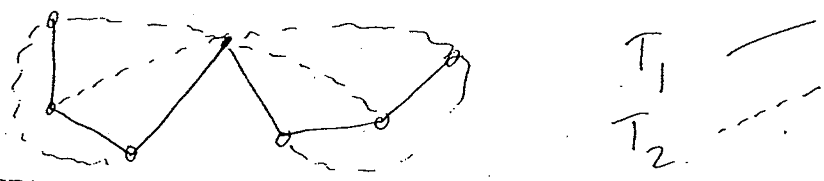
   (c)

$$
\begin{aligned}
\min \quad & cx \\
& x(\delta^+(i)) = 1 \\
& x(\delta^-(i)) = 1 \qquad\qquad i = 1, \ldots, n \\
& x \in \{0,1\}^A \\
& x(P(j,i)) \leq |P(j,i)| - 1 \quad \text{for all paths } P(j,i) \text{ from } j \text{ to } i \text{ and all pairs } [i,j] \text{ such that } i \text{ has to precede } j.
\end{aligned}
$$

The last set of inequalities can be strengthened to

$$x(j, Q) + x(Q, Q) + x(Q, i) \leq |Q| \text{ for all } Q \subset N \setminus \{1, i, j\}.$$

6. (a) FALSE. There are several small examples, e.g.,




(b) This is TRUE.

Note that the union of the edges colored with any two of the three colors is a collection of disjoint cycles of $G$. Further, for any node subset $S$, the number of edges of a cycle with exactly one endpoint in $S$ is even. Thus, for any pair of distinct colors $i, j$ and for any node subset $S$, $\delta_i(S) + \delta_j(S)$ is even. applying this to the two pairs (1,2) and (1,3) and arguing about parity shows the result.

(c) This is TRUE. (only if) Given that $T(G)$ is Eulerian, we can infer that $G$ is connected and therefore, so is $L(G)$. Furthermore, $L(G)$ is simply the induced subgraph of $T(G)$ on the nodes corresponding only to the edges of $G$. Since $T(G)$ is Eulerian, the degree of every node in it is even. In going from $T(G)$ to $L(G)$, the degree of a node corresponding to an edge in $G$ reduces by exactly two (corresponding to deleting the nodes representing its two endpoints that appear in $T(G)$). Thus, for every such node, the degree continues to stay even. We have thus shown that $L(G)$ is connected with all nodes having even degree, and hence it is Eulerian.

(if) This is the reverse operation from the previous para. To go from $L(G)$ to $T(G)$, we add in the nodes corresponding to the vertices of $G$, and connect them to their adjacent edges and nodes. Note that there is a simple bijection between the node and edge neighbors of any vertex of $G$, and so the degree of this vertex in $T(G)$ is even. We also increase the degree of every node in $T(G)$ that corresponds to an edge of $G$ by two (connecting it to its two endpoints). This leaves the degree of such nodes even. Finally, it is easy to see that $T(G)$ is also connected, showing that it is Eulerian.

**Ravi's Networks Solution** [10 points]

7. (a) Let $G = (V, A)$ be a directed acyclic graph (DAG) with positive weights $w : A \rightarrow \mathbb{R}^+$ on the arcs. Furthermore, let $r \in V$ be a node such that there is a directed path from $r$ to every node $v \in V$. A *directed spanning tree* of $G$ rooted at $r$ is defined as an (arc-) minimal subgraph of $G$ in which there is a directed path from $r$ to every node $v \in V$. Note that this tree has $|V| - 1$ arcs and every node in this tree has indegree exactly one, except for $r$ which has zero indegree.

   Give a linear time ($O(|A|)$) algorithm to find a directed spanning tree of $G$ of minimum total weight. (Partial credit for algorithms with higher running time.)

   (b) Let $G = (V, E)$ be a connected undirected graph with positive lengths $w : E \rightarrow \mathbb{R}^+$ on the edges. Recall that a *shortest path tree* of $T$ rooted at a node $r \in V$ is a spanning tree $T_r$ of $G$ such that for any node $v \in V$, the length of the path in $T_r$ from $v$ to $r$ equals that of a shortest length path from $r$ to $v$ in $G$.

   i. Given an example of a graph and a root node for which the shortest path tree rooted at this node is not unique.

   ii. Design a polynomial-time algorithm to find a shortest path tree of an undirected graph rooted at a given node with the minimum *total* length. (The total length is the sum of lengths of all edges in the spanning tree).

## John Hooker's Nonlinear Programming Solution

1. (a) Noting that every node except $r$ has exactly one incoming arc, we do the following for every node in the graph except the root $r$: scan all the incoming arcs and pick the lowest weight among these (breaking ties arbitrarily) to include in the solution.

   To show that the arcs picked form a directed tree rooted at $r$, it suffices to show that there is a path from $r$ to every node $v$. This can be shown "backwards" by following the trail of incoming arcs picked starting at $v$. Since there are no cycles ($G$ is a DAG), the only node at which this trail can end is at $r$.

   Since every node except $r$ has indegree exactly one in any directed spanning tree, and we have picked the cheapest such arc, our tree has minimum total weight.

   The running time is trivially linear in the number of edges: we scan the incoming arcs at every node retaining the minimum. Each arc is scanned exactly once in this procedure, and the minimum for each node can be maintained on the fly.

   (b) From the graph $G$ we form a DAG $G_r$ containing all edges that are in a shortest path from $r$ to some node $v$. We direct such an edge in the shortest path from $r$ to $v$, along the direction of this path from $r$ to $v$.

   Let $l_v$ represent the length of a shortest path from $r$ to $v$. Note that $l_r = 0$. Furthermore, if $G_r$ contains an arc $(x, y)$ note that $l_y = l_x + l(xy)$. In particular, since all lengths are positive, we have $l_y > l_x$.

   It is easy to see $G_r$ contains no cycles since this would give a contradiction following the last derived inequality around the cycle. Also $G_r$ contains at least one directed path from $r$ to $v$. Moreover, *any* path from $r$ to any node $v$ in $G_r$ is a shortest length path from $r$ to $v$. So our task is reduced to finding a minimum-length directed spanning tree in $G_r$. We can now apply the answer to the previous part of this question to accomplish this.

1997

# Ph.D. Qualifying Exam

## Algorithms, Combinatorics & Optimization

## GSIA portion

4 questions

Time: 4 hours

1. Assume that the optimal solutions to the two linear programs

$$\text{(i)}\quad \begin{array}{ll} \text{Max} & cx \\ \text{subject to} & Ax \le b \\ & x \ge 0 \end{array} \qquad\qquad \text{(ii)}\quad \begin{array}{ll} \text{Max} & dx \\ \text{subject to} & Ax \le b \\ & x \ge 0 \end{array}$$

have the same basis. Consider the problem

$$\text{(iii)}\quad \begin{array}{ll} \text{Max} & (\lambda_1 c + \lambda_2 d)x \\ \text{subject to} & Ax \le b \\ & x \ge 0, \end{array}$$

where $\lambda_1 + \lambda_2 = 1$ and $\lambda_1, \lambda_2 \ge 0$.

If $u^*$ and $v^*$ are optimal dual solutions to (i) and (ii), show that $\lambda_1 u^* + \lambda_2 v^*$ is an optimal dual solution to (iii).

2. (a) Show that if a convex set C in $R^d$ is covered by a finite family of halfspaces (each may be either open or closed) then C is covered by some d + 1 or fewer of these halfspaces.

(b) Consider the following statement:

*A set A in $R^d$ is contained in the boundary of conv A if and only if B is contained in the boundary of conv B for each subset B of at most f(d) elements from A.*

(i) Let $f^*(d)$ be the smallest value of f(d) for which this statement is always true. What is $f^*(d)$?

(ii) Give examples showing $f^*(d)$ could not be smaller.

(iii) Prove that the value of $f^*(d)$ you gave in answer to part (i) is always sufficient to ensure that the statement is true.

3. Define $$K = \{x \in \mathbb{R}^n : Ax \geq b,\ 0 \leq x \leq 1\}$$
$$= \{x \in \mathbb{R}^n : \tilde{A}x \geq \tilde{b}\}$$
and $$K^0 = \text{conv}\ (K \cap \{0,1\}^n).$$

Consider the following procedure:

*0.* *Select an index $j \in \{1,\dots,n\}$*

*1.* *Multiply $\tilde{A}x \geq \tilde{b}$ with $1 - x_j$ and $x_j$ to obtain the non-linear system*

$$\begin{aligned} (1 - x_j)(\tilde{A}x - \tilde{b}) &\geq 0 \\ x_j(\tilde{A}x - \tilde{b}) &\geq 0. \end{aligned} \tag{1}$$

*2.* *Linearize (1) by substituting $y_i$ for $x_i x_j$, $i = 1,\dots,n$, $i \neq j$, and $x_j$ for $x_j^2$. Call the polytope defined by the resulting system $M_j(K)$.*

*3.* *Project $M_j(K)$ onto the x-space by eliminating $y_i$, $i = 1,\dots,n$, $i \neq j$. Call the resulting polytope $P_j(K)$.*

(a) Show that $K^0 \subseteq P_j(K)$.

(b) Show that $P_j(K) \subseteq K$.

(c) Assume that $K \cap \{x : x_j = 0\} = \emptyset$. Show that $P_j(K) \subseteq K \cap \{x : x_j = 1\}$. (Hint: The inequality $x_j - \varepsilon \geq 0$ is valid for K for some $\varepsilon > 0$.)

(d) Let $K = \{x \in \mathbb{R}^2$: $$\begin{aligned} -2x_1 + x_2 &\leq 0 \\ 2x_1 + x_2 &\leq 2 \\ 0 \leq x_j &\leq 1,\ j = 1,2\}. \end{aligned}$$
Describe $K^0$, $P_1(K)$ and $P_2(K)$.

(e) For the polytope K of part (d), show that the Chvátal-Gomory procedure cannot produce $K^0$ in one iteration. What is the Chvátal rank of the inequality $x_2 \leq 0$?

4. Give polynomial time algorithms for the following two problems:

(a) In a connected undirected graph $G = (V,E)$ with strictly positive edge weights find a maximum-weight edge set $E'$ that contains no cutset of $G$.

(b) In a connected acyclic digraph $D = (N,A)$ find a maximum-cardinality arc set $A'$ such that no two members of $A'$ are contained in the same directed path.

SOLUTION

Because the optimal basis to (i) and (ii) is the same they both have the same primal solution $x^*$. Because $u^*$ is optimal for (i) we have

$$u^* \geq 0, \quad u^*A \geq c, \quad \text{and} \quad u^*b = cx^*. \tag{1}$$

Because $v^*$ is optimal for (ii) we have

$$v^* \geq 0, \quad v^*A \geq d, \quad \text{and} \quad v^*b = dx^*. \tag{2}$$

Because of the first two properties in (1) and (2) we have

$$\lambda_1 u^* + \lambda_2 v^* \geq 0, \quad (\lambda_1 u^* + \lambda_2 v^*)A = \lambda_1 u^*A + \lambda_2 v^*A \geq \lambda_1 c + \lambda_2 d .$$

Because of the last property in (1) and (2) we have

$$\begin{aligned}(\lambda_1 u^* + \lambda_2 v^*)b &= \lambda_1 u^*b + \lambda_2 v^*b \\ &= \lambda_1 cx^* + \lambda_2 dx^* \\ &= (\lambda_1 c + \lambda_2 d)x^*\end{aligned}$$

so that $\lambda_1 u^* + \lambda_2 v^*$ is an optimal dual solution to (iii).

Question 2

a. This is an application of Helly's Theorem. Recall that Helly's Theorem states that a family of $n$ convex sets in $R^d$ has non-empty intersection if each subfamily of at most $d+1$ sets does.

Suppose that the convex set $C$ in $R^d$ is covered by the $n$ halfspaces $H_i$, $i = 1, \ldots, n$. Consider the complementary halfspaces $H_i^C$, $i = 1, \ldots, n$. Since the halfspaces cover $C$, the family of convex sets $H_i^C \cap C$, $i = 1, \ldots, n$ must have empty intersection. By Helly's Theorem there must be a subfamily of at most $d+1$ sets with empty intersection. We may assume, without loss of generality then that the family $H_i^C \cap C$, $i = 1, \ldots, d+1$ has empty intersection. But then the halfspaces $H_i$, $i = 1 \ldots, d+1$ cover $C$.

b. This is an application of Steinitz's Theorem. Steinitz's Theorem states that if a point $x$ is in the interior of *conv* $A$ for some $A \subseteq R^d$, then it is in the interior of *conv* $T$ for some $T \subseteq A$ with $|T| \leq 2d$.

With Steinitz's Theorem in hand, the proof is simple. If $A \subseteq R^d$ is not contained in the interior of *conv* $A$ then there is a point $x \in A$ in the interior of *convA*. By Steinitz Theorem, $x$ is in the interior of *conv* $T$ for some subset $T$ of at most $2d$ elements of $A$. Choose $B$ to be $T \cup \{x\}$.

Now, to prove Steinitz's Theorem.

We prove this by induction on $d$. For $d = 1$ the theorem is trivial. Assume Steinitz's Theorem holds for $d < k$ for some $k \geq 2$.

Consider a point $x$ in the interior of *conv* $A$ for some subset $A$ of $R^k$. First, we establish that $x$ is in the interior of *conv* $T$ for some finite subset $T$ of $A$. Since $x$ is in the interior of *conv* $A$, $x$ is in the interior of some simplex in *convA*. By Caratheodory's Theorem, we may express each vertex of the simplex as a convex combination of at most $k+1$ points of $A$, thus taking the $k+1$ points of $A$ for each of the $k+1$ vertices of the simplex, we get a set $T$ of at most $(k+1)^2$ points in $A$ with $x$ in the interior of *conv* $T$.

Now, choose a minimal subset $B$ of $T$ with the property that $x$ is in the interior of *conv* $B$. Choose a point $b$ in $B$. Since $B$ is minimal, $x$ is not in the interior of $conv(B - \{b\})$ and so there is a hyperplane $H$ through $x$ such that all of $B - \{b\}$ lies on one side of $H$. Since $x$ is in the interior of *conv* $B$, it follows that $H$ must separate $b$ from $conv(B - \{b\})$ and there must be a point $b'$ in $B - \{b\}$ not in $H$.

Now, consider the mapping $f$ of points in $B - \{b\}$ into the hyperplane $H$ defined in the following way. For each point $p \in B - \{b\}$ define $f(p)$ to be the point at which the line segment $conv\{p, b\}$ meets the hyperplane $H$. Note that each point $p$ must map into a unique and distinct point in $H$ for, if $p$ and $p'$ map into a common point, the three points $b$, $p$ and $p'$ are colinear and, we may assume, that $p'$ lies between $p$ and $b$. It is easy to see that in this case we may remove $p'$ from $B$ without changing $conv\ B$, contradicting the minimality of $B$.

Further, it is easy to see that $x$ must lie in the (relative) interior of $conv\{f(p) : p \in B - \{b\}\}$ (i.e., the interior of the set in the $k-1$ dimensional affine space $H$).

Now, by the inductive hypothesis, we may select a subset $S$ of at most $2k-2$ points from $B$ so that $x$ is in the interior of $conv\{f(p) : p \in B - \{b\}\}$. Further, it is easy to see that $x$ is in the interior of the convex hull of the at most $2k$ points $S \cup \{b, b'\}$.

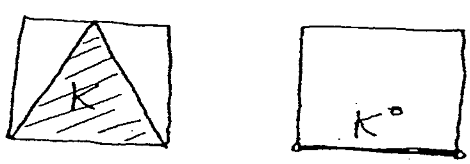
This competes the proof.

3 (a) Consider $\bar{x} \in K \cap \{0,1\}^n$. Clearly, $\bar{x}$ satisfies (1). Let $\bar{y}_i = \bar{x}_i \bar{x}_j$ for $i \neq j$. Then $(\bar{x}, \bar{y}) \in M_j(K)$. So $\bar{x} \in P_j(K)$. It follows that $K^\circ \subseteq P_j(K)$.

(b) The inequalities obtained by linearizing $(1-x_j)(\tilde{A}x - \tilde{b}) \geq 0$ are valid for $M_j(K)$. So are those obtained by linearizing $x_j(\tilde{A}x - \tilde{b}) \geq 0$. Therefore the sum of these linearized inequalities is valid for $M_j(K)$, i.e. $\tilde{A}x - \tilde{b} \geq 0$ is valid for $M_j(K)$ and thus for $P_j(K)$.

(c) $x_j - \varepsilon \geq 0$ can be obtained as a positive combination of $\tilde{A}x - \tilde{b} \geq 0$. Therefore (1) implies $(1-x_j)(x_j - \varepsilon) \geq 0$. It follows that $x_j - x_j^2 - \varepsilon + \varepsilon x_j \geq 0$. Linearizing, we get that $-\varepsilon + \varepsilon x_j \geq 0$ is valid for $M_j(K)$. Thus $x_j \geq 1$ is valid for $P_j(K)$.

(d)




$P_1(K) = K^\circ$
$P_2(K) = K$

(e) To get $x_2 \leq 0$ in one iteration of the C-G procedure, we would need to have $u, v, a, b, c, d \geq 0$ s.t. $x_2 \leq 0$ is obtained by rounding

$$u(-2x_1 + x_2) + v(2x_1 + x_2) + ax_1 + bx_2 - cx_1 - dx_2 \leq 2v + a + b$$

i.e. $2v + a + b < 1$ (2)

$-2u + 2v + a - c \geq 0$ (3) and $u + v + b - d \geq 1$. (4).

$\frac{1}{2}(3) + (4)$ yields $2v + b + \frac{1}{2}a - d - \frac{1}{2}c \geq 1$ contradicting (2).

So $x_2 \leq 0$ has Chvatal rank at least 2.

$-x_1 + x_2 \leq 0$ (5) has Chvatal rank 1 (Add $\frac{1}{2}(-2x_1 + x_2 \leq 0) + \frac{1}{2}(x_2 \leq 1)$ and round)

Similarly $x_1 + x_2 \leq 1$ (6) has Chvatal rank 1. Now adding $\frac{1}{2}(5) + \frac{1}{2}(6)$ and rounding shows that $x_2 \leq 0$ has Chvatal rank 2.

## Solution

(a) Find a minimum weight spanning tree $T$ in $G$. Then $E' := E \setminus T$ contains no cutset of $G$, since every cutset has an edge in $T$. Also, $E'$ is a maximum weight edge set with respect to this property, since $T$ is minimum-weight.

(b) Add nodes $s$ and $t$ to $D$, and arcs $(s,i),(i,t)$ for all $i \neq s,t$. Assign lower bounds $\ell_{ij}$ and upper bounds $u_{ij}$ on the flow in arc $(i,j)$, as follows:

$$\ell_{ij} = \begin{cases} 1 & (i,j) \in A' \\ 0 & (i,j) \in A \setminus A' \end{cases}$$

$$u_{ij} = \infty \qquad (i,j) \in A.$$

A minimum-value s-t flow in $D$ yields the minimum number of directed paths required to cover all arcs in $A'$. Its dual yields a cut containing a maximum number of arcs of $A'$. This is also the maximum number of arcs of $A'$ no two of which are contained in the same path.

1995

# Ph.D. Qualifying Exam
# Algorithms, Combinatorics & Optimization
# GSIA portion

6 questions

Time: 5 hours

Saturday, January 21, 1995

Convex Analysis

Solve one of the following problems.

1. Let $K \subseteq \mathcal{R}^n$ be a closed, pointed, full-dimensional cone, and let

$$K^* = \{y \in \mathcal{R}^n | y^T x \leq 0, \ \forall x \in K\}.$$

Let $A$ be an $m \times n$ matrix, $b$ an $m$-vector, and assume that $(y^T A)^T \in$ *int* $K^*$ for some $y$. ? Prove that the set $S = K \cap \{x | Ax = b\}$ is bounded. *Hint.* Consider *rec* $S$.

2. Let $a \in \mathcal{R}^n$, and assume $a_1 > \ldots > a_n$. Let $e$ be the $n$-vector with all components equal to 1. Consider the set

$$P = \{(z, u) | z \in \mathcal{R}, u \in \mathcal{R}^n, u \geq 0, ze + u \geq a\}.$$

Prove that $(z^*, u^*)$ is an extreme point of $P$ if and only if for some $k \in \{1, ..., n\}$,

$$z^* = a_k$$
$$u_j^* = \begin{cases} a_j - a_k & \text{for } j = 1, \ldots, k \\ 0 & \text{for } j = k+1, \ldots, n \end{cases} \tag{6}$$

## IP Question 1

Consider the 3-index assignment problem $AP_3$

$$min \sum_i \sum_j \sum_k c_{ijk} x_{ijk}$$

s.t.

$$\begin{aligned}
\sum_j \sum_k x_{ijk} &= 1 \qquad i \in I \\
\sum_i \sum_k x_{ijk} &= 1 \qquad j \in J \\
\sum_i \sum_j x_{ijk} &= 1 \qquad k \in K \\
x_{ijk} \in \{0,1\}, &\qquad i \in I, j \in J, k \in K,
\end{aligned}$$

where $|I| = |J| = |K| = n$, and its generalized version $GAP_3$ in which the coefficients, including those on the right hand side, are arbitrary positive integers instead of 1.

1. Formulate different types of Lagrangean duals for $AP_3$ and compare the strength of the bounds they provide.
2. Do the same thing for $GAP_3$.

## IP Question 2

Let $P := \cup_{i \in T} P_i$, where

$$P_i := \{x \in \mathcal{R}^n : Ax = d^i, x \geq 0\}, \qquad i \in T, \ \ |T| \geq 2$$

where $A$ is an $m \times n$ matrix and each $d^i$, $i \in T$, is an $m$-vector.

1. Give a linear description of $C := cl\ conv\ P$.
2. Show that $C \subseteq Q$, where $Q$ is the set of those $x \in \mathcal{R}^n$ that have an extension $(x, \lambda) \in \mathcal{R}^{n+t}$, $t = |T|$, satisfying

$$\begin{aligned} Ax - \sum_{i \in T} d^i \lambda_i &= 0 \\ \sum_{i \in T} \lambda_i &= 1 \\ x \geq 0, \lambda_i \geq 0, i \in T & \end{aligned}$$

3. Show that $Q = C$ if for every $m \times m$ nonsingular submatrix $B$ of $A$ and every convex combination $d(\lambda) := \sum_{i \in T} d^i \lambda_i$ (where $\sum_{i \in T} \lambda_i = 1$), $B^{-1} d(\lambda) \geq 0$ implies $B^{-1} d^i \geq 0$ for all $i \in T$ such that $\lambda_i > 0$. [Hint: if the condition holds, then from any basic feasible solution to the system in part 2 your should be able to construct a basic feasible solution to the system in part 1 which describes $C$.]

## Linear Programming Theory and Algorithms

Consider the linear programming problem,

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & 0 \le x \le m, \end{aligned} \tag{1}$$

which is assumed to be feasible. Define,

$$S(u) = \max\{c^T x | u^T Ax \le u^T b,\ 0 \le x \le m\}. \tag{2}$$

Thus $S(u)$ is the maximum value of $c^T x$ subject to a *surrogate constraint*; i.e., a nonnegative linear combination of the constraints in (1). The upper bounds $m$ are used in (1) to ensure that $S(u)$ has a finite value. The *surrogate dual* of (1) is the problem,

$$\begin{aligned} \min \quad & S(u) \\ \text{s.t.} \quad & u \ge 0. \end{aligned} \tag{3}$$

a) Prove weak duality for the surrogate dual; i.e.,

$$\max\{c^T x | Ax \le b,\ 0 \le x \le m\} \le \min\{S(u) | u \ge 0\}.$$

b) Prove strong duality for the surrogate dual; i.e.,

$$\max\{c^T x | Ax \le b,\ 0 \le x \le m\} = \min\{S(u) | u \ge 0\}.$$

c) The optimal value of the linear programming problem (1) could conceivably be found by solving the surrogate dual. It suffices to find a local minimum, because $S(u)$ is a quasiconvex function. Prove that $S(u)$ is indeed quasiconvex (i.e.,

$$S(\alpha u + (1-\alpha)v) \le \max\{S(u), S(v)\}$$

for any $u, v \ge 0$ and any $\alpha \in [0,1]$).

## Graph Theory

(a) A *signed* graph is a (undirected) graph with edge weights $+1$ or $-1$. A cycle of a signed bipartite graph is *quad* if the sum of its edge weights is a multiple of 4. A *hole* is a chordless cycle.

Show that a signed bipartite graph has a quad cycle if and only if it has a quad hole.

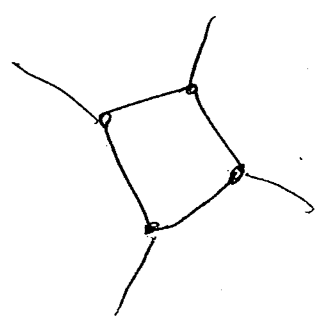
(b) A signed bipartite graph in which no cycle is quad is said to be *unbalanced*. Let $G$ be a bipartite graph which can be signed to be unbalanced. Show that the following algorithm produces such a signing.

Pick a spanning tree $T$ of $G$ and sign its edges arbitrarily. Sign every edge $e$ not in $T$ so that the unique cycle of $T \cup \{e\}$ is not quad.

(c) A *wheel* $(C, x)$ is defined by a hole $C$ and a node $x$ not in $C$ but having at least three neighbors in $C$. A *3-path configuration* is defined by three internally disjoint paths $P_1$, $P_2$ and $P_3$ (i.e. no common intermediate nodes) connecting two nodes $u$ and $v$. If $P_1$, $P_2$ and $P_3$ are chordless (hence $u$ and $v$ are nonadjacent) and no edge connects nodes in distinct paths, the 3-path configuration is said to be *induced*. If $G$ is bipartite and $u$ and $v$ belong to the same side of the bipartition, the 3-path configuration is said to be *homogeneous*.

Let $G$ be a bipartite graph which can be signed to be unbalanced. Show that $G$ contains no induced homogeneous 3-path configuration and no wheel.

(d) Show that a bipartite graph $G$ contains an induced homogeneous 3-path configuration or a wheel if and only if $G$ contains a homogeneous 3-path configuration.



# Network Flows

Consider a directed network with a cost and a capacity associated with every arc, and a supply or demand associated with every node. All data are integer.

(a) Two minimum cost flow problems $P'$ and $P''$ are *capacity adjacent* if $P''$ differs from $P'$ only in one arc capacity and by 1 unit. Given an optimal solution of $P'$, describe an efficient method for solving $P''$. Justify your answer.

(b) Let $U$ be the largest arc capacity. Let $K = \lceil log_2 U \rceil$ and suppose that we represent each arc capacity as a $K$-bit binary number, adding leading zeros if necessary to make each capacity $K$ bits long. Then the minimum cost flow problem $P_k$ considers the capacity of each arc as the $k$ leading bits in its binary representation. Given an optimal solution of $P_k$, how would you obtain an optimal solution of $P_{k+1}$ by solving only capacity adjacent problems? What is the running time of the resulting algorithm for solving the minimum cost flow problem $P = P_K$?

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# OR qualifying exam: Part II

## Three hours

Answer all questions. You are expected to spread your time evenly between NLP, DP and Proba (one hour each).

## Nonlinear Programming

Answer both questions.

1. Consider the linear programming problem,

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{1}$$

 a) Write the Lagrangian dual of (1), dualizing only the equality constraints.

 b) Show that the Lagrangian dual is equivalent to the linear programming dual.

2. Consider a linear programming problem in the following form,

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = 0 \\ & e^T x = 1 \\ & x \geq 0, \end{aligned} \tag{2}$$

 where $e$ is a vector of $n$ 1's. It can be written as a nonlinear problem by the change of variable $x = Y^2 e$, where $Y = \text{diag}(y_1, \ldots, y_n)$. The resulting problem is,

$$\begin{aligned} \min \quad & c^T Y^2 e \\ \text{s.t.} \quad & AY^2 e = 0 \\ & e^T Y^2 e = 1 \end{aligned} \tag{3}$$

 Note that we have got rid of the inequality constraints, which may simplify solution.
 Let's solve (3) with a sequential linear programming approach. At each iteration the LP subproblem is,

$$\begin{aligned} \min \quad & c^T Y_k y \\ \text{s.t.} \quad & AY_k y = 0 \\ & e^T Y_k y = 1, \end{aligned} \tag{4}$$

 where $y^k$ is the current iterate and $Y_k = \text{diag}(y_1^k, \ldots, y_n^k)$.

 a) Show that if the solution of the subproblem is $y = y^k$ (i.e., the sequential algorithm has converged), then the current iterate $y^k$ satisfies the K-T necessary conditions for optimality of (3).

 b) To ensure that the LP subproblem is bounded at each iteration, one can add a trust region constraint to (4), resulting in the subproblem below.[1]

[1] A sequential algorithm that uses this subproblem is equivalent to the affine scaling version of Karmarkar's interior point algorithm. With a slight modification in the normalization constraint $e^T Y_k y = 1$, it is equivalent to Karmarkar's original projective scaling method.

$$
\begin{aligned}
\min \quad & c^T Y_k y \\
\text{s.t.} \quad & A Y_k y = 0 \\
& e^T Y_k y = 1 \\
& \|Y_k^{-1}(y - y^k)\| \le \delta.
\end{aligned}
\tag{5}
$$

Show that if the solution of the subproblem is $y = y^k$, then $y^k$ satisfies the K-T necessary conditions for optimality of (3).

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# Ph.D. Qualifying Exam

## Part I (5 hours):

Answer 6 questions out of 7.

## Linear Programming 1

1. The well known Klee-Minty (KM) example of size $n$ can be stated as:

$$\begin{array}{lrcll} \text{Maximize} & \displaystyle\sum_{j=1}^{n} 10^{n-j} x_j & & & \\ \text{Subject to} & \displaystyle\{2\sum_{j=1}^{i-1} 10^{i-j} x_j\} + x_i & \leq & 100^{i-1} & (i = 1, 2, \ldots, n) \\ & x_j & \geq & 0 & (j = 1, 2, \ldots, n). \end{array}$$

(a) Write the KM problem for $n = 3$.

(b) Show that the maximum $z$-change (maximum objective function change) pivoting rule will find the optimal solution in one simplex pivot.

(c) Write the dual problem to the KM problem for $n = 3$. Find the optimal solution to the dual problem by inspection. Show that knowledge of this solution essentially reduces the KM example to a one-by-one problem

(d) Indicate why the results in (b) and (c) hold for the KM problem for any $n$.

2. The aim is to solve the linear programming problem

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0. \end{aligned} \tag{1}$$

It will be solved by first solving the more restricted problem,

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax = b \\ & x \geq d \end{aligned} \tag{2}$$

for some nonnegative $d \neq 0$. The bounds $d$ will then be gradually reduced, and (2) repeatedly reoptimized, until an optimal solution of the original problem is obtained. The motivation for this strategy is that the more restricted problem (2) excludes many extreme points that might have been visited if (1) were solved first. The solution of (2) may be close to the optimum and therefore allow one to avoid visiting these extreme points when the bound $d$ is relaxed. Assume for the sake of argument that this motivation is valid.

(a) It may be possible to terminate the algorithm even before $d = 0$. State a condition that is sufficient for the algorithm to be terminated. *Hint.* Solve (2) after the change of variable $y = x - d$, and let $B$ be the optimal basis.

(b) Write an expression for a value of $x$ that is optimal in (1) at termination.

(c) What type of simplex algorithm would you recommend for reoptimization? Why?

(d) If the termination condition in part (a) is not satisfied, the algorithm replaces $d$ with $d'$ (where $0 \leq d' \leq d$), and (2) is reoptimized. Give a sufficient condition on $d'$ for which $B$ is still optimal. The new bound $d'$ is chosen so as to violate this condition.

(e) Let $d'$ be chosen as above. Show that $Ad' < Ad$. *Hint.* Does $cy$ increase or decrease?

(f) Assume that (2) is solved and reoptimized with a simplex algorithm that is guaranteed to terminate. Prove that the algorithm described above terminates with an optimal solution after finitely many iterations, if an optimal solution exists.

## Convex Analysis [10 points]

3. Identify which of the following sets are convex and which are not. Explain why.

   (a) [2 points] A point $x$ in $\mathbb{R}^{\{1,2,\ldots,n\}\times\{1,2,\ldots,n\}}$ is said to be *metric* if it obeys the following conditions.

      i. $x_{ij} \geq 0$ for all $i, j \in \{1, 2, \ldots, n\}$.

      ii. $x_{ii} = 0$ for all $i \in \{1, 2, \ldots, n\}$.

      iii. $\forall i, j \in \{1, 2, \ldots, n\}, \ x_{ij} = x_{ji}$.

      iv. $\forall i, j, k \in \{1, 2, \ldots, n\}$ the triangle inequality $x_{ij} \leq x_{ik} + x_{kj}$ holds.

      Is the set of all metric points convex?

   (b) [4 points] Define a point $x$ in $\mathbb{R}^{\{1,2,\ldots,n\}\times\{1,2,\ldots,n\}}$ to be a *cycle metric* if it is metric and in addition, we have

      v. for any pair $i, j$ not adjacent in the cyclic order $1, 2, \ldots, n$, the value of $x_{ij}$ equals the shorter of the two alternative distances (using the $x$-values as lengths) between $i$ and $j$ obtained by using the paths in the cycle numbered $1, 2, \ldots, n$ in the clockwise and counter-clockwise directions respectively.

      Note that a cycle metric is uniquely defined by the positive values assigned to the $n$ entries $x_{12}, x_{23}, \ldots, x_{n1}$.

      Is the set of all cycle metric points convex?

   (c) [4 points] Define a point $x$ in $\mathbb{R}^{\{1,2,\ldots,n\}\times\{1,2,\ldots,n\}}$ to be a *tree metric* if it is metric and in addition,

      vi. there is a spanning tree $T$ on the node set $\{1, 2, \ldots, n\}$ such that for any pair $i, j$, the value of $x_{ij}$ is the sum of the $x$-values of all the edges in the unique path in $T$ between $i$ and $j$.

      Is the set of all tree metric points convex?

## Integer Programming

4. Consider the set covering problem

$$\min\{cx : Ax \geq 1,\ x \in \{0,1\}^n\},$$

where $A$ is $m \times n$ with $\sum_j a_{ij} = 3$, and the system $Ax \geq 1$ is of the form

$$x_{j_1(i)} + x_{j_2(i)} + x_{j_3(i)} \geq 1, \quad i = 1, \ldots m.$$

Describe a lift and project procedure for generating strong cuts from the disjunctions

$$x_{j_1(i)} = 1 \ \vee\ x_{j_2(i)} = 1 \ \vee\ x_{j_3(i)} = 1. \tag{1}$$

In particular:

(a) Describe by a linear system in $I\!R^{4n+3}$ the convex hull of the union of the three polyhedra obtained from the LP relaxation by imposing (1);

(b) Show how to project this linear system onto the subspace $I\!R^n$ of the $x$ variables;

(c) Discuss briefly the properties of the cuts obtained.

5. The precedence-constrained ATSP asks for a minimum-cost tour which satisfies the following constraints: given the home city, 1, where the tour starts and ends, for a specified list of ordered pairs $[i, j]$, $i \neq 1 \neq j$, each city $i$ is visited by the tour before the corresponding city $j$.

   (a) Which arcs are unusable in any tour (which variables are forced to 0) because of the precedence constraints?

   (b) Define the precedence graph $G^*$ as having a node for every city other than 1, and an arc $(i, j)$ for every pair such that $i$ has to precede $j$. Use $G^*$ to give a necessary and sufficient condition for the problem to be feasible.

   (c) Formulate the precedence-constrained ATSP as a 0-1 program in the arc variables only. Can the inequalities that you are using be strengthened?

## Graph Theory

6. Prove or disprove.

   (a) Any undirected graph that is the union of two edge-disjoint spanning trees is biconnected (2-vertex connected).

   (b) Recall that a dominating set of an undirected graph $G$ is a subset of nodes such that every node of $G$ is adjacent to at least one node from this subset. We assume that every node is adjacent to itself.

   There exist a pair of node-disjoint dominating sets in any connected undirected graph.

   (c) Let $G$ be a cubic (3-regular) graph with a proper 3-edge coloring using colors 1,2 and 3. For any node subset $S$, let $\delta_i(S)$ denote the number of edges colored $i$ that have exactly one endpoint in $S$. For any node subset $S$, all three numbers $\delta_1(S), \delta_2(S)$ and $\delta_3(S)$ have the same parity.

Networks and Matchings [10 points]

7. (a) Let $G = (V, A)$ be a directed acyclic graph (DAG) with positive weights $w : A \rightarrow \mathbb{R}^+$ on the arcs. Furthermore, let $r \in V$ be a node such that there is a directed path from $r$ to every node $v \in V$. A *directed spanning tree* of $G$ rooted at $r$ is defined as an (arc-) minimal subgraph of $G$ in which there is a directed path from $r$ to every node $v \in V$. Note that this tree has $|V| - 1$ arcs and every node in this tree has indegree exactly one, except for $r$ which has zero indegree.

   Give a linear time ($O(|A|)$) algorithm to find a directed spanning tree of $G$ of minimum total weight. (Partial credit for algorithms with higher running time.)

   (b) Let $G = (V, E)$ be a connected undirected graph with positive lengths $w : E \rightarrow \mathbb{R}^+$ on the edges. Recall that a *shortest path tree* of $T$ rooted at a node $r \in V$ is a spanning tree $T_r$ of $G$ such that for any node $v \in V$, the path in $T_r$ from $v$ to $r$ is a shortest length path from $v$ to $r$ in $G$.

   i. Give an example of a graph and a root node for which the shortest path tree rooted at this node is not unique.

   ii. Design a polynomial-time algorithm to find a shortest path tree of an undirected graph rooted at a given node with the minimum *total* length. (The total length is the sum of lengths of all edges in the spanning tree).

**Jerry Thompson's LP1 Solution**

1. (a)

$$\begin{array}{lrcrcrcl} \text{Maximize} & 100x_1 & + & 10x_2 & + & x_3 & & \\ \text{Subject to} & x_1 & & & & & \leq & 1 \\ & 20x_1 & + & x_2 & & & \leq & 100 \\ & 200x_1 & + & 20x_2 & + & x_3 & \leq & 10,000 \\ & & & & & x_1, x_2, x_3 & \geq & 6 \end{array}$$

Bringing $x_3$ into the basis gives $z = 10,000$ and the reduced costs of $x_1$ and $x_2$ become negative, $-100, -10$ so no improvements are possible. Hence solution is $x_1 = x_2 = 0$, $x_3 = 10,000$, $z = 10,000$. If we bring in any other variable on the first step a lesser change in the objective function occurs.

(b)

$$\begin{array}{lrcrcrcl} \text{Minimize} & v_1 & + & 100v_2 & + & 10,000v_3 & & \\ \text{subject to} & v_1 & + & 20v_2 & + & 200v_3 & \geq & 100 \\ & & & v_2 & + & 20v_3 & \geq & 10 \\ & & & & & v_3 & \geq & 1 \end{array}$$

Clearly $v_1 = 0$, $v_2 = 0$, $v_3 = 1$ satisfies all of the inequalities and $z = 10,000$. Since the first two constraints are oversatisfied they can be ignored.

(c) For (a) the reduced costs of variable $j$ after the pivot are $-10^{n-i}$ and $z = 100^{n-1}$. for (b) the surplus variable for the $j$-th constraint when $v_n = 1$ and $v_1 = v_2 = \cdots = v_{n-1} = 0$ is $100^{n-j}$.

**John Hooker's LP2 Solution**

(a) After the change of variable $y = x - d$, (2) becomes

$$\begin{aligned} \min \quad & cy + cd \\ \text{s.t.} \quad & Ay = b - Ad \\ & y \geq 0. \end{aligned} \tag{2}$$

Let $B$ be the optimal basis, so that the optimal solution is $(y_B, y_N) = (B^{-1}(b - Ad), 0)$. (2) is identical to (1) when $d = 0$. So the algorithm can be terminated if $B$ is optimal when $d = 0$. Because $d$ does not affect the reduced costs, it is enough for $B$ to be feasible, i.e.,

$$B^{-1}b \geq 0.$$

(b) $(x_B, x_N) = (B^{-1}b, 0)$ is optimal in (1).

(c) It suffices that $B^{-1}(b - Ad') \geq 0$, or

$$Ad' \leq B^{-1}b.$$

(d) The dual simplex method, because the dual solution $c_B B^{-1}$ remains feasible in the dual.

(e) Let $y, y'$ be the solutions of (2) corresponding to $d, d'$. Because $B$ becomes infeasible in (2) when $d'$ replaces $d$, the corresponding dual solution becomes suboptimal. The new optimal dual value is therefore strictly greater than the previous one, which implies by strong duality that $cy' > cy$. Meanwhile the optimal value of (2) cannot increase because $d' \leq d$. So $cy' + Ad' \leq cy + Ad$, which implies $Ad' < Ad$.

(f) Because $cy$ strictly increases in each iteration, an optimal basis $B$ that becomes infeasible in (2) cannot become feasible again. Because there are a finite number of bases, the algorithm terminates with an optimal solution.

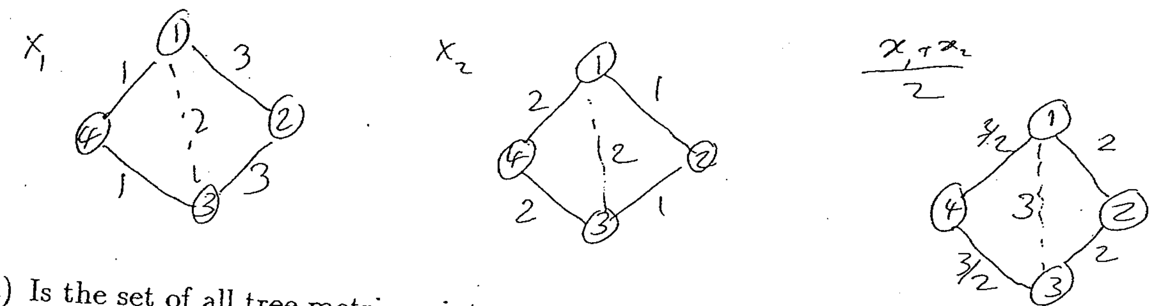
## Ravi's Convex Analysis Solution

(a) Is the set of all metric points convex?

YES. There are a couple of obvious ways to show this. The most direct is to observe that this set is described as the intersection of the finite set of closed halfspaces given by (*i*) through (*iv*). The more roundabout way is to verify convexity directly by showing that all points in the line between any pair of metric points are metric.

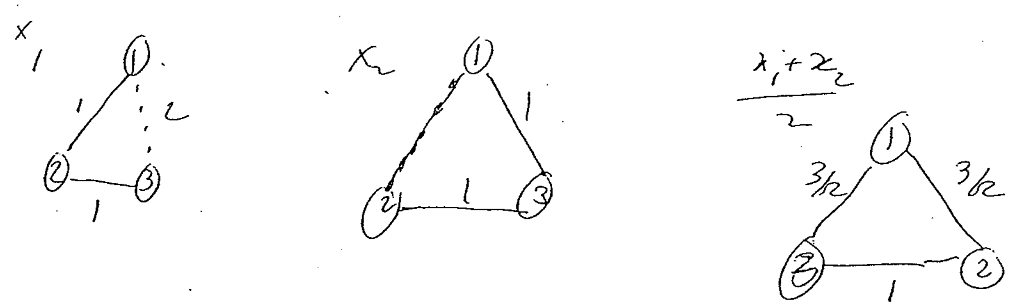
(b) Is the set of all cycle metric points convex?

NO. Here is an example, where $x_1$ and $x_2$ are cycle metrics, but $x = \frac{x_1+x_2}{2}$ is not one.




(c) Is the set of all tree metric points convex?

NO. Here is an example of two points $x_1$ and $x_2$ both tree metrics but their midpoint is not.



## Balas' Linear Programming 1 Solution

4. (a) Let the system $Ax \geq 1$, $x \geq 0$, $-x \geq -1$, be written as $\widetilde{A}x \geq \widetilde{1}$. Then the convex hull is the set of $x \in I\!R^n$ for which there exist vectors $(y^k, y_0^k) \in I\!R^{n+1}$, $k = 1,2,3$, satisfying

$$\begin{array}{rcrcrcll}
x & - & \sum_{k=1,2,3} y^k & & & = & 0 & \\
 & & \widetilde{A}y^k & - & \widetilde{1}y_0^k & \geq & 0 & \\
 & & y_{j(ik)}^k & - & y_0^k & = & 0 & k = 1,2,3 \\
 & & & & \sum_{k=1,2,3} y_0^k & = & 1 & \\
 & & & & y_0^k & \geq & 0, & k = 1,2,3
\end{array}$$

(b) The projection cone $W$ is the set of $(\alpha, \{u^k, u_0^k\}_{k=1,2,3}, \beta) \in I\!R^{n+3(m+2n+1)+1}$ satisfying

$$\begin{array}{rcrcrcrcll}
\alpha & - & u^k\widetilde{a}_j & & & & & = & 0, & j \in \{1,\ldots,n\} - j(ik) \\
\alpha & - & u^k\widetilde{a}_{j(ik)} & - & u_0^k & & & = & 0 & \\
 & & u^k\widetilde{1} & + & u_0^k & - & \beta & \geq & 0, & k = 1,2,3,
\end{array}$$

where $\tilde{a}_j$ is the $j$-th column of $\widetilde{A}$.

(c) The extreme rays of $W$ give rise to cuts $\alpha x \geq \beta$ that are facets of the convex hull of the union of the three polyhedra.

## Balas' Linear Programming 2 Solution

5. (a) Unusable arcs:

   $(1, j)$ for any $j$ that has a predecessor
   $(i, 1)$ for any $i$ that has a successor
   $(j, i)$ for any pair $[i, j]$ such that $i$ has to precede $j$
   $(i, j)$ for any pair $[i, j]$ for which there exists $k \neq 1$ such that $i$ has to precede $k$ and $k$ has to precede $j$.

   For any other arc, one can exhibit a feasible tour containing it.

   (b) There exists a feasible tour if and only if the precedence graph $G^*$ is acyclic.
   $\Rightarrow$ obvious $\Leftarrow$ if $G^*$ is acyclic, any tour visiting the cities in the topological order defined on $G^*$ is feasible.

   (c)

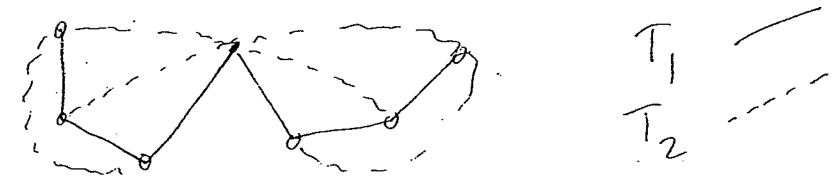
$$
\begin{aligned}
\min \quad & cx \\
& x(\delta^+(i)) = 1 \\
& x(\delta^-(i)) = 1 \qquad\qquad i = 1, \ldots, n \\
& x \in \{0,1\}^A \\
& x(P(j,i)) \leq |P(j,i)| - 1 \quad \text{for all paths } P(j,i) \text{ from } j \text{ to } i \text{ and all pairs } [i,j] \text{ such that } i \text{ has to precede } j.
\end{aligned}
$$

The last set of inequalities can be strengthened to

$$x(j, Q) + x(Q, Q) + x(Q, i) \leq |Q| \text{ for all } Q \subset N \setminus \{1, i, j\}.$$

## Ravi's Graph Theory Solution

6. (a) FALSE. There are several small examples, e.g.,




(b) This is TRUE.

Note that the union of the edges colored with any two of the three colors is a collection of disjoint cycles of $G$. Further, for any node subset $S$, the number of edges of a cycle with exactly one endpoint in $S$ is even. Thus, for any pair of distinct colors $i, j$ and for any node subset $S$, $\delta_i(S) + \delta_j(S)$ is even. applying this to the two pairs (1,2) and (1,3) and arguing about parity shows the result.

(c) This is TRUE. (only if) Given that $T(G)$ is Eulerian, we can infer that $G$ is connected and therefore, so is $L(G)$. Furthermore, $L(G)$ is simply the induced subgraph of $T(G)$ on the nodes corresponding only to the edges of $G$. Since $T(G)$ is Eulerian, the degree of every node in it is even. In going from $T(G)$ to $L(G)$, the degree of a node corresponding to an edge in $G$ reduces by exactly two (corresponding to deleting the nodes representing its two endpoints that appear in $T(G)$). Thus, for every such node, the degree continues to stay even. We have thus shown that $L(G)$ is connected with all nodes having even degree, and hence it is Eulerian.

(if) This is the reverse operation from the previous para. To go from $L(G)$ to $T(G)$, we add in the nodes corresponding to the vertices of $G$, and connect them to their adjacent edges and nodes. Note that there is a simple bijection between the node and edge neighbors of any vertex of $G$, and so the degree of this vertex in $T(G)$ is even. We also increase the degree of every node in $T(G)$ that corresponds to an edge of $G$ by two (connecting it to its two endpoints). This leaves the degree of such nodes even. Finally, it is easy to see that $T(G)$ is also connected, showing that it is Eulerian.

**Ravi's Networks Solution [10 points]**

7. (a) Let $G = (V, A)$ be a directed acyclic graph (DAG) with positive weights $w : A \to \mathbb{R}^+$ on the arcs. Furthermore, let $r \in V$ be a node such that there is a directed path from $r$ to every node $v \in V$. A *directed spanning tree* of $G$ rooted at $r$ is defined as an (arc-) minimal subgraph of $G$ in which there is a directed path from $r$ to every node $v \in V$. Note that this tree has $|V| - 1$ arcs and every node in this tree has indegree exactly one, except for $r$ which has zero indegree.

   Give a linear time $(O(|A|))$ algorithm to find a directed spanning tree of $G$ of minimum total weight. (Partial credit for algorithms with higher running time.)

   (b) Let $G = (V, E)$ be a connected undirected graph with positive lengths $w : E \to \mathbb{R}^+$ on the edges. Recall that a *shortest path tree* of $T$ rooted at a node $r \in V$ is a spanning tree $T_r$ of $G$ such that for any node $v \in V$, the length of the path in $T_r$ from $v$ to $r$ equals that of a shortest length path from $r$ to $v$ in $G$.

   i. Given an example of a graph and a root node for which the shortest path tree rooted at this node is not unique.

   ii. Design a polynomial-time algorithm to find a shortest path tree of an undirected graph rooted at a given node with the minimum *total* length. (The total length is the sum of lengths of all edges in the spanning tree).

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# Ph.D. Qualifying Exam

## Part II (3 hours):

Answer all 4 questions.

## Nonlinear Programming

1. (a) Let $A$ be an $n \times m$ matrix with full row rank. The orthogonal projection of a vector $c \in \mathbb{R}^n$ onto $\{x \mid Ax = 0\}$ is

$$d = c - A^T(AA^T)^{-1}Ac.$$

Use first and second-order optimality conditions to show that $d$ is the closest point to $c$ in $\{x \mid Ax = 0\}$. That is, show $x = d$ is the unique solution of

$$\begin{aligned} \min \quad & \|x - c\|^2 \\ \text{s.t.} \quad & Ax = 0. \end{aligned}$$

(b) We wish to solve the linear programming problem

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

by solving the problem

$$\begin{aligned} \min \quad & cx - \alpha \sum_j \ln x_j \\ \text{s.t.} \quad & Ax = b \end{aligned} \tag{1}$$

We are given a starting solution $x^0 > 0$ that is feasible in (1). In iteration $k$ of the algorithm, the current iterate is $x^k$. One step of Newton's method is applied to the unconstrained problem,

$$\min \; cx - \alpha \sum_j \ln x_j,$$

using $x^k$ as the starting point, to obtain the point $\bar{x}$. Then the orthogonal projection $d$ of $\bar{x} - x^k$ onto $\{x \mid Ax = 0\}$ is computed. The next iterate is

$$x^{k+1} = x^k + d.$$

The parameter $\alpha$ is gradually reduced to zero as the algorithm proceeds.

i. Given that the current iterate is $x^k$, write an expression for $\bar{x}$. It is convenient to let $D = \text{diag}(x_1^k, \ldots, x_n^k)$, so that $x^k = De$, where $e$ is a vector of ones, and the gradient[1] $\nabla(\sum_j \ln x_j)$ evaluated at $x = x^k$ is $D^{-1}e$.

ii. Let $d_p$ be the orthogonal projection of $-D^2c$ onto $\{x \mid Ax = 0\}$, and let $d_c$ be the orthogonal projection of $x^k$ onto $\{x \mid Ax = 0\}$. Show that the direction $d$ is a linear combination of $d_p$ and $d_c$.

Notes: The above method is a *projected Newton barrier method* for solving an LP. The directions $d_c$, $d_p$ are *predictor* and *corrector* steps. Note that the corrector is given less and less weight as the *barrier parameter* $\alpha$ goes to zero. For appropriate $\alpha$'s, the method is identical to Karmarkar's famous projective scaling method for solving LP's.

[1] The gradient $\nabla f(x)$ is $(\frac{\partial}{\partial x_1} f(x), \ldots, \frac{\partial}{\partial x_n} f(x))$.

# PhD Qualifying Exam

# ACO

# and OR Part I

6 questions

4-1/2 hours

January 17, 1998

8:30 am - 1:00 pm

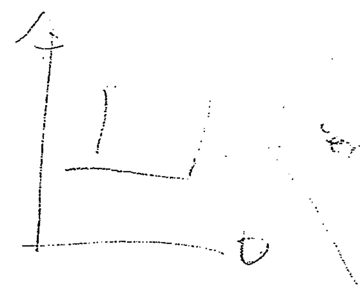
Room 151

no sol'n 5,6

1. Consider the following linear program:

$$
\begin{aligned}
&\text{Maximize } cx \\
&\text{Subject to} \\
&\qquad\qquad Ax \leq b \\
&\qquad\qquad\ \ x \geq 0
\end{aligned}
\tag{1}
$$

where $A$ is $m \times n$ and $b$, $c$, and $x$ have consistent dimensions. Let $e$ be a $1 \times n$ vector each of whose entries is 1. Show that the set $X = \{x | Ax \leq b,\ x \geq 0\}$ is a bounded convex set if and only if the linear program obtained from (1) by replacing $c$ by $e$ has a finite solution.



## Linear Programming

2. Consider the two linear programming problems,

$$\begin{array}{ll} \min/\max & cx \\ \text{s.t.} & Ax \geq a. \end{array} \tag{1}$$

For simplicity, assume that the feasible set is nonempty and bounded. By strong duality,

**Theorem 1** *$\bar{z}$ is the minimum value of (1) if and only if there is a $\bar{u} \geq 0$ such that $\bar{u}A = c$ and $\bar{u}a = \bar{z}$.*

A similar theorem holds for the maximum.

Consider next a set $S$ of vectors of the form $(x, y)$. The *projection* of $S$ onto $y$ is $\{y \mid (x, y) \in S \text{ for some } x\}$. The following is well known.

**Theorem 2** *The projection of the polyhedron $\{(x, y) \mid Ax + By \geq a\}$ onto $y$ is*

$$\{y \mid u^i By \geq u^i a \text{ for all } i \in I\},$$

*where $\{u^i \mid i \in I\}$ is the set of extreme rays*[1] *of the polyhedral cone $\{u \geq 0 \mid uA = 0\}$.*

The problem (1) can be written,

$$\begin{array}{ll} \text{opt} & cx \\ \text{s.t.} & Ax \geq a. \end{array} \tag{2}$$

The *solution* of (2) is defined to be the projection of $\{(z, x) \mid z = cx,\ Ax \geq a\}$ onto the scalar variable $z$. So the solution is an interval $[z_1, z_2]$, where $z_1$ is the minimum value of (1) and $z_2$ is the maximum value of (1).

(a) Use Theorem 2 to prove Theorem 1.

(b) Consider the generalized optimization problem,

$$\begin{array}{ll} \text{opt} & Cx \\ \text{s.t.} & Ax \geq a, \end{array} \tag{3}$$

where $z \in R^k$ and the feasible set is again assumed to be nonempty and bounded. The solution of (3) is the projection of $\{(z, x) \mid z = Cx,\ Ax \geq a\}$ onto $z$. For $w \in R^k$ let $z_w$ be the minimum value of

$$\begin{array}{ll} \min & wz \\ \text{s.t.} & z \geq Cx \\ & Ax \geq a. \end{array} \tag{4}$$

Use Theorems 1 and 2 and weak LP duality[2] to show that the solution of (3) is

$$\{z \mid wz \geq z_w \text{ for all } w \in R^k\}.$$

[1] Extreme rays that are positive scalar multiples of each other are considered the same extreme ray.
[2] i.e., if $x$ is feasible in a minimization problem (1) and $u$ is feasible in its dual, then $cx \geq ua$.

3. Let $K_{m,m,m}$ be the complete 3-partite graph on $3m$ vertices, with $m$ vertices in every part, where $m$ is even.

   i. Determine the size of a
      - maximum edge matching
      - maximum vertex packing (stable set)
      - minimum vertex covering (of edges)
      - minimum edge covering (of vertices)
      - minimum vertex coloring
      - maximum clique
      - minimum clique covering (of vertices)

      in $K_{m,m,m}$.

   ii. What is the number of maximal cliques in $K_{m,m,m}$?

   iii. Is $K_{m,m,m}$ perfect?. (Prove your answer.)

4. Consider the following problem $P$:

Given an undirected edge-weighted graph $G(w) = (V, E)$ with edge lengths $w_{ij}$, $(i, j) \in E$, and an edge set $F \subset E$, find a shortest postman tour, i.e. a shortest (in terms of the edge-lengths) closed walk that traverses every edge at least once, subject to the constraint that no edge in $F$ can be traversed more than once.

   i. Describe a polynomial time algorithm for solving $P$ and give its complexity.

   ii. Apply the algorithm to the example in figure 1, where the numbers represent edge lengths, for the cases:

   (b1) $F = \emptyset$

   (b2) $F = \{(2, 5), (3, 6), (5, 6)\}$.

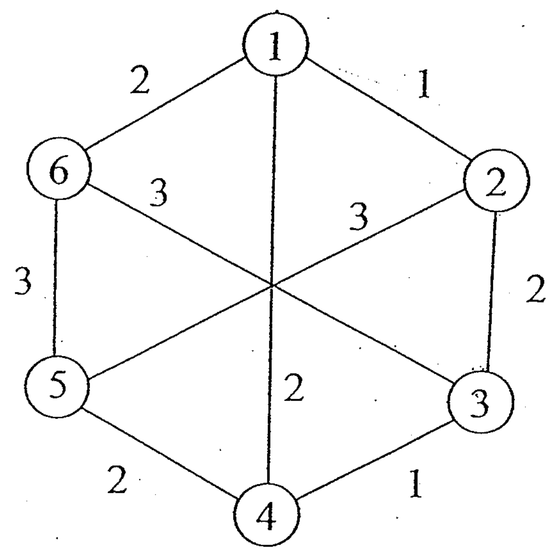



Figure 1

5. For an undirected graph $G = (V, E)$, consider the polytope

$$\begin{aligned} \sum_{ij \in E: i \in S,\ j \notin S} x_{ij} &\geq 2 \text{ for every } S \subset V,\ S \neq \emptyset \\ 0 \leq x_{ij} \leq 1 & \quad\quad \text{for every } ij \in E. \end{aligned} \qquad (P)$$

The polytope (P) is known as the subtour elimination polytope.

(a) Is it possible to solve the linear program $\min\{cx : x \in P\}$ in time which is polynomial in the size of a binary encoding of the vector $c$ and of the graph $G$? Explain briefly your answer.

(b) Given $G$ and $c$, GTSP denotes the problem of finding a minimum cost cycle which goes at least once through each node of $G$. Show that the inequality $\sum_{e \in E} x_e \geq 10$ is valid for GTSP for the following graph $G$:

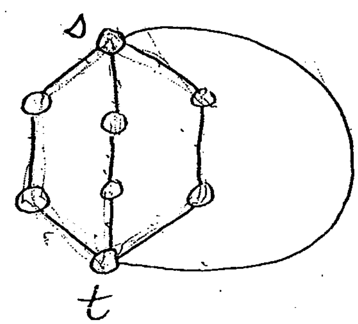



Figure 2

Can this inequality be obtained from (P) as a Chvatal-Gomory inequality? Explain briefly your answer.

(c) Can the inequality in (b) be generalized to graphs with 3 $st$-paths of length $k$, for any $k \geq 3$?

(d) Define the strength of a valid inequality $\alpha x \geq \beta$ for GTSP as

$$\max_{\substack{c \geq 0 \\ c \neq 0}} \frac{\min\{cx : x \in P\}}{\min\{cx : x \in P \cap \{x : \alpha x \geq \beta\}\}}.$$

Can you guess the strength of the inequalities discussed in (b) and (c)? No proof is required for this question.

6. Consider the following integer program

$$\min \quad \sum_i \sum_j \sum_k c_{ijk} x_{ijk}$$

$$\sum_k a_k x_{ijk} = b_{ij} \quad \text{for all } i, j \tag{1}$$

$$\sum_i x_{ijk} = 1 \quad \text{for all } j, k \tag{2}$$

$$\sum_j x_{ijk} = 1 \quad \text{for all } i, k \tag{3}$$

$$x_{ikj} = 0, 1 \quad \text{for all } i, j, k,$$

and the five lagrangian relaxations obtained by relaxing

- (1)
- (2)
- (1),(2)
- (2), (3)
- (1), (2), (3)

Compare the strengths of the five corresponding lagrangian duals and discuss their merits in terms of ease of solution, for use in a branch and bound algorithm.

1998

1998

1. Consider the following linear program:

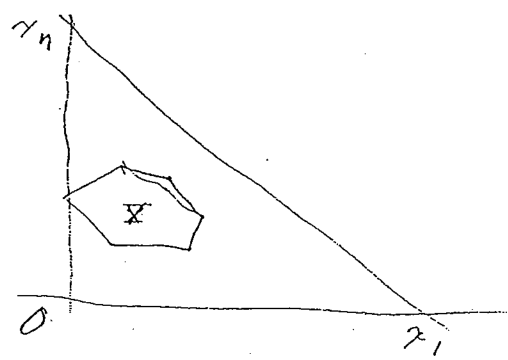
Maximize cx

Subject to

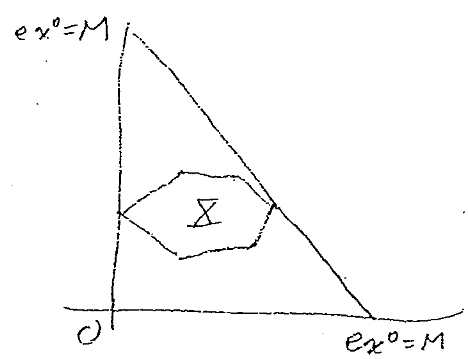
$$Ax <= b \tag{1}$$

where A is mxn and b, c, and x have consistent dimensions. Let e be a 1xn vector each of whose entries is 1. Show that the set X = {x | Ax <= b} is a bounded convex set if and only if the linear program obtained from (1) by replacing c by e has a finite solution.

If $X$ is bounded then for large enough $M$, $X$ is contained in the convex set $0 \le ex \le M$, $x \ge 0$




If Max ex is finite then for M = max ex
$X$ is contained in, and even touches $0 \le ex^0 \le M$, $x \ge 0$



2

## Linear Programming Solution

(a) Show that $\bar{z}$ is the minimum value of (1) if and only if there is a $\bar{u} \geq 0$ such that $\bar{u}A = c$ and $\bar{u}a = \bar{z}$.

First suppose $\bar{z}$ is the minimum value of (1). The constraint set can be written,

$$\begin{bmatrix} -c \\ A \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} z \geq \begin{bmatrix} 0 \\ a \end{bmatrix}. \tag{5}$$

By Theorem 2, the projection of this set onto $z$ is

$$\{z \mid v_i z \geq u^i a \text{ for all } i \in I\},$$

where $\{(v_i, u^i) \mid i \in I\}$ is the set of extreme rays of $\{(v, u) \geq 0 \mid uA = vc\}$. Because there are finitely many extreme rays and $\bar{z}$ is the minimum value of (2), $(u^i/v_i)a = \bar{z}$ for some $i \in I$. So if we set $\bar{u} = u^i/v_i$, $\bar{u}A = c$ and $\bar{u}a = \bar{z}$.

For the converse, suppose $\bar{u}$ satisfies (a). Then Theorem 2 implies that any feasible $z$ satisfies $z \geq \bar{u}a$. But because $\bar{z} = \bar{u}a$, $\bar{z}$ is the minimum value of (2).

(b) 2 Let $P$ be the solution of (3). Show that $P = P'$, where

$$P' = \{z \mid wz \geq z_w \text{ for all } w \in R^k\}.$$

First let $\bar{z} \in P$ and show that $w\bar{z} \geq z_w$ for any $w \in R^k$. Because $z_w$ is the minimum value of (4), Theorem 1 implies that there are $(\bar{v}, \bar{u}) \geq (0, 0)$ such that $z_w = \bar{u}a$, $\bar{u}A = \bar{v}C$ and $\bar{v} = w$. Thus $\bar{u}A = wC$ as well. Because $\bar{z} \in P$, Theorem 2 implies that $w\bar{z} \geq \bar{u}a$. This and the fact that $z_w = \bar{u}a$ imply that $w\bar{z} \geq z_w$, as desired.

For the converse, let $\bar{z} \in P'$. Take any $\hat{u} \geq 0$ and any $\hat{v}$ with $\hat{u}A = \hat{v}C$. By Theorem 2 it suffices to show $\hat{v}\bar{z} \geq \hat{u}a$. But because $\bar{z} \in P'$, $\hat{v}\bar{z} \geq z_{\hat{v}}$, and Theorem 1 implies that there is a $\bar{u} \geq 0$ with $z_{\hat{v}} = \bar{u}a$ and $\bar{u}A = \hat{v}C$. So $\hat{v}\bar{z} \geq z_{\hat{v}} = \bar{u}a \geq \hat{u}a$, where the last inequality follows from weak LP duality and the feasibility of $(\hat{v}, \bar{u})$ in the dual of (4).

## Solutions

**Question ~~1~~ 3**

(i) (a) Let $S_1 := \{i_1, \ldots, i_m\}$, $S_2 := \{j_1, \ldots, j_m\}$ and $S_3 := \{k_1, \ldots, k_m\}$ be the three parts of $K_{m,m,m}$.

- A maximum matching is provided by the edge set $\{(i_1, j_1), \ldots, (i_{m/2}, j_{m/2});$ $(i_{(m/2)+1}, k_1), \ldots, (i_m, k_{m/2});$ $(j_{(m/2)+1}, k_{(m/2)+1}), \ldots, (j_m, k_m)\}$. Its cardinality is $3m/2$.
- Each $S_i$, $i = 1, 2, 3$, is a maximum stable set, and $|S_i| = m$.
- Since $S_1$ is a maximum vertex packing, $S_2 \cup S_3$ is a minimum vertex covering (of edges). Its cardinality is $2m$.
- The maximum matching specified above covers all the vertices, hence it is a minimum edge covering (of vertices). Its cardinality is $3m/2$.
- $-(S_1, S_2, S_3)$ is a minimum vertex coloring with cardinality 3.
- Each vertex set of the form $(i, j, k)$, with $i \in S_1$, $j \in S_2$, $k \in S_3$, is a clique. Since its size is the same as that of a minimum coloring, namely 3, it is a maximum clique.
- The set of triangles $\{(i_1, j_1, k_1), \ldots, (i_m, j_m, k_m)\}$ covers all nodes. Since its cardinality is $m$, the same as that of a maximum vertex packing, it is a minimum clique covering (of vertices).

(ii) (b) Since each triangle $(i, j, k)$ with $i \in S_1$, $j \in S_2$, $k \in S_3$, is a maximal clique, the number of such cliques is $|S_1| \times |S_2| \times |S_3| = m^3$.

(iii) (c) $K_{m,m,m}$ is ~~not~~ perfect: the edge set $\{(i_1, j_1, (j_1, i_2), (i_2, j_2), (j_2, k_1), (k_1, i_1)\}$ is an odd cycle. wrong!

Question 4

(a) The algorithm, an easy modification of the standard Chinese postman algorithm, is as follows.

If all vertices of $G$ have even degree, the solution is the (unique) Euler tour in $G$, which can be found by Fleury's algorithm.

If $V^*$ is the set of odd-degree vertices, find shortest paths $P_{ij}$ between all pairs of vertices $i, j \in V^*$, in the graph $G(\tilde{w})$ obtained from $G(w)$ by replacing the edge lengths $w_{ij}$ with $W > \sum(w_{ij} : (i,j) \in E)$ for all $(i,j) \in F$.

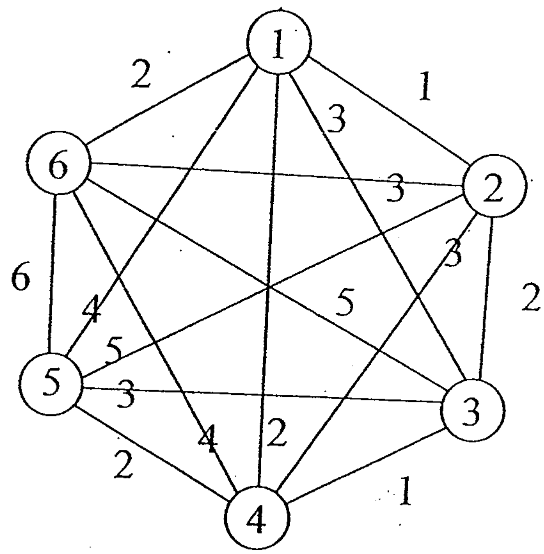
Next find a minimum-cost perfect matching $M$ in the complete undirected graph with vertex set $V^*$ and edge-weights $\tilde{w}(P_{ij}) := \sum(\tilde{w}_{k\ell} : (k,\ell) \in P_{ij})$. If $\tilde{w}(M) > W$, the problem has no solution. Otherwise duplicate the edges of the shortest paths in $G(w)$ defining the weight of $M$. An Euler tour in the resulting (Eulerian) multigraph is the solution.

The complexity of the procedure is the same as that of the standard postman algorith, $O(n^3)$.

(b) All vertices have odd degree, $V^* = V$.

(b1) Application of the standard Chinese postman algorithm leads to the duplication of arcs $(1,2), (3,4), (5,6)$.

(b2) The above described algorithm looks for a minimum-weight perfect matching in the graph below and finds $M := \{(1,2), (3,6), (4,5)\}$, or, alternatively, $M' = \{(1,6), (2,5), (3,4)\}$, with $\tilde{w}(M) = \tilde{w}(M') = 8$. An optimal solution is obtained by duplicating the edges of $M$ or $M'$.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# PhD Qualifying Exam

## OR Part II

4 questions

3 hours

January 17, 1998

2:00 pm - 5:00 pm

Room 151

2. Consider the $m \times n$ matrix $A = [a_1, a_2, \ldots, a_m]^T$, i.e., $a_i$'s are column vectors that correspond to the rows of $A$.

   a. (2) Consider the problem

$$\min_x \max_i \|x - a_i\|_2^2,$$

   where $a_i$'s are given points in $\mathbb{R}^n$. This problem arises, for example, when one tries to choose a location for a fire station; $a_i$'s denote the coordinates of several fire-prone buildings, and the objective is to minimize the maximum distance from the fire station to any of the buildings. This problem can be written as an $(n+1)$-dimensional minimization problem as follows:

$$\begin{array}{lrcl} \min & x_{n+1} & & \\ \text{s.t.} & & & \\ & x_{n+1} & \geq & \|x - a_i\|_2^2, \quad \forall i \end{array}$$

   Show that the feasible region of the above problem is convex.

   b. (3) Let $\lambda_i$, $i = 1, \ldots, m$, be such that $\lambda_i \geq 0$, $\forall i$, and that $\sum_i \lambda_i = 1$. Show that $\hat{x} := \sum_i \lambda_i a_i$ solves

$$\min_x \sum_i \lambda_i \|x - a_i\|_2^2.$$

   c. (5) Construct the Lagrangian function $\mathcal{L}(x, \lambda)$ of the $(n+1)$-dimensional minimization problem of part a and the Wolfe dual of the problem given by:

$$\begin{array}{lrcl} \max_{x,\lambda} & \mathcal{L}(x,\lambda) & & \\ \text{s.t.} & & & \\ & \nabla_x \mathcal{L}(x,\lambda) & = & 0 \\ & \lambda & \geq & 0. \end{array}$$

   Suppose that you have an algorithm for minimizing a convex quadratic function subject to linear constraints. Show how one can solve the problem given in part a using this algorithm and the Wolfe dual above?

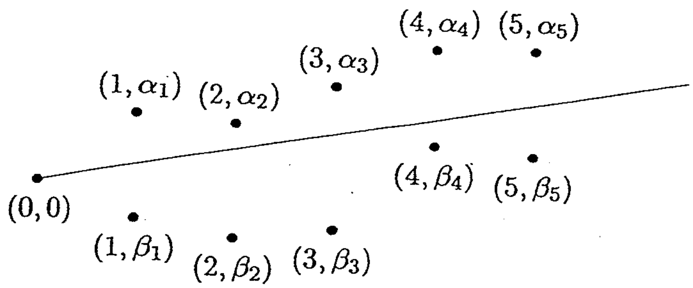
# PhD Qualifying Exam

## Part I (5 hours)

Saturday January 9, 1999

9:00 am - 2:00 pm

1. A ship enters the channel in the figure below at a point that has coordinates $(x, y) = (0, 0)$. There are buoys at coordinates $(i, \alpha_i)$ on one side of the channel, and at coordinates $(i, \beta_i)$ on the other side, for $i = 1 \ldots, m$.




The captain wishes to follow a straight line path between the buoys that maximizes the closest distance (measured vertically) at which the ship passes a buoy. If the ship's path is represented by the equation $y = ux$, the desired path is given by the optimal value of $u$ in the linear programming problem,

$$\begin{array}{ll} \max & \Delta \\ \text{s.t.} & \Delta \leq \alpha_i - iu, \ i = 1, \ldots, m \quad (\lambda_i) \\ & \Delta \leq iu - \beta_i, \ i = 1, \ldots, m \quad (\mu_i). \end{array} \tag{1}$$

(Note that $\Delta$ and $u$ are not restricted to be nonnegative.) If slack variables are inserted, the problem can be written,

$$\begin{array}{ll} \max & \Delta \\ \text{s.t.} & \Delta + iu + s_i = \alpha_i, \ i = 1, \ldots, m \\ & \Delta - iu + t_i = -\beta_i, \ i = 1, \ldots, m \\ & s_i, t_i \geq 0, \ i = 1, \ldots, m. \end{array} \tag{2}$$

It suffices to consider only basic solutions of (2) in which $\Delta$ and $u$ are basic. Thus exactly two slack variables are nonbasic.

(a) If $s_j$ and $s_k$ ($j \neq k$) are nonbasic, what are the values of $u$ and $\Delta$ in the corresponding basic solution (expressed in terms of the problem data $\alpha_i, \beta_i$)?

(b) What are the reduced profits of $s_j$ and $s_k$ in this solution (expressed in terms of the problem data)?[1] Hint: Write profit in terms of the nonbasics.

(c) Use these reduced profits to show that this solution cannot be optimal.

(d) If $s_j$ and $t_k$ are nonbasic, what are the values of $u$ and $\Delta$ in the corresponding basic solution (expressed in terms of the problem data)?

(e) What are the reduced profits of $s_j$ and $t_k$ in this solution?

[1] The reduced profit in a maximization problem is the counterpart of the reduced cost in a minimization problem.

(f) Use these reduced profits to show that this solution is optimal if it is feasible.

(g) Because finding a basic feasible solution solves the problem, it may be more convenient to solve the dual. Associate dual variables $\lambda_i, \mu_i$ with the constraints as shown in (1). If $\lambda_j, \mu_k$ are basic in the dual, use the reduced profits of $s_j, t_k$ to deduce the values of $\lambda_j, \mu_k$ in this basic solution.

(h) Write expressions for the reduced costs of any nonbasic $\lambda_i$ and any nonbasic $\mu_i$ in this basic dual solution. [Because closed-form expressions are available for the basic solutions and reduced costs, the simplex method is greatly accelerated. The starting basic feasible solution is obtained by choosing any basic pair $\lambda_j, \mu_k$.]

2. Consider the polytope

$$P = \{x \in I\!R^n : \sum_{j=1}^{n} \varepsilon_j^i x_j \leq 1 \text{ for } i = 1, \ldots, 2^n\}$$

where $\varepsilon^i$, $i = 1, \ldots, 2^n$, are all the $n$-dimensional vectors with components equal to $+1$ or $-1$.

(a) Describe the vertices of $P$, say $v^1, \ldots, v^K$.

(b) Describe the vertices of $Q = \{x \in I\!R^n : \sum_{j=1}^{n} v_j^k x_j \leq 1 \text{ for } k = 1, \ldots, K\}$.

(c) Describe the facets of $P$ and $Q$. Are they isomorphic to polytopes that you know?

(d) Describe the edges of $P$ and $Q$. How many are there?

(e) What are the diameters of $P$ and $Q$?

3. Consider the mixed 0-1 programming problem

$$\min\{cx + dy : Ax + Gy = b, \quad x \in \{0,1\}^n, \quad y \geq 0\} \tag{P}$$

where $A$ is $m \times n$ and $G$ is $m \times p$, with $p \geq m$, and its linear programming relaxation (LP), in which the condition $x \in \{0,1\}^n$ is replaced by $0 \leq x_j \leq 1$, $j = 1, \ldots, n$.

Show that any optimal solution to (LP) in which at least $m$ components of $y$ are basic is an optimal solution to (P).

4. Consider the asymmetric traveling salesman polytope $P$ on a complete digraph $G = (N, A)$, defined as the convex hull of 0-1 vectors $x$ with components $x_{ij}$, $(i, j) \in A$, satisfying the degree equations and the cycle inequalities

$$\sum(x_{ij} : (i,j) \in C) \leq |C| - 1, \quad C \in \mathcal{C}, \quad 2 \leq |C| \leq n - 2$$

where $\mathcal{C}$ is the set of directed cycles of $G$.

(a) Find all the sequential liftings of the inequality $x_{12} + x_{23} + x_{31} \leq 2$ and show that they are of Chvatal rank 1 by giving their Chvatal derivation.

(b) Show that $x_{12}+x_{23}+x_{31} \leq 2$ also has a nonsequential lifting, and give the Chvatal derivation and Chvatal rank of this lifting.

5. (10 points) Let $G = (V, E)$ be a loopless, connected, undirected multigraph with node set $V$ and edge set $E$.

Let $T$ denote a spanning tree on the nodes $V$ of $G$, not necessarily a subgraph of $G$, i.e., the edges of $T$ are not necessarily a subset of $E$. A cut-labeling of such a spanning tree $T$ with respect to $G$ is defined as follows: for every edge $e = (x, y)$ in the tree $T$, the nodes of $T - e$ are partitioned into two parts $V_x$ and $V_y$, those containing $x$ and $y$ respectively; Label the edge $(x, y)$ in $T$ by the value of the cut in $G$ defined by the partition $(V_x, V_y)$, namely, the number of edges in $G$ with one endpoint in $V_x$ and the other endpoint in $V_y$.

(a) (2 points) Suppose $T$ is a star - a tree with $|V| - 1$ leaves attached to a single center vertex. Suppose that every label in its cut labeling with respect to $G$ is an even number. Prove that $G$ is Eulerian.

(b) (3 points) Now suppose that $T$ is a "two-sided broom": namely, two center nodes $c_1$ and $c_2$ are connected by a tree edge, and all the other nodes are attached as leaves by edges to one of the two centers. Suppose again that every label in the cut labeling of this $T$ with respect to $G$ is an even number. Prove that $G$ is Eulerian.

(c) (5 points) Now suppose $T$ is any fixed spanning tree on the nodes $V$. And suppose that every label in the cut labeling of this $T$ with respect to $G$ is an even number. Prove that $G$ is Eulerian.

6. Let $G$ be a directed graph, $u, v$ two nonadjacent nodes of $G$ and $k$ a positive integer. A *directed walk* of length $k$ is a sequence $x_0, e_1, x_1, e_2, \ldots, e_k, x_k$ of nodes and arcs such that $e_i = x_{i-1}x_i$ for all $i = 1, \ldots, k$. A *directed path* is a directed walk without repetition of nodes. A *chord* is an arc $x_ix_j$ where $j \geq i + 2$.

   Can the following problems be solved in polynomial time? Briefly justify your answers.

   (a) Find a shortest chordless directed path from $u$ to $v$.

   (b) Find a longest directed path from $u$ to $v$.

   (c) Find a directed path of length $k$ from $u$ to $v$.

   (d) Find $k$ edge-disjoint directed paths from $u$ to $v$.

   (e) Find whether all the directed trails from $u$ to $v$ have even length.

   (f) Find whether there exist $k$ arcs such that all directed paths from $u$ to $v$ go through at least one of them.

   (g) Find whether there exists a node $w \neq u, v$ such that all the shortest directed paths from $u$ to $v$ go through node $w$.

1999

# Solutions

1. Solution to Linear Programming question

(a) If $s_j, s_k$ are nonbasic, the corresponding inequalities in (1) are satisfied as equations. Solving them yields,

$$u = \tfrac{\alpha_j - \alpha_k}{j-k} - \tfrac{1}{j-k}s_j + \tfrac{1}{j-k}s_k \quad \Delta = \tfrac{j\alpha_k - k\alpha_j}{j-k} - \tfrac{j}{j-k}s_k + \tfrac{k}{j-k}s_j \tag{3}$$

The solution values $u = \frac{\alpha_j - \alpha_k}{j-k}$ and $\Delta = \frac{j\alpha_k - k\alpha_j}{j-k}$ are obtained by setting $s_j = s_k = 0$ in (3).

(b) The reduced costs of $s_j, s_k$ are their coefficients in the expression for the objective function $\Delta$ in (3).

(c) Because one of the two reduced costs is positive, this basic solution can never be optimal.

(d) If $s_j, t_k$ are nonbasic, then solution of the corresponding equations yields,

$$u = \tfrac{\alpha_j + \beta_k}{j+k} - \tfrac{1}{j+k}s_j + \tfrac{1}{j+k}s_k \quad \Delta = \tfrac{k\alpha_j - j\beta_k}{j+k} - \tfrac{k}{j+k}s_j - \tfrac{j}{j+k}t_k \tag{4}$$

The solution values $u = \frac{\alpha_j + \beta_k}{j+k}$ and $\Delta = \frac{k\alpha_j - j\beta_k}{j+k}$ are obtained by setting $s_j = t_k = 0$ in (4).

(e) The reduced profits of $s_j, t_k$ are their coefficients in the expression for $\Delta$ in (4).

(f) Because both reduced profits are negative, this basic solution is optimal if it is feasible.

(g) The dual of (1) is,

$$\begin{array}{lll} \min & \sum_i \alpha_i\lambda_i - \sum_i \beta_i\mu_i & \\ \text{s.t.} & \sum_i \lambda_i + \sum_i \mu_i = 1 & (\Delta) \\ & \sum_i i\lambda_i - \sum_i i\mu_i = 0 & (u) \\ & \lambda_i, \mu_i \geq 0,\ i = 1, \ldots, m. & \end{array}$$

The reduced profits of $s_j, t_k$ are precisely the slacks in the dual constraints $\lambda_j \geq 0$, $\mu_k \geq 0$. So the solution values are,

$$\lambda_j = \tfrac{k}{j+k} \quad \mu_k = \tfrac{j}{j+k}$$

(h) The reduced cost of $\lambda_i$ is the surplus in the corresponding primal constraint,

$$\Delta + iu - \alpha_i = \tfrac{(k+i)\alpha_j - (j-i)\beta_k}{j+k}.$$

The reduced cost of $\mu_k$ is similarly obtained,

$$\Delta - iu + \beta_i = \tfrac{(k-i)\alpha_j - (j+i)\beta_k}{j+k}$$

2. Solution to Convex Polytopes Question

   (a) The vertices of $P$ are the $2n$ unit vectors $e_j$ and their negatives $-e_j$.

   (b) $\varepsilon^i$ for $i = 1, \ldots, 2^n$.

   (c) $P$ has $2^n$ facets each containing $n$ vertices. So each facet is a simplex. $Q$ has $2n$ facets each containing $2^{n-1}$ vertices. Each facet of $Q$ is an $(n-1)$-dimensional hypercube.

   (d) Each vertex of $P$ is adjacent to all but one vertex, so $P$ has $2n(n-1)$ edges. Each vertex of $Q$ is adjacent to $n$ vertices. So $Q$ has $n2^{n-1}$ edges.

   (e) $P$ has diameter 2 and $Q$ has diameter $n$.

3. Solution to first IP question

The system

$$\begin{array}{ccccccc} Ax & + & Gy & & & = & b \\ x & & & + & z & = & 1 \\ x & , & y & , & z & \geq & 0 \end{array}$$

has $m+n$ rows, hence a basis has at most $m+n$ columns. If $y$ has at least $m$ basic components (which is the most it can have), then $x$ and $z$ together have at most $n$ basic components. But since every row of the form $x_j + z_j = 1$ has to have either $x_j$ or $z_j$ basic and there are $n$ such rows, none of the pairs $x_j, z_j$ can have both of its members basic. Hence for $j = 1, \ldots, n$, either $x_j$ is basic (with value 1), or $z_j$ is basic (with value 1, hence $x_j = 0$).

[illegible] E will be [illegible].

4. Solution to second IP Question

(a) The sequential liftings of $x_{12} + x_{23} + x_{31} \leq 2$ are:

$$\begin{array}{ccccccccc} x_{12} & + & x_{23} & + & x_{31} & + & 2x_{13} & \leq & 2 \\ x_{12} & + & x_{23} & + & x_{31} & + & 2x_{32} & \leq & 2 \\ x_{12} & + & x_{23} & + & x_{31} & + & 2x_{21} & \leq & 2 \end{array}$$

The first of the above inequalities can be obtained by adding

$\frac{2}{3}$ times the outdegree equation for node 1,
$\frac{2}{3}$ times the indegree equation for node 3,
$\frac{2}{3}$ times the 2-cycle inequality $x_{12} + x_{31} \leq 2$, and
$\frac{1}{3}$ times the 3-cycle inequality $x_{12} + x_{23} + x_{31} \leq 2$, to obtain

$$x_{12} + x_{23} + x_{31} + 2x_{13} + \frac{2}{3}(\sum_{i=4}^{n} x_{1j} + \sum_{i=4}^{n} x_{i3}) \leq \frac{8}{3},$$

and then rounding down the coefficients on both sides.
The second and third inequalities can be obtained the same way.

(b) The subtour elimination inequality on the node set $\{1, 2, 3\}$ is a lifting of the cycle inequality on the same node set, which cannot be derived sequentially because any chord when lifted first gets a coefficient of 2. The Chvatal derivation of the subtour elimination inequality on $\{1, 2, 3\}$ consists of adding up $\frac{1}{3}$ times the above three sequentially lifted inequalities and $\frac{1}{3}$ times the 3-cycle inequality $x_{13}+x_{32}+x_{21} \leq 2$, and rounding down the righthand side of the resulting inequality.
Since only rank 0 and rank 1 Chvatal inequalities were used in its derivation, the subtour elimination inequality on $\{1, 2, 3\}$ has Chvatal rank at most 2. In fact, it has Chvatal rank 2, as can be seen from the fact that the linear program that maximizes $x_{12} + x_{21} + x_{13} + x_{31} + x_{23} + x_{32}$ subject to $0 \leq x_{ij} \leq 1$ for all $i, j$, the degree equations, and the cycle inequalities, has value at least 3. Indeed, any solution to the above constraints that has $x_{ij} = \frac{1}{2}$ for $i, j \in \{1, 2, 3\}$ has an objective function value of 3. But then the dual variables associated with this primal solution provide multipliers for a nonnegative linear combination of the constraints that minimizes the resulting righthand side, and this minimum is $\geq 3$: hence the righthand side of the resulting inequality cannot be rounded to 2. This proves that the Chvatal rank of our inequality is strictly greater than 1.

5. Solution to Graph Theory question

(a) $G$ is Eulerian iff it is connected and every vertex has even degree. Since the label on any leaf edge of the star $T$ equals the degree of the leaf node, all these nodes have even degree. It remains to prove that the center of the star also has even degree. However, this degree equals the total degree of all the nodes in the graph (an even number) minus twice the degree of all the edges that are not incident to the center, which is an even number as well.

(b) In this case, we must prove that both $c_1$ and $c_2$ have even degree (since the leaves have even degree which equals the cut-label of their incident tree edge in $T$). We do this by showing first that $c_1$ has even degree. Note that the edge $(c_2, c_1)$ has even label representing the number of edges in the cut defined by the bipartition of $T - (c_1, c_2)$. We form a smaller multigraph $G'$ by contracting all the nodes in the partition containing $c_2$ into a single node $C_2$, and deleting self-loops. The analogous contraction in $T$ results in a spanning tree $T'$ for $G'$ with valid cut-labels. Since $C_2$ is now an even degree node in the graph $G'$ after the contraction, we can apply the result of the previous part to conclude that the center $c_1$ has even degree. Performing the contraction on the partition containing $c_1$ and applying the above argument shows that $c_2$ also has even degree and hence, $G$ is Eulerian.

(c) The answer to the previous question can be generalized to show that if any spanning tree $T$ has cut-labels with respect to $G$ being all even, then $G$ is Eulerian. Trivially, all the leaf nodes in $T$ have even degree as demonstrated by the cut-label on the edges incident on them in $T$. Consider any internal node $c$, with neighbors $v_1, v_2, \ldots, v_k$ in the tree $T$. We can now contract the $k$ subtrees incident on $c$, (namely, the $i^{th}$ subtree is the partition of $T - (c, v_i)$ containing $v_i$) to form a contracted multigraph $G'$ from $G$ and a corresponding tree $T'$ from $T$ with its labels being valid for $G'$. Each of the contracted nodes has even degree in $G'$ since these cuts in $G$ have even degree as labeled in $T$. Applying the result of part 1 to this star tree $T'$ leads us to conclude that $G$ is Eulerian. Since $c$ is any arbitrary internal node of $T$, we have shown that all internal nodes of $T$ have even degree, and thus $G$ is Eulerian.

6. Solution to Networks question

(a) Polynomial algorithm since a shortest path from $u$ to $v$ is always chordless.

(b) NP-hard since the Hamilton path problem can be reduced to this question.

(c) Same answer by taking $k = |V|$.

(d) Polynomial by the max flow algorithm with all edges of capacity 1.

(e) Polynomial since this amounts to checking that $G$ is bipartite.

(f) Polynomial since this amounts to checking whether a min cut has cardinality $k$ or less.

(g) Polynomial since this amounts to finding a shortest path from $u$ to $v$ in $G$ and $G \setminus w$, for each $w \neq u, v$.

2000 2010

# ACO Qualifying Exam
# OR Qualifying Exam, Part I

(5 hours)

January 15th

## Linear Programming Problem

Recall that if Dantzig-Wolfe decomposition is applied to a problem of the form

$$\begin{array}{ll} \min & cx \\ \text{s.t.} & Ax = a \\ & Bx \le b \\ & x \ge 0 \end{array} \tag{1}$$

the master problem is

$$\begin{array}{lll} \min & \sum_{k=1}^{K} (cy^k)\alpha_k & \\ \text{s.t.} & \sum_{k=1}^{K} (Ay^k)\alpha_k = a & (u) \\ & \sum_{k=1}^{K} \alpha_k = 1 & (u_0) \\ & \alpha_k \ge 0,\ k = 1, \ldots, K & \end{array} \tag{2}$$

where $y^1, \ldots, y^K$ are the extreme points so far generated. The next extreme point $y^{K+1}$ is an optimal basic solution $y$ of the subproblem

$$\begin{array}{ll} \min & (c - uA)y \\ \text{s.t.} & By \le b \\ & y \ge 0 \end{array} \tag{3}$$

where $(u, u_0)$ is the dual solution of the previous master. It is assumed that (3) is always bounded and feasible.

(a) To solve a 0-1 problem

$$\begin{array}{ll} \min & cx \\ \text{s.t.} & Ax = a \\ & x_j \in \{0, 1\},\ j = 1, \ldots, n \end{array}$$

it is useful to solve its linear programming relaxation

$$\begin{array}{ll} \min & cx \\ \text{s.t.} & Ax = a \\ & 0 \le x_j \le 1,\ j = 1, \ldots, n \end{array} \tag{4}$$

If there are a very large number of variables it may be advantageous to apply Dantzig-Wolfe decomposition to (4). Formulate the master problem and subproblem and explain why this approach might be useful.

(b) What is the reduced cost of $\alpha_{K+1}$ when this variable is added to (2)?

(c) Write an expression for $y_j^{K+1}$, $j = 1, \ldots, n$, in terms of $u$.

(d) Show that if (1) is feasible, $a$ can be written as a convex combination of vectors, each of which is a sum of a subset of columns of $A$.

## Question on Integer Programming

Consider $P = \{x \in R^n : Ax \leq b\}$ and $P_I = \{x \in Z^n : Ax \leq b\}$, where $A$ is an $m \times n$ matrix and $b$ is an $m$-column vector. Assume that $A$ and $b$ have integral entries and that the constraints $Ax \leq b$ contain the inequalities $0 \leq x \leq 1$.

Chvátal cuts are defined by $\lfloor uA \rfloor x \leq \lfloor ub \rfloor$ for any row vector $u \in R^m_+$, where $\lfloor uA \rfloor$ denotes the vector obtained from the vector uA by rounding down every component to an integer. The elementary closure $P_C$ is the convex set obtained as the intersection of all Chvátal cuts.

The lift-and-project closure $P_{L\&P}$ is obtained as follows. For fixed $j \in \{1, \ldots, n\}$, form the nonlinear system

$$\begin{aligned} (1 - x_j)(b - Ax) &\geq 0 \\ x_j(b - Ax) &\geq 0 \end{aligned}$$

and linearize it by substituting $x_j$ for $x_j^2$ and $y_i$ for $x_i x_j$, $i \neq j$. This higher dimensional polyhedron is then projected back on the $x$-space. Let $P_j$ denote the resulting polyhedron. Define $P_{L\&P} = \cap_j P_j$.

The disjunctive closure is obtained as follows. For any $(\pi, \pi_0) \in Z^{n+1}$, let $P(\pi, \pi_0)$ be the convex hull of the union of the two polyhedra $P \cap \{\pi x \leq \pi_0\}$ and $P \cap \{\pi x \geq \pi_0 + 1\}$. Define $P_D = \cap_{(\pi,\pi_0) \in Z^{n+1}} P(\pi, \pi_0)$.

(a) (4 points) Describe explicitely the polytopes $P_C$, $P_{L\&P}$ and $P_D$ for the following instance

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ x_1 + x_2 &\leq \frac{3}{2} \\ 0 \leq x_1 &\leq 1 \\ 0 \leq x_2 &\leq 1. \end{aligned}$$

(b) (3 points) In general, is it true that $P_D \subseteq P_{L\&P}$? [Hint: you may appeal to results from the literature, textbooks, etc).

(c) (3 points) Do you know other types of cuts or other strengthening procedures for $P$? Can you compare the corresponding elementary closures to some of the convex sets $P_C$, $P_{L\&P}$ and $P_D$?

Question on Advanced Integer Programming

**Problem (10 points):**

Let $B = (b_1, b_2, \ldots, b_n)$ denote the basis of an integer lattice $\mathcal{L}$, i.e., $\mathcal{L}$ is the set of all integer combinations of the integer column vectors $b_1, \ldots, b_n$. In short,

$$\mathcal{L} = \{\sum_{i=1}^{n} \lambda_i b_i : (\lambda_1, \ldots, \lambda_n) \in Z^n\} = \{B \cdot \lambda^T : \lambda \in Z^n\}.$$

Consider the two basic problems on lattices: (1) The shortest lattice vector problem (SVP) is to find the smallest length nonzero vector (i.e, nonzero vector with minimum $L_2$ norm) in $\mathcal{L}$ and (2) the closest lattice vector problem (CVP) is given a nonzero integral vector, $x$ say, to find a vector in $\mathcal{L}$ that is closest (in $L_2$-norm) to the given vector $x$. The goal of this question is to show that the SVP is no harder than the CVP.

1. (2 points) Suppose $s^* = B \cdot (\lambda^*)^T$ is a shortest lattice vector in $\mathcal{L}$. Show that all the multipliers $\lambda_1^*, \lambda_2^*, \ldots \lambda_n^*$ cannot be even integers.

2. (3 points) Suppose then without loss of generality that the co-efficient $\lambda_1^*$ is odd in the solution to SVP($\mathcal{L}$). Consider the modified lattice $\mathcal{L}^1$ defined by the modified basis $B^1 = (2b_1, b_2, \ldots, b_n)$. Show that there is a solution to the CVP on the lattice $\mathcal{L}^1$ from the point $b_1$ whose distance from $b_1$ is at most $||s^*||$ (i.e., the length of a shortest lattice vector in the original lattice $\mathcal{L}$). [Hint: Consider the vector $B^1 \cdot (\frac{\lambda^*+1}{2}, \lambda_2^*, \ldots, \lambda_n^*)^T$.]

3. (3 points) On the other hand, suppose we are given a solution $\lambda'$ to the CVP on the modified lattice $\mathcal{L}^1$ from the vector $b_1$. Let $\ell$ denote the distance of this solution from

$b_1$, i.e., $\ell = ||B^1 \cdot \lambda^T - b_1||$. Then show that length of a solution to the SVP in the original lattice $\mathcal{L}$ is at most $\ell$. [Hint: Consider the vector $B \cdot (2\lambda'_1 - 1, \lambda'_2, \lambda'_3, \ldots, \lambda'_n)^T$.]

4. (2 points) Use the above to design an algorithm for the shortest vector problem in $\mathcal{L}$ using polynomially many calls to a procedure for computing the closest vector given a lattice and an input vector.

# Question on *Graph Theory*

A simple directed graph $G = (N, A)$ is *assignable* if it admits a *cycle decomposition*, i.e. a spanning union of node-disjoint directed cycles.

(a) Give a necessary and sufficient condition for $G$ to be assignable.

[Hint: Construct an undirected bipartite graph $G^*$ whose edges are in 1-1 correspondence with the arcs of $G$, and interpret the problem on $G^*$.]

(b) Putting nonnegative weights on the nodes of $G$, formulate the problem of finding a maximum weight assignable subgraph in $G$ as a linear program. Prove the validity of your formulation.

Question on Networks and Matchings

**Problem (10 points):**

Let $(S, \bar{S})$ be a min $s-t$ cut in a capacitated undirected graph $G = (V, E)$ (assume all capacities in this question are nonnegative). We now add to $G$ a new node, $z$, with capacitated edges $(i, z)$ $i \in V$ to obtain a new graph $G'$.

1. (5 points) Prove that there exists a new min $s-t$ cut, $(S', \bar{S}')$, such that either $z \in S'$ and $S \subset S'$ or $z \in \bar{S}'$ and $\bar{S} \subset \bar{S}'$.

2. (2 points) Use this property to find a new min cut by solving 2 small problems instead of a big one. Is the computational effort reduced? Can you use the property to find a max flow in $G'$ by solving two small problems?

3. (1 point) Use the above to construct an algorithm for computing min cuts.

4. (2 points) Is the assumption that the graph is undirected essential? If it is, state where it has been used. Else, state a possible simplification when the graphs considered are directed but acyclic.

Question on Convex Polyhedra

(a) Consider the polyhedron

$$P := \{x \in I\!R^n : Ax \le b\},$$

where $A$ is $m \times n$ and $b$ is $m \times 1$, and its polars defined as

$$\begin{aligned} P^{0+} &:= \{y \in I\!R^n : xy \le 1 \text{ for all } x \in P\} \\ P^{0-} &:= \{y \in I\!R^n : xy \ge 1 \text{ for all } x \in P\} \\ P^{*} &:= \{(y, y_0) \in I\!R^{n+1} : xy \ge y^0 \text{ for all } x \in P\}. \end{aligned}$$

(a1) Show that

$$P^{0+} = \{y \in I\!R^n : y = uA \text{ for some } u \ge 0 \text{ such that } ub \le 1\}$$

(a2) Find corresponding expressions for $P^{0-}$ and $P^*$ and prove their validity.

(b) Given the polyhedra $P_i := \{x \in I\!R^n : A^i x \le b^i\}$, $i \in Q$, where each $A^i$ is $m_i \times n$ and each $b^i$ is $m_i \times 1$, let $P_D := \text{conv}\,(\cup_{i\in Q} P_i)$, where conv denotes the convex hull.

(b1) Show that

$$P_D^{0+} = \{y \in I\!R^n : \exists u^i \; y = u^i A^i, \boxed{\forall\, i \in Q} \;\&\; \cancel{\text{for some}}\; u^i \ge 0 \;\&\; \cancel{\text{such that}}\; u^i b^i \le 1 \; \cancel{i \in Q}\}.$$

(b2) Find corresponding expressions for $P_D^{0-}$ and $P_D^*$.

(b3) Show that

$$? \; \text{conv}\, P_D = \{x \in I\!R^n : yx \ge y_0 \text{ for all } (y, y_0) \in P_D^*\}.$$

4

SOLUTIONS PART I OR/ACO QUAL EXAM JAN 2000

## Linear Programming Solution

(a) The master problem is (2) and the subproblem is

$$\begin{array}{ll} \min & (c-uA)y \\ \text{s.t.} & y \le e \\ & y \ge 0 \end{array} \tag{5}$$

The original problem has size $m \times n$ for very large $n$. In D-W, the master problem has size $m \times K$. If $K << n$, it may be easier to solve $K$ master problems than the original problem. The subproblem is trivial because it completely decouples.

(b) The reduced cost of $\alpha_{K+1}$ is $(c-uA)y^{K+1} + u_0$.

(c) The subproblem solution is obviously

$$y_j = \begin{cases} 1 & \text{if } c_j < uA_j \\ 0 & \text{otherwise} \end{cases}$$

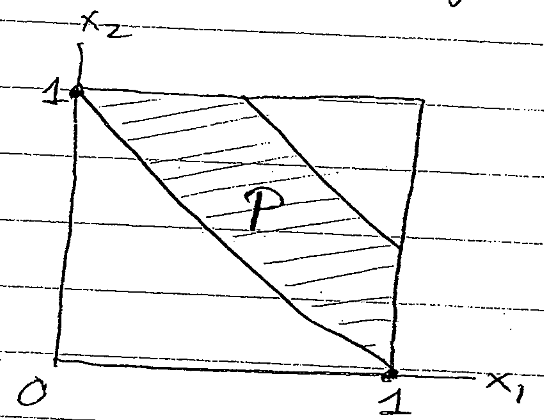
(d) Because each $y^k$ is a 0-1 vector, (2) implies

$$a = \sum_k \left( \sum_{j \in J_k} A_j \right) \alpha_k$$

where $\sum_k \alpha_k = 1$ and $\alpha_k \ge 0$ and $J_k \subset \{1, \ldots, n\}$.

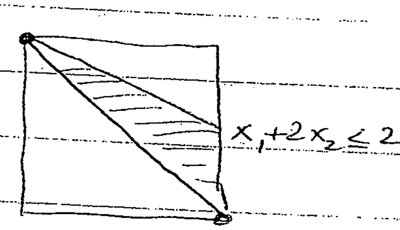
Solution to question on integer programming

(a)

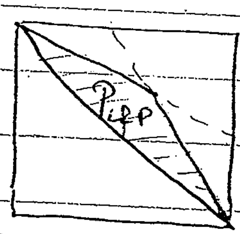



$P_C = P_I$ since $x_1 + x_2 \le \lfloor \frac{3}{2} \rfloor = 1$ is a Chvatal cut.

$P_1 = \text{conv}\{P \cap \{x_1 = 0\} \cup P \cap \{x_1 = 1\}\}$




Similarly $P_2 = \text{conv}\{P \cap \{x_2 = 0\} \cup P \cap \{x_2 = 1\}\}$




$$P_{L\&P} = \left\{ x \in \mathbb{R}^2 : \begin{array}{l} x_1 + x_2 \ge 1 \\ x_1 + 2x_2 \le 2 \\ 2x_1 + x_2 \le 2 \end{array} \right\}$$

$$P_D = P_I = P \cap \{x_1 + x_2 \le 1\} \cup \underbrace{P \cap \{x_1 + x_2 \ge 2\}}_{\emptyset}$$

(b) In general, $P_D \subseteq P_{L\&P}$ since $P_{L\&P} = \bigcap_j P_j$ and $P_j$ is obtained from the special disjunction $x_j \le 0$ or $x_j \ge 1$ which corresponds to $\pi$ being a unit vector in $P(\pi, \pi_0)$.

(c) This is an open ended question. Gomory fractional cuts can be viewed as Chvatal cuts (see Nemhauser Wolsey). Lovasz – Schrijver propose several procedures that are at least as strong as the lift-and-project procedure described here. In fact, the Lovász-Schrijver strengthening can be strictly stronger (see Balas, Ceria, Cornuejols). Gomory's mixed integer cuts are at least as strong as Gomory's fractional cuts (and they can be strictly stronger). In addition, there are special purpose cuts for the T.S.P, knapsack problem etc. The lifted cover inequalities for the knapsack problem, for example, can be useful for general integer programs.

Question on Advanced Integer Programming

**Problem (10 points):**

Let $B = (b_1, b_2, \dots, b_n)$ denote the basis of an integer lattice $\mathcal{L}$, i.e., $\mathcal{L}$ is the set of all integer combinations of the integer column vectors $b_1, \dots, b_n$. In short,

$$\mathcal{L} = \{\sum_{i=1}^{n} \lambda_i b_i : (\lambda_1, \dots, \lambda_n) \in Z^n\} = \{B \cdot \lambda^T : \lambda \in Z^n\}.$$

Consider the two basic problems on lattices: (1) The shortest lattice vector problem (SVP) is to find the smallest length nonzero vector (i.e, nonzero vector with minimum $L_2$ norm) in $\mathcal{L}$ and (2) the closest lattice vector problem (CVP) is given an integral vector, $x$ say, to find a vector in $\mathcal{L}$ that is closest (in $L_2$-norm) to the given vector $x$. The goal of this question is to show that the SVP is no harder than the CVP.

1. (2 points) Suppose $s^* = B \cdot (\lambda^*)^T$ is a shortest lattice vector in $\mathcal{L}$. Show that all the multipliers $\lambda_1^*, \lambda_2^*, \dots \lambda_n^*$ cannot be even integers.

2. (3 points) Suppose then without loss of generality that the co-efficient $\lambda_1^*$ is odd in the solution to SVP($\mathcal{L}$). Consider the modified lattice $\mathcal{L}^1$ defined by the modified basis $B^1 = (2b_1, b_2, \dots, b_n)$. Show that there is a solution to the CVP on the lattice $\mathcal{L}^1$ from the point $b_1$ whose distance from $b_1$ is at most $||s^*||$ (i.e., the length of a shortest lattice vector in the original lattice $\mathcal{L}$). [Hint: Consider the vector $B^1 \cdot (\frac{\lambda^*+1}{2}, \lambda_2^*, \dots, \lambda_n^*)^T$.]

3. (3 points) On the other hand, suppose we are given a solution $\lambda'$ to the CVP on the modified lattice $\mathcal{L}^1$ from the vector $b_1$. Let $\ell$ denote the distance of this solution from

$b_1$, i.e., $\ell = ||B^1 \cdot \lambda'^T - b_1||$. Then show that length of a solution to the SVP in the original lattice $\mathcal{L}$ is at most $\ell$. [Hint: Consider the vector $B \cdot (2\lambda'_1 - 1, \lambda'_2, \lambda'_3, \ldots, \lambda'_n)^T$.]

4. (2 points) Use the above to design an algorithm for the shortest vector problem in $\mathcal{L}$ using polynomially many calls to a procedure for computing the closest vector given a lattice and an input vector.

**Solutions**

1. Suppose for a contradiction this is the case, then $B \cdot (\frac{\lambda^*}{2})^T = s^*/2$ is a shorter nonzero vector in $\mathcal{L}$.

2. The difference vector of the proposed solution and $b_1$ is $(B^1 \cdot (\frac{\lambda_1^*+1}{2}, \lambda_2^*, \ldots, \lambda_n^*)^T) - b_1 = \frac{\lambda_1^*+1}{2} \cdot (2b_1) + \sum_{i=2}^n \lambda_i^* \cdot b_i - b_1 = B \cdot (\lambda^*)^T = s^*$. We note that this is a legitimate candidate solution to the CVP since $\lambda_1^*$ is odd and hence $\frac{\lambda_1^*+1}{2}$ is an integer.

3. The proposed vector $B \cdot (2\lambda'_1 - 1, \lambda'_2, \lambda'_3, \ldots, \lambda'_n)^T$ lies in the original lattice $\mathcal{L}$ since all multipliers are integer since the $\lambda'$s are integers. This vector can be expanded as follows.

   $B \cdot (2\lambda'_1 - 1, \lambda'_2, \lambda'_3, \ldots, \lambda'_n)^T = (2\lambda'_1 - 1)b_1 + \sum_{i=2}^n \lambda'_i b_i = \lambda'_1(2b_1) + \sum_{i=2}^n \lambda'_i b_i - b_1 = B^1 \cdot (\lambda')^T - b_1$. Thus the length of this vector is $\ell$. This is a feasible solution to the SVP in $\mathcal{L}$ since $2\lambda'_1 - 1$ is nonzero for any integer $\lambda'_1$. Thus the shortest vector in $\mathcal{L}$ has length at most $\ell$.

4. For i$= 1, \ldots, n$, compute the length of the solution to the CVP on $B^i$ from the vector $b_i$, and determine the minimum, say $i = k$ with solution $\lambda(k)$ of multipliers for the

lattice $\mathcal{L}^k$. Return the vector $B \cdot (\lambda(k)_1, \ldots, 2\lambda(k)_k - 1, \ldots, \lambda(k)_n)^T$ as the shortest lattice vector in $\mathcal{L}$. Correctness follows from the above answers.

## Answer to Question on Graph Theory

(a) For all $S \subset N$, denote by $G(S)$ the subgraph of $G$ induced by $S$, and define

$$\Gamma(S) := \{j \in N : (i,j) \in A \text{ for some } i \in S\}$$

Then $G$ is assignable if and only if

$$|S \setminus \Gamma(S)| \leq |\Gamma(S) \setminus S| \text{ for all } S \subseteq N$$

**Proof.** Let $G^* := (V^*, E^*)$, where $V^* = V_1 \cup V_2$ with $|V_1| = |V_2| = |N|$, $V_1 \cap V_2 = \emptyset$; for every $i \in N$ there is a vertex $f(i) \in V_1$ and a vertex $g(i) \in V_2$; and

$$E^* := \{(f(i), g(j)) : (i,j) \in A\}.$$

Then $G$ admites a cycle decomposition if and only if $G^*$ has a perfect matching. From the König-Hall theorem, this is the case if and only if $|V_1| = |V_2|$ (which is true here) and $|X| \leq |N(X)|$ for all $X \subseteq V_1$, where $N(X) := \{j \in V : (i,j) \in E \text{ for some } i \in X\}$. But $X \subset V_1$ satisfies this condition if and only if $S := \{i \in N : f(i) \in X\}$ satisfies $|S \setminus \Gamma(S)| \leq |\Gamma(S) \setminus S|$.

(b) The problem can be stated as

$$\max \sum (w_i y_i : i \in N)$$

such that

$$\begin{aligned}
\sum(x_{ij} : j \in \Gamma(i)) - y_i &= 0 \quad i \in N \\
\sum(x_{ij} : i \in \Gamma^{-1}(j)) - y_j &= 0 \quad j \in N \\
x_{ij} \geq 0, (i,j) \in A, 0 \leq y_i \leq 1, &\quad i \in N.
\end{aligned}$$

This formulation is valid because the coefficient matrix of the constraint set is easily seen to be totally unimodular, and thus the basic solutions are integer. Further, since the $y_i$ are 0-1, so are the $x_{ij}$.

# Question on Networks and Matchings for Qualifying exams - January 2000

**Author: Rafi Hassin** (hassin@math.tau.ac.il)

**Problem (10 points):**

Let $(S, \bar{S})$ be a min $s-t$ cut in a capacitated undirected graph $G = (V, E)$ (assume all capacities in this question are nonnegative). We now add to $G$ a new node, $z$, with capacitated edges $(i, z)$ $i \in V$ to obtain a new graph $G'$.

1. (5 points) Prove that there exists a new min $s-t$ cut, $(S', \bar{S}')$, such that either $z \in S'$ and $S \subset S'$ or $z \in \bar{S}'$ and $\bar{S} \subset \bar{S}'$.

2. (2 points) Use this property to find a new min cut by solving 2 small problems instead of a big one. Is the computational effort reduced? Can you use the property to find a max flow in $G'$ by solving two small problems?

3. (1 point) Use the above to construct an algorithm for computing min cuts.

4. (2 points) Is the assumption that the graph is undirected essential? If it is, state where it has been used. Else, state a possible simplification when the graphs considered are directed but acyclic.

**Solutions**

1. The question is based on a paper by Donald M. Topkis where the proofs use theory of minimizing sub-additive functions over a lattice. The proofs here are different:

   Suppose $z \in S'$ and $S \setminus S' \neq \emptyset$. The incentive to add $S \setminus S'$ to $S'$ in $G'$ is greater than it was in $G$ since we also save edges between $z$ and $S \setminus S'$ and edges between $S \setminus S'$ and

$\bar{S} \cap S'$. By minimality of $S$ in $G$ it follows that this change doesn't add to the capacity of $(S', \bar{S}')$.

Alternative proof: Let $f$ be a maximum flow in $G$. Use the residual capacities and compute in $G'$ a maximum $s - z$ flow in the graph induced by $S \cup \{z\}$. Denote it (and its value) $g$. There is an $s - z$ cut saturated by the flow $f + g$. Similarly, compute a maximum $z - t$ flow $h$ using the residual capacities in the graph induced by $\bar{S} \cup \{z\}$. Suppose $g \leq h$, then $g + f$ is an $s - t$ maximum flow in $G'$ saturating a cut $(S', \bar{S}')$ such that $\bar{S} \subset \bar{S}'$ and $z \in \bar{S}'$. The other case obtains when $g \geq h$.

2. A min cut can be obtained by finding min cuts once in the graph obtained by contracting $S \cup \{z\}$ into a single node $s'$, and then in the graph obtained by contracting $\bar{S} \cup \{z\}$ into a node $t'$. Suppose we use an algorithm that requires $cn^3$ elementary steps for a graph with $n$ nodes. Then instead of $c(n+1)^3$ for $G'$ we will have $c(k+1)^3 + c(n-k+1)^3$. Alternatively, use the second proof to compute the flows $g$ and $h$. This will also give the max flow in $G'$.

3. Add a node at a time.

4. Everything applies to digraphs as well. If the final graph is acyclic, we can add nodes according to the topological order induced by it, and then only one subproblem needs to be solved at each step.

## Answer to Question on Convex Polyhedra

(a1) We have to show that for a given $y$, the inequality

$$yx \leq 1$$

is a consequence of the system

$$Ax \leq b$$

if and only if

$$y = uA$$

for some $u \geq 0$ such that $ub \leq 1$. But this follows from the nonhomogeneous version of the Farkas lemma.

(a2) An analogous reasoning shows that $yx \geq 1$ is a consequence of the system $-Ax \geq -b$ if and only if

$$y = u(-A)$$

for some $u \geq 0$ such that $u(-b) \geq 1$, and $yx \geq y_0$ is a consequence of $-Ax \geq -b$ if and only if

$$y = u(-A), \quad y_0 \geq u(-b)$$

for some $u \geq 0$. Therefore

$$P^{0-} = \{y \in \mathbb{R}^n : y = u(-A) \text{ for some } u \geq 0 \text{ such that } u(-b) \geq 1\}$$

and

$$P^* = \{(y, y_0) \in \mathbb{R}^{n+1} : y = u(-A),\ y_0 \geq u(-b) \text{ for some } u \geq 0\}.$$

(b1) The inequality $yx \leq 1$ is satisfied by all $x \in P_D$ if and only if it is satisfied by all $x \in P_i$ for each $i \in Q$. From the answer to (a1), this means that $yx \leq 1$ for all $x \in P_D$ if and only if

$$y = uA^i,\ i \in Q,$$

for some $u^i \geq 0$ such that $u^i b^i \leq 1$, $i \in Q$.

(b2) From the answer to (a2), the corresponding expressions are

$$
\begin{aligned}
P_D^{0-} &= \{y \in \mathbb{R}^n : y = u^i(-A^i),\ i \in Q,\ \text{for some } u^i \geq 0 \text{ such that } u^i(-b^i) \geq 1,\ i \in Q\} \\
P_D^* &= \{(y, y_0) \in \mathbb{R}^{n+1} : y = u^i(-A^i),\ y_0 \geq u^i(-b^i) \text{ for some } u^i \geq 0,\ i \in Q\}
\end{aligned}
$$

(b3) $P_D^*$ is the set of all $(y, y_0) \in \mathbb{R}^{n+1}$ such that $yx \geq y_0$ is a valid inequality for $P_D$. But the set of all valid inequalities for $P_D$ defines the convex hull of $P_D$.

2000

# OR Qualifying Exam

## Part II

(3 hours)

January 16th

**Nonlinear Programming Question**

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, $b \neq 0$ be given.

Consider the problem

$$\begin{array}{rcl} \min & & \|x\| \\ Ax & = & b \\ x & \geq & 0. \end{array} \qquad (P)$$

(Here $\|\cdot\|$ denotes the Euclidean norm, i.e., $\|x\| = \sqrt{x^T x}$.)

(a) Write down the Lagrangian function and Lagrangian dual problem.

(b) Prove that for any given $c \in \mathbb{R}^n$

$$\inf_{x \in \mathbb{R}^n} (\|x\| - x^T c) = \begin{cases} 0 & \text{if} \quad \|c\| \leq 1 \\ -\infty & \text{if} \quad \|c\| > 1 \end{cases}$$

(c) Show that the Lagrangian dual problem in (a) can be written as

$$\begin{array}{rc} \max & b^T y \\ & \|A^T y + w\| \leq 1 \\ & w \geq 0. \end{array} \qquad (D)$$

(d) Suppose $\overline{x}$ is an optimal solution of (P). Using the KKT conditions, show that there exist $\overline{y}, \overline{w} \geq 0$ feasible for (D) and such that $b^T \overline{y} = \|\overline{x}\|$ (i.e., strong duality holds).

Solution NON LINEAR PROGRAMMING

(a) $L(x,u,v) = \|x\| + v^T(b-Ax) + u^T(-x)$

$$(P) \iff \min_{x} \max_{\substack{u \geq 0 \\ v}} L(x,u,v)$$

$$(D) \iff \max_{\substack{u \geq 0 \\ v}} \min_{x} L(x,u,v) \iff$$

$$\max_{\substack{u \geq 0 \\ v}} \min_{x} \left[\|x\| + v^T(b-Ax) - u^T x\right]$$

(b) By Cauchy-Schwarz,

$$|x^T c| \leq \|x\| \, \|c\|.$$

Thus $\|c\| \leq 1 \Rightarrow |x^T c| \leq \|x\|$, so

$$\|x\| - x^T c \geq \|x\| - \|x\| = 0.$$

It then follows that

$$\inf_{x} (\|x\| - x^T c) = 0.$$

On the other hand if $\|c\| > 1$, taking $x = \lambda c$ with $\lambda > 0$, we get

$$\|x\| - x^T c = \lambda\|c\| - \lambda\|c\|^2 = \lambda\|c\|(1-\|$$

$$\to -\infty \quad \text{as } \lambda \to +\infty$$

because $1 - \|c\| < ($

So $\inf_{x} (\|x\| - x^T c) = -\infty$

(c) From (a) we have

(D) $$\max_{\substack{u \geq 0 \\ v}} \min_x \left[ \|x\| + v^T(b - Ax) - u^T x \right]$$

$$= \max_{\substack{u \geq 0 \\ v}} b^T v + \min_x \left[ \|x\| - x^T (A^T v + u) \right]$$

But by part (b)

$$\min_x \left[ \|x\| - x^T (A^T v + u) \right] = \begin{cases} 0 & \text{if } \|A^T v + u \\ -\infty & \text{otherwise} \end{cases}$$

Thus (D) $\Leftrightarrow$ $$\max \; b^T v$$
$$\|A^T v + u\| \leq 1$$
$$u \geq 0.$$

(d) If $\bar{x}$ is a solution then KKT hold (because the constraints are linear), so there exist $\bar{u} \geq 0$, $\bar{v}$ s.t.

$$\nabla f(\bar{x}) + \nabla g(\bar{x}) \bar{u} + \nabla h(\bar{x}) \bar{v} = 0$$
$$\bar{u} \geq 0 \qquad \bar{u}^T g(\bar{x}) = 0.$$

In this case

$$\nabla f(\bar{x}) = \frac{\bar{x}}{\|\bar{x}\|}, \quad \nabla g(\bar{x}) = -I, \quad \nabla h(\bar{x}) = A^T$$

So KKT $\Leftrightarrow$ $\dfrac{\bar{x}}{\|\bar{x}\|} - \bar{u} + A^T \bar{v} = 0, \quad \bar{u}^T \bar{x} = 0$

Multiplying by $\bar{x}^T$, we get

$$0 = \|\bar{x}\| - \bar{x}^T \bar{u} + \bar{x}^T A^T \bar{v} = \|\bar{x}\| + b^T \bar{v}$$

So $(\bar{y}, \bar{w}) = (-\bar{v}, \bar{u})$ is feasible for (P) and $b^T \bar{y} = \|\bar{x}\|$.

# OR Ph. D. Qualifying Examination
## Part I

Graduate School of Industrial Administration
Carnegie Mellon University
Pittsburgh, PA 15213

January 2001

Jan 12

Room 153

| | | |
|---|---|---|
| OR – ACO | All 6 questions | 12 – 5 pm |
| OM | 4 questions | 1 – 5 pm |

Answer all six questions.

1. (Linear Programming)

   Consider the family of linear functions $f_i(x) = a^i x + b_i$ for $i = 1, \ldots, m$. Let the *bottleneck LP* be the problem of minimizing $\max_i\{f_i(x)\}$ over all $x \geq 0$ for $x \in R^n$. Let $C$ be the convex hull of the points $a^1, \ldots, a^m$. Show that the bottleneck LP is bounded (has a finite optimal value) if and only if $C$ intersects the nonnegative orthant of $R^n$.

2. (Convex polytopes)

   Recall the following:

   - A $d$-polytope is a *simplex* if it is the convex hull of $d+1$ affinely independent points.
   - A polytope is *simplicial* if each of its facets is a simplex.
   - A $d$-polytope is *simple* if each of its vertices is contained in exactly $d$ facets.

   Extend the notions of simplicial and simple as follows: A $d$-polytope is $k$-simplicial ($k \leq d-1$) if each of its $k$-faces is a simplex. A $d$-polytope is $k$-simple ($k \leq d-1$) if each of its $(d-k-1)$-faces is contained in exactly $k+1$ facets.

   (a) (2pts) Let $P$ be a $d$-polytope, and without loss of generality assume $0 \in \mathrm{int}(P)$. Prove that $P$ is $k$-simple if and only if $P^\Delta$ is $k$-simplicial.

   (b) (1pt) Prove that every $d$-polytope ($d \geq 2$) is 0-simple, 1-simple, 0-simplicial, and 1-simplicial.

   (c) (1pt) Prove that if a polytope is $k$-simple ($k$-simplicial) then it is also $h$-simple ($h$-simplicial) for $h \leq k$.

   (d) (1pt) Let $P$ be a polytope and $F$ be a face of $P$. Show that if both $P/F$ and $F$ are simplexes then so is $P$.

   (e) (3pts) Assume $d \geq 3$. Prove that if a $d$-polytope $P$ is both $k_1$-simple and $k_2$-simplicial with $k_1 + k_2 \geq d+1$ then $P$ is a simplex.

   (f) (2pts) Give an example (with $d \geq 3$) to show that the previous statement is not true (in general) if the condition $k_1 + k_2 \geq d+1$ is weakened to $k_1 + k_2 \geq d$.

3. (Graph Theory)

   Identify a class of regular, perfect graphs for which the number of maximal cliques grows exponentially with the number of vertices. What is the smallest number $n_0$ of vertices for which a graph in the class has more maximal cliques than edges? Is the instance for which $n_0$ is attained the unique graph (up to isomorphism) with this property?

4. (Networks and Matchings)

Consider the following problem:

Given an arc-weighted connected digraph $G = (N, A)$, find a minimum-weight closed walk in $G$ that traverses every arc at least once.

(a) Describe an algorithm for solving $(P)$, and give its complexity.

(b) Consider now a modification $(P')$, in which $k$ arcs of $G$, where $1 \leq k < |A|$ is independent of $|V|$, are undirected. Can you modify your algorithm to solve $(P')$? What happens to its complexity?

5. (Integer Programming)

Derive as many facets of the 0-1 programming polytope

$$P := \mathrm{conv}\{x \in \{0,1\}^7 : 6x_1 + 5x_2 + 3x_3 + 2x_4 + 2x_5 + 2x_6 + x_7 \geq 15\}$$

as you can. Justify your procedure.

6. (Advanced Integer Programming)

Compare the standard (Dantzig, Johnson, Fulkerson) formulation of the traveling salesman problem with the Miller, Tucker, Zemlin formulation, which replaces the subtour elimination constraints

$$x(S, S) \leq |S| - 1, \quad S \subset N, \ |S| \geq 2 \tag{1}$$

with the constraints

$$u_i - u_j + nx_{ij} \leq n - 1 \quad \text{for all } i, \ j \in N \setminus \{1\}. \tag{2}$$

Project the constraints (2) onto the $x$-space to find out which formulation yields a tighter LP relaxation.

# Solutions to OR Ph. D. Qualifying Examination
# Part I

Graduate School of Industrial Administration
Carnegie Mellon University
Pittsburgh, PA 15213

January 2001

- (Linear Programming – John Hooker)

  The bottleneck LP can be reformulated,

$$\begin{aligned} \min\ & z \\ \text{s.t.}\ & z - a^i x \geq b_i,\ i = 1, \ldots, m \\ & x \geq 0 \end{aligned} \tag{1}$$

  The dual of (1) is

$$\begin{aligned} \max\ & \sum_i b_i u_i \\ \text{s.t.}\ & \sum_i u_i = 1 \\ & -\sum_i a^i u_i \leq 0 \\ & u \geq 0 \end{aligned}$$

  Problem (1) is bounded if and only if the dual problem is feasible. But the dual is feasibile if and only if some convex combination $\sum_i a^i u_i$ is nonnegative. This is true if and only if the convex hull of $a^1, \ldots, a^m$ intersects the nonnegative orthant.

- (Convex polytopes – Javier Pena)

  (a) Recall that the map $F \mapsto F^\diamond$ defines a one to one correspondence between the $k$-faces of $P$ and the $(d-k-1)$-faces of $P^\Delta$ such that

$$F \subseteq G \Leftrightarrow G^\diamond \subseteq F^\diamond.$$

  The equivalence readily follows: $F$ belongs to exactly $d$ facets if and only if $F^\diamond$ contains exactly $d$ vertices, i.e., if and only if $F^\diamond$ is a simplex.

  (b) A point is a 0-simplex and a segment is a 1-simplex. Hence all vertices and edges of any polytope are simplices. Thus any polytope is 0-simplicial and 1-simplicial. In addition, for any polytope $P$, $P^\Delta$ is 0-simplicial and 1-simplicial, thus by part (a) $P$ is 0-simple and 1-simple.

  (c) Suppose $P$ is $k$-simplicial. Any $h$-face of $P$ is contained is a $k$-face of $P$. Since all faces of a simplex are simplices and $P$ is $k$-simplicial then any $h$-face is also a simplex, i.e., $P$ is $h$-simplicial. The statement for $k$-simple follows from (a) by going through the polar as in (b).

  (d) If both $F$ and $P/F$ are simplexes then both lattices $L(F)$ and $L(P/F)$ are boolean and hence so is $L(P)$. Thus $P$ must be a simplex.

(e) By part (c), we can assume $k_1 + k_2 = d + 1$ and $k_1 \geq k_2$. Let $F$ be a $(k_2 - 1)$-face of $P$. $F$ is a simplex because $P$ is $k_2$-simplicial.

Now consider the quotient polytope $P/F$. This is a $\dim(P) - \dim(F) - 1 = d - (k_2 - 1) - 1 = (k_1 - 1)$-polytope. Furthermore, $L(P/F)$ is isomorphic to the opposite of $L(F^\circ)$. Since $P^\Delta$ is $k_1$-simplicial (part (a)), $F^\circ$ is a simplex and thus so is $P/F$. From part (d) we conclude that $P$ is a simplex.

(f) The cyclic polytope $C(n, d)$ is $(d-1)$-simplicial and 1-simple, but it is *not* isomorphic to a simplex for $n > d + 1$.

- (Graph Theory – Balas)

  $G(k)$ belongs to the class if it is the complement of $kK_m$ ($k$ copies of the complete graph on $m$ vertices). The number of maximal cliques in $G(k)$ is $k^{\frac{|V|}{k}}$, while the number of edges is $(k^2/2)(|V|/k)((|V|/k) - 1)$.

  The smallest number of vertices for which $G(k)$ has more maximal cliques than edges is $n_0 = 12$, attained for $k = 2, 3$ and 4. Indeed, for $|V| = 12$, $G(2)$ has $2^6 = 64$ maximal cliques and 60 edges, $G(3)$ has $3^4 = 81$ maximal cliques and 54 edges, and $G(4)$ has $4^4 = 64$ maximal cliques and 48 edges.

- (Networks and Matchings – Balas)

  1. The algorithm duplicates arcs as needed to obtain $\deg^-(v) = \deg^+(v)$ for all $v \in N$, then finds a directed Euler tour in the resulting directed Eulerian multidigraph. To obtain the degree conditions, change $G$ into a network $G^*$ with supplies and demands equal to $\deg^-(v) - \deg^+(v)$, $v \in V$, and find a minimum cost integer feasible flow $x$ in $G^*$. Then $x_{ij}$ is the number of copies of arc $(i, j)$ that have to be added to $G$ in order to obtain the desired multigraph.
  2. For each undirected arc, both directions have to be considered separately; i.e. the complexity of the procedure becomes exponential in $k$.

- (Integer Programming – Balas)

  By complementing all the variables one obtains a knapsack inequality, and using known lifting procedures one obtains a number of lifted cover inequalities. A known necessary and sufficient condition tells which ones are facets.

- (Advanced Integer Programming – Balas)

  The extreme rays of the projection cone are the incidence vectors of directed cycles, and so the projection of (2) onto the $x$-space yields

$$\sum_{(i,j)\in C} x_{ij} \leq \frac{n-1}{n}|C| \text{ for all } C \in \mathcal{C}, \qquad (2')$$

where $C$ is the collection of directed cycles not involving node 1. Clearly, the inequalities (2′) are dominated by (1).

# OR Ph. D. Qualifying Examination
# Part II

Graduate School of Industrial Administration
Carnegie Mellon University
Pittsburgh, PA 15213

January 2001

Jan 19

1 – 5 pm

Room 227

Answer all four questions.

1. (Nonlinear programming)

   The goal of this question is to use the KKT conditions to solve (recursively) the problem

$$\text{(P)}\quad \begin{array}{ll} \min & f(x) := \sum_{j=1}^{n}(c_j x_j + K_j/x_j) \\ \text{s.t.} & 0 < x_1 \leq x_2 \leq \cdots \leq x_n, \end{array}$$

   where $c_j$ and $K_j$ are positive for each $j$. It is easy to see that (P) has an optimal solution. (You *do not* need to prove this.)

   (a) (2pts) Let (UP) be the "unconstrained" problem of minimizing $f(x)$ subject only to each $x_j$ being positive. Find the optimal solution $x^U$ of (UP).

   (b) (2pt) Write down the KKT conditions for (P).

   (c) (1pt) Suppose that $x_j^U \geq x_{j-1}^U$ for $j = 2, \ldots, n$. What is the optimal solution of (P)?

   (d) (3pts) Now suppose that $x_j^U < x_{j-1}^U$ for some $j = 2, \ldots, n$. Prove that any optimal solution $x^*$ of (P) satisfies $x_j^* = x_{j-1}^*$.

   (e) (2pts) Now describe a method to solve (P) recursively.

2. (Stochastic processes)

   There follows two definitions of a stochastic process $\{Z_n : n \geq 0\}$.

   (a) $\{Z_n : n \geq 0\}$ defined on the positive integers: At any time $n$ the process moves an amount $X_n$ where $X_n$ are a family of iid integral random variables that are also independent of $Z_n$ with $\mathrm{E}[X] < 0$ and $\sigma_X^2 < \infty$. If the process would drop equal to or below 0 for any $n$, it is "bounced" back a strictly positive iid random distance $B_n$ onto the positive integers, *immediately*. (Assume $B_n$ is non-lattice, independent of and $X_n$ and $Z_n$, and $\mathrm{E}[B^2] < \infty$). So the process *never* is less than or equal to zero.

   For example, if $Z_n = 4$ and $X_n = -4$, then the process would be brought to 0. In this case it is bounced out ($B_n = 26$ say), so that $Z_{n+1} = 26$. Similarly, if $Z_n = 4$ and $X_n = -6$, then the process would be brought to $-2$. In this case it is bounced out ($B_n = 26$ say), so that $Z_{n+1} = 26$.

   i. Prove that $Z_n$ converges to a stationary distribution, $Z$, as $n \to \infty$.

   ii. Give an expression for $\mathrm{P}(Z = k)$, for any positive integer $k$, in terms of time averages.

   iii. If the long run average probability that $Z_n = k$ is $\alpha$, the long run probability that $X_n \leq -Z_n$ is $\beta$, and the probability that $B_n = k$ is $\gamma$, what is the probability that the (stationary) process was bounced last period, given that it is equal to $k$ this period?

# Solutions to OR Ph. D. Qualifying Examination Part II

Graduate School of Industrial Administration
Carnegie Mellon University
Pittsburgh, PA 15213

January 2001

1. (Nonlinear programming – Javier Pena)

   (a) $f(x)$ is convex and $C^1$ on $\mathbf{R}^n_{++}$ so the minimum is attained where its gradient vanishes, i.e., where

   $$\frac{\partial f(x)}{\partial x_j} = c_j - \frac{K_j}{x_j^2} = 0.$$

   That is, at the point $x^U$ defined componentwise as

   $$x_j^U = \sqrt{K_j/c_j} \text{ for } j = 1, \ldots, n.$$

   (b) We shall include only the constraints $x_{j-1} \leq x_j$ for $j = 2, \ldots, n$ as the nonnegativity constraint must be satisfied strictly. Rewrite these constraints as

   $$g(x) = (x_1 - x_2,\ x_2 - x_3,\ \ldots,\ x_{n-1} - x_n)^T \leq 0.$$

   The KKT conditions for (P) are $\nabla f(x) + \nabla g(x)u = 0$, $u^T g(x) = 0$, which correspond to

   $$c_1 - \frac{K_1}{x_1^2} = u_1,$$

   and

   $$c_j - \frac{K_j}{x_j^2} = u_{j-1} - u_j,\ u_{j-1}(x_{j-1} - x_j) = 0 \text{ for } j = 2, \ldots, n.$$

   (c) In such case $x^U$ is feasible for (P) and consequently it is an optimal solution to (P).

(d) Proceed by contradiction: Because $x^*$ is feasible, $x_j^* \geq x_{j-1}^*$. If $x_j^* > x_{j-1}^*$ then by part (c) $u_{j-1}^* = 0$, and also

$$c_{j-1} - \frac{K_{j-1}}{(x_{j-1}^*)^2} = u_{j-2}^* - u_{j-1}^*,$$

and

$$c_j - \frac{K_j}{(x_j^*)^2} = u_{j-1}^* - u_j^*.$$

(Define $u_0^* = 0$.)
Since $u_{j-1}^* = 0$,

$$c_{j-1} - \frac{K_{j-1}}{(x_{j-1}^*)^2} = u_{j-2}^* \geq 0 \Rightarrow (x_{j-1}^*)^2 \geq \frac{K_{j-1}}{c_{j-1}},$$

and

$$c_j - \frac{K_j}{(x_j^*)^2} = -u_j^* \leq 0 \Rightarrow (x_{j-1}^*)^2 \leq \frac{K_j}{c_j}.$$

But $x_j^U < x_{j-1}^U \Rightarrow \frac{K_j}{c_j} < \frac{K_{j-1}}{c_{j-1}}$. Therefore,

$$(x_{j-1}^*)^2 \geq \frac{K_{j-1}}{c_{j-1}} > \frac{K_j}{c_j} \geq (x_j^*)^2$$

which contradicts $x_j^* > x_{j-1}^*$.

(e) Proceed inductively as follows: Solve the unconstrained problem. If $x_j^U \geq x_{j-1}^U$ for $j = 2, \ldots, n$ then we are done. Otherwise, choose some $j = 2, \ldots, n$ such that $x_j^U < x_{j-1}^U$. Get the solution by solving the smaller problem

$$(\text{P'}) \quad \begin{array}{ll} \min & \overline{f}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \\ \text{s.t.} & 0 < x_1 \ldots x_{j-1} \leq x_{j+1} \leq \cdots \leq x_n, \end{array}$$

where

$$\begin{aligned} \overline{f}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) &= \textstyle\sum_{i=1}^{j-2}(c_i x_i + K_i/x_i) \\ &\quad + (c_{j-1} + c_j)x_{j-1} + (K_{j-1} + K_j)/x_{j-1} \\ &\quad + \textstyle\sum_{i=j+1}^{n}(c_i x_i + K_i/x_i) \end{aligned}$$

2. (Stochastic Processes – Alan Scheller-Wolf)

(a) i. As $Z_n$ can easily be shown to be a Markov chain on the integers, proving the existence of a stationary distribution can be achieved by showing the existence of some state $s$, which is positive recurrent. Given the characteristics of

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# PhD Qualifying Exam

OR and ACO students

January 10, 2002
8:00am to 1:00pm: 5 hours

Answer six questions: Questions 1-4 and two of Questions 5-7

# Question 1: linear programming

Given an $m \times n$ matrix $A \in \mathbb{R}^{m \times n}$. Let

$$v(A) := \min_{x \geq 0, e_n^T x = 1} \left( \max_{i=1,\dots,m} (Ax)_i \right),$$

where $e_n = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T$ is the vector of all one's in $\mathbb{R}^n$. In other words, $v(A)$ minimizes the largest entry of $Ax$ over all $x$ such that $x \geq 0$, $\sum_{j=1}^n x_j = 1$.

(a) (3 points) Using *only* $A$ and suitable constant vectors in $\mathbb{R}^n$, $\mathbb{R}^m$, write a linear program whose optimal value is $v(A)$. You must succinctly justify that the optimal value of your linear program is indeed attained and equals $v(A)$.

(b) (2 points) Write down the dual of your linear program in (a).

(c) (3 points) Prove that

$$v(A) = v^*(A^T) := \max_{y \geq 0, e_m^T y = 1} \left( \min_{j=1,\dots,n} (A^T y)_j \right),$$

where $e_m = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T$ is the vector of all one's in $\mathbb{R}^m$.

(d) (2 points) Prove *von Neumann's Minimax Theorem:*

$$\min_{x \geq 0, e_n^T x = 1} \left( \max_{y \geq 0, e_m^T y = 1} (y^T A x) \right) = \max_{y \geq 0, e_m^T y = 1} \left( \min_{x \geq 0, e_n^T x = 1} (y^T A x) \right).$$

# Question 2: graph theory

Let $G = (V, E)$ be an undirected graph.

1. Describe a method for finding a cutset $(S, V \setminus S)$ in $G$ such that $\min\{w_e : e \in (S, V \setminus S)\}$ is maximized over all cutsets of $G$ and outline a proof of its correctness.

2. Describe a method for finding a cutset $(S, V \setminus S)$ in $G$ such that $\max\{w_e : e \in E \setminus (S, V \setminus S)\}$ is minimized over all cutsets of $G$ and outline a proof of its correctness.

# Question 3: networks and matchings

Let $G = (V, E)$ be an undirected graph and for $i = 1, 2$, let $K_i$ be the vertex set of a clique in $G$. Describe an algorithm for finding a maximum clique in $G(K_1 \cup K_2)$, the subgraph of $G$ induced by $K_1 \cup K_2$.
[Hint: look at $\bar{G}$.]

# Question 4: integer programming

Let $F_j$ be the set of vectors $x \in R^n$ such that $j$ components of $x$ are $\frac{1}{2}$ and the remaining $n-j$ components are equal to 0 or 1. For example in $R^2$, the set $F_1$ contains the four vectors $(\frac{1}{2}, 0), (\frac{1}{2}, 1), (0, \frac{1}{2}), (1, \frac{1}{2})$.

(a) Given $j \in \{1, \ldots, n-1\}$, consider any $\pi \in Z^n$, $\pi_0 \in Z$ such that $\pi x < \pi_0$ for every $x \in F_j$ (strict inequality). Show that $\pi x \leq \pi_0$ for every $x \in F_{j+1}$. Hint: Let $v \in F_{j+1}$ and consider the following two cases:

(a1) (2 points) First prove the result when $\pi v$ is an integer.

(a2) (3 points) Assume now that $\pi v \notin Z$. Without loss of generality, assume that $v_1 = \frac{1}{2}$ and $\pi_1 \neq 0$ and let $v^1, v^2 \in F_j$ be equal to $v$ except for the first component which is 0 or 1 respectively. Show that, for $i = 1, 2$, each of the line segments $[v, v^i]$ contains a point $\tilde{v}^i$ such that $\pi \tilde{v}^i \in Z$. Deduce that $\pi v \leq \pi_0$.

(b) For any $n \geq 1$, let

$$P = \{x \in R^n : \; 0 \leq x \leq 1, \; \sum_{j \in J} x_j + \sum_{j \notin J} (1 - x_j) \geq \tfrac{1}{2} \text{ for all } J \subseteq \{1, 2, \ldots, n\}.$$

(b1) (2 points) Show that $P \cap Z^n = \emptyset$ and $F_1 \subset P$.

(b2) (3 points) Deduce from (a) that the Chvátal rank of $P$ is at least $n$.

# Question 5: advanced integer programming

Let $M = (S, \mathcal{I})$ be a matroid. Define $\mathcal{I}^* = \{J \subseteq S : r(S - J) = r(S)\}$. Then $M^* = (S, \mathcal{I}^*)$ is a matroid (you are not asked to prove this).

(a) (5 points) Show $(M^*)^* = M$.

(b) (5 points) Show that the following algorithm finds a maximum-weight basis of $S$. Begin with $J = S$. While $J \notin \mathcal{I}$, find $e \in J$ with $r(J - \{e\}) = r(S)$ and such that $c_e$ is minimum, and replace $J$ by $J - \{e\}$.

# Question 6: convex polyhedra

Consider a polyhedron $P := \{x \in \mathbb{R}^n : Ax \geq b\} \neq \emptyset$. The *dominant* of $P$ is defined as $P^+ := \{y \in \mathbb{R}^n : y \geq x \text{ for some } x \in P\}$.

1. Show that $P^+ = P + \mathbb{R}^n_+$. What is the dimension of $P^+$ if (a) $\dim P = n$, (b) $\dim P = n - 1$?

2. Express $P^+$ in terms of $y$ only.

   [Hint: replace "for some $x \in P$" in the definition of $P^+$ by the system defining $P$, and project onto the $y$-space.]

# Question 7: convex analysis

Let $E$, $Y$ be Euclidean spaces, $f : E \to (-\infty, +\infty]$ be a convex function, and $A : E \to Y$ be a linear map.

For any given $y \in Y$ consider the problem

$$\text{(P)} \quad \inf\{f(x) : x \in E,\ Ax = y\}.$$

(a) (2 points) Show that the Fenchel dual of (P) is

$$\text{(D)} \quad \sup\{\langle y, \phi\rangle - f^*(A^*\phi) : \phi \in Y\}.$$

(b) (2 points) Define the *value* function $v : Y \to [-\infty, +\infty]$ as

$$v(y) := \inf\{f(x) : x \in E,\ Ax = y\}.$$

Prove that $v$ is convex.

(c) (3 points) Suppose $\bar{y} \in Y$ is such that both (P) and (D) attain their optimal values at $\bar{x} \in E$ and $\bar{\phi} \in Y$ respectively. Furthermore, assume that these optimal values are the same.

Prove that

$$\bar{\phi} \in \partial v(\bar{y}).$$

(d) (3 points) Suppose $\bar{y} \in Y$, $\bar{x} \in E$, and $\bar{\phi} \in Y$ are such that the conditions in (c) hold. Prove that

$$A^*\bar{\phi} \in \partial f(\bar{x}).$$

# PhD Qualifying Exam

OR students

January 12, 2002
9:00am to 1:00pm: 4 hours

Answer all three questions

# Question on nonlinear programming

(a) (2 points) Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ be given. Assume the sets

$$\{x : Ax = b, x > 0\}, \ \{(y,s) : A^T y + s = c, s > 0\}$$

are nonempty.

Relying on the existence of the primal-dual central path for linear programs, prove that at least one of these two sets must be unbounded (that is, it contains vectors of arbitrarily large norm).

Questions (b) through (d) below refer to the following situation: Suppose $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ are such that the system

$$Ax = b, \ x \geq 0$$

does *not* have a solution. In the statements below $\|\cdot\|$ denotes the Euclidean 2-norm, that is, $\|u\| = \sqrt{u^T u}$.

(b) (3 points) Consider the problem

$$\text{(P)} \quad \min\{\|Ax - b\| : x \geq 0\}.$$

(The solution to this is, in some sense, the "closest" we can satisfy the system of constraints.)

Show that (P) can be cast as a conic optimization problem using the cone $\mathbb{R}^n_+ \times K$ for a suitable second-order cone $K$. Your formulation can *only* use $A, b$ and constant vectors.

(c) (2 points) Write down the dual of your conic program in (b). Does strong duality hold? Justify your answer.

(d) (3 points) Let

$$\alpha := \min\{\|Ax - b\| : x \geq 0\}$$

and

$$\beta := \min\{\|y\| : A^T y \leq 0, \ b^T y = 1\}$$

Prove that $\alpha\beta = 1$. You may assume that both minima defining $\alpha$ and $\beta$ are attained.

SOLUTIONS QUAL 2002 PART I

# Solution to question on linear programming

(a) Consider the following linear program

$$\begin{array}{ll} \min & t \\ \text{s.t.} & Ax - te_m \leq 0 \\ & e_n^T x = 1 \\ & x \geq 0. \end{array}$$

This linear program is clearly feasible: pick $x$ as any unitary vector and $t$ sufficiently large. It is also bounded because $\{Ax : x \geq 0,\ e_n^T x = 1\}$ is the image under a linear map of a bounded set, and hence bounded. Therefore the optimal is attained. At the optimal solution necessarily $t = \max_{i=1,\dots,m}(Ax)_i$. Consequently the minimum yields $v(A)$.

(b) The dual is

$$\begin{array}{ll} \max & \tau \\ \text{s.t.} & A^T v + \tau e_n \leq 0 \\ & -e_m^T v = 1 \\ & v \leq 0. \end{array}$$

(c) Like in (a), we can rewrite $v^*(A^T)$ as

$$\begin{array}{ll} \max & \tau \\ \text{s.t.} & A^T y - \tau e_n \geq 0 \\ & e_m^T y = 1 \\ & y \geq 0, \end{array}$$

which is (replacing $v$ by $-y$) precisely the dual problem in (b). Strong duality then implies that $v(A) = v^*(A^T)$.

(d) Given any $x \in \mathbb{R}^n$, notice that

$$\max_{y \geq 0, e_m^T y = 1} (y^T A x) = \max_{i=1,\dots,m} (Ax)_i.$$

Likewise, given any $y \in \mathbb{R}^m$, we have

$$\min_{x \geq 0, e_n^T x = 1} (y^T A x) = \min_{j=1,\dots,n} (A^T y)_j.$$

Therefore,

$$\begin{aligned} \min_{x\geq 0, e_n^T x=1}\left(\max_{y\geq 0, e_m^T y = 1}(y^T A x)\right) &= \min_{x\geq 0, e_n^T x=1}\left(\max_{i=1,\dots,m}(Ax)_i\right) \\ &= v(A) \\ &= v^*(A^T) \\ &= \max_{y\geq 0, e_m^T y=1}\left(\min_{j=1,\dots,n}(A^T y)_j\right) \\ &= \max_{y\geq 0, e_m^T y=1}\left(\min_{x\geq 0, e_n^T x=1}(y^T A x)\right). \end{aligned}$$

Answer to Question 2 (Graph Theory)

1. Find a minimum-weight spanning tree $T$ in $G$ by Kruskal's or Prim's greedy algorithm, and let $(i_*; j_*)$ be a maximum-weight edge of $T$. Since $T$ is minimum-weight, $(i_*, j_*)$ is a minimum-weight edge of the cutset $(S^*, V \setminus S^*)$ defined by the pair $(T, (i_*, j_*))$. Since every cutset intersects $T$ in at least one edge and $(i_*, j_*)$ is a maximum-weight edge of $T$, $(i_*, j_*)$ maximizes the minimum edge-weight of a cutset among all cutsets of $G$.

2. Find a maximum-weight spanning tree $T^*$ in $G$, and let $[S, T]$ be the bipartition of $V$ induced by $T^*$. Then the cutset $(S, T)$ minimizes the maximum edge-weight among all edges not in the cutset. For suppose $(k, \ell)$ is the heaviest edge not in $(S, T)$, and let $C$ be the cycle formed by adding $(k, \ell)$ to $T^*$, then

$$w_{ij} \geq w_{k\ell} \text{ for all } (i, j) \in C,$$

since $T^*$ is a maximum-weight spanning tree. Since $k$ and $\ell$ are both in $S$ or both in $T$, the path in $T$ joining $k$ to $\ell$ is even, hence $C$ is an odd-length cycle. But then any bipartition $(R, V \setminus R)$ of $V$ must contain both endpoints of at least one edge of $C$ either in $R$, or in $V \setminus R$, hence there exists no better partition than $[S, T]$.

**Answer to Question 3** (Networks and Matchings)

W.l.o.g., assume $|K_1| \leq |K_2|$. A maximum clique must contain $K_1 \cap K_2$, plus the vertex set of a maximum clique in $G(K_1' \cup K_2')$, where $K_1' = K_1 \setminus K_2$ and $K_2' = K_2 \setminus K_1$. Now a maximum clique in $G(K_1' \cup K_2')$ is a maximum stable set in $\overline{G(K_1' \cup K_2')}$, which is bipartite. The following algorithm finds such a set:

1. Find a maximum matching $M$ in $\overline{G(K_1' \cup K_2')}$. If $M$ covers all vertices in $K_1'$, stop: $K_2'$ is a maximum stable set. Otherwise

2. Starting from the vertices of $K_1'$ covered by $M$, label all vertices reachable via alternating paths. If $L$ is the set of labeled vertices, then $(K_2' \setminus L) \cup (K_1' \cap L)$ is a maximum stable set.

Let $K''$ be the maximum stable set in $\overline{G(K_1' \cup K_2')}$ found by the algorithm. Then the vertex set $(K_1' \cap K_2') \cup K''$ induces a maximum clique in $G(K_1 \cup K_2)$.

# Solution to question on integer programming

(a1) Any vector $v \in F_{j+1}$ is the convex combination of two vectors $v^1, v^2 \in F_j$. Since $\pi v^1 < \pi_0$ and $\pi v^2 < \pi_0$, it follows that $\pi v < \pi_0$. Now $\pi v$ integral implies $\pi v \leq \pi_0$.

(a2) Assume now $\pi v \notin Z$. Then $\pi v = k + \frac{1}{2}$ for some $k \in Z$ and there exists a component, say the first, where $v_1 = \frac{1}{2}$ and $\pi_1 \neq 0$. Let $v^1, v^2 \in F_j$ be two vectors equal to $v$ except for the first component which is 0 or 1 respectively. Since $\pi_1 \neq 0$, $\pi v^1 \leq k$ or $\geq k+1$. Since $\pi x$ is a continuous function which takes the value $k + \frac{1}{2}$ when $x = v$ and a value $\leq k$ or $\geq k+1$ when $x = v^1$, there exists $\bar{v}^1 \in [v, v^1]$ such that $\pi \bar{v}^1$ is integer. Similarly, there exists $\bar{v}^2 \in [v, v^2]$ such that $\pi \bar{v}^2$ is integer. Since $\pi v^1 < \pi_0$ and $\pi v^2 < \pi_0$, every $x \in [v^1, v^2]$ satisfies $\pi x < \pi_0$. This implies $\pi \bar{v}^1 \leq \pi_0$ and $\pi \bar{v}^2 \leq \pi_0$. Thus every $x \in [\bar{v}^1, \bar{v}^2]$ satisfies $\pi x \leq \pi_0$. In particular $\pi v \leq \pi_0$.

(b1) Consider any 0,1 vector $x$ and let $J = \{j : x_j = 0\}$. Then $\sum_{j \in J} x_j + \sum_{j \notin J}(1 - x_j) = 0$. So $x \notin P$.

Consider any $x \in F_1$. Since $x$ has one component equal to $\frac{1}{2}$, the quantity $\sum_{j \in J} x_j + \sum_{j \notin J}(1 - x_j)$ is greater than or equal to $\frac{1}{2}$ for every $J$.

(b2) Every Chvátal cut for $P$ can be written in the form $\pi x \leq \pi_0$, where $\pi \in Z^n$, $\pi_0 \in Z$ and $\pi x < \pi_0$ for every $x \in P$. Since $F_1 \subset P$, it follows from (a) that $F_2$ is contained in the first Chvátal closure of $P$. By induction, (a) implies that $F_n$ is contained in the $(n-1)^{st}$ Chvátal closure of $P$. Since $F_n \neq \emptyset$ and $P \cap Z^n = \emptyset$, at least one more iteration of the Chvátal procedure is needed.

# Advanced Integer Programming

Let $M = (S, \mathcal{I})$ be a matroid. Define $\mathcal{I}^* = \{J \subseteq S : r(S - J) = r(S)\}$. Then $M^* = (S, \mathcal{I}^*)$ is a matroid (you are not asked to prove this).

(a) (5 points) Show $(M^*)^* = M$.

**Claim 0.1** *Let $r$ be the rank function of $M$. The rank function of $M^*$ is $r^*(A) = |A| + r(S - A) - r(S)$.*

**Proof:** Let $J \subseteq A$ be maximal among sets in $\mathcal{I}^*$ contained in $A$. Thus $r^*(A) = r^*(J) = |J|$. Proof is by induction on $|A - J|$. If $A \in \mathcal{I}^*$, then $r(S-A) = r(S)$ so that $r^*(A) = |A| = |A| + r(S-A) - r(S)$. Now suppose statement holds for all $A'$ such that $|A' - J'| < k$ for $J'$ maximal among independent sets in $A'$ and that $|A - J| = k$. Then $r(S - A) < r(S - J)$ and there must be an $e \in A - J$ such that $r(S - A + \{e\}) = r(S - A) + 1$. By induction, $r^*(A - \{e\}) = |A - \{e\}| + r(S - A + \{e\}) - r(S) = |J|$, since $J \subseteq A - \{e\}$ and is maximal among independent sets in $A - \{e\}$. Thus we have that $|J| = |A| - 1 + r(S - A) + 1 - r(S) = |A| + r(S - A) - r(S)$. ∎

$$
\begin{aligned}
(\mathcal{I}^*)^* &= \{I \subseteq S : \ r^*(S - I) = r^*(S)\} \\
&= \{I \subseteq S : \ |S - I| + r(I) - r(S) = |S| + r(\emptyset) - r(S)\} \\
&= \{I \subseteq S : \ r(I) = |S| - |S - I| = |I|\} \\
&= \mathcal{I}
\end{aligned}
$$

∎

The matroid $M^*$ is called the *dual* matroid of $M$.

(b) (5 points) Show that the following algorithm finds a maximum-weight basis of $S$. Begin with $J = S$. While $J \notin \mathcal{I}$, find $e \in J$ with $r(J - \{e\}) = r(S)$ and such that $c_e$ is minimum, and replace $J$ by $J - \{e\}$.

**Claim 0.2** *$A$ is a basis of $M$ if and only if $S - A$ is a basis of $M^*$.*

**Proof:**

$$
\begin{aligned}
A \text{ a basis } &\leftrightarrow \ r(A) = r(S) \text{ and } r(A - \{e\}) < r(S), \ \forall e \in A \\
&\leftrightarrow \ S - A \in \mathcal{I}^* \text{ and } S - A + e \notin \mathcal{I}^*, \ \forall e \in A \\
&\leftrightarrow \ S - A \text{ is a basis of } M^*
\end{aligned}
$$

∎

**Claim 0.3** *$A$ a maximum (minimum) weight basis of $M$ if and only if $S - A$ is a minimum (maximum) weight basis of $M^*$.*

**Proof:** Follows from Claim 2 and fact that $c(S) = c(S - A) + c(A)$ for all $A \subset S$. ■

If apply the greedy algorithm to $M^*$ with weights $w_e = c(S) - c_e$ then greedy finds a maximum $w$-weight basis of $M^*$, which is a minimum $c$-weight basis of $M^*$. The greedy algorithm applied to $M^*$ is precisely the algorithm stated in (b).

Answer to Question 6 (Convex Polyhedra)

1. $x \in P^+ \Rightarrow \exists x' \in P$ with $x \geq x'$; hence $x = x' + (x - x')$, where $x - x' \geq 0$, i.e. $x \in P + \mathbb{R}^n_+$. Conversely, $x \in P + \mathbb{R}^n_+ \Rightarrow \exists x' \in P : x = x' + a$ for $a \in \mathbb{R}^n_+$, i.e. $\exists x' \in P$ with $x \geq x'$. The dimension of $P^+$ is $n$ in both cases.
2. $P^+ = \{y \in \mathbb{R}^n : y - x \geq 0,\ Ax \geq b\}$. Let $A$ be $m \times n$. The projection cone is $W = \{(u, v) \in \mathbb{R}^{n+m} : u + vA = 0, (u, v) \geq 0\}$ hence

$$P^+ := \{y \in \mathbb{R}^n : -uy \geq vb, \text{ for all } (u, v) \geq 0 \text{ such that } u = vA\}$$
$$= \{y \in \mathbb{R}^n : vA \geq vb \text{ for all } v \geq 0\}.$$

Wrong.



max 0
$-Ax \leq -b$
$x - y \leq 0$

→ max 0
$-Ax \leq -b$
$Ix \leq Iy$

dual
min $-vb + uy$
$-vA + uI = 0$
$v, u \geq 0$

Proj cone $W = \{(v, u) \in \mathbb{R}^{m+n} : -vA + uI = 0,\ v, u \geq 0\}$

$\boxed{-vb + uy \geq 0}$ for $u, v$ in $W$

# Solution to question on convex analysis

(a) Rewrite (P) as

$$\text{(P)} \quad \inf\{f(x) + g(Ax) : x \in E\}$$

where $g(u) = \delta_{\{y\}}(u)$ the indicator function of the set $\{y\}$.
The Fenchel dual problem is, by definition,

$$\text{(D)} \quad \sup\{g^*(\phi) - f^*(A^*\phi) : \phi \in Y\}.$$

But

$$g^*(\phi) = \sup\{\langle v, \phi\rangle - g(v) : v \in Y\} = \langle y, \phi\rangle.$$

Thus we can rewrite (D) as

$$\text{(D)} \quad \sup\{\langle y, \phi\rangle - f^*(A^*\phi) : \phi \in Y\}.$$

(b) Let $y_1, y_2 \in Y$ and $\lambda \in (0,1)$ be given. We want to show

$$v(\lambda y_1 + (1-\lambda)y_2) \leq \lambda v(y_1) + (1-\lambda)v(y_2).$$

Indeed, for any $x_1, x_2$ such that $Ax_1 = y_1, Ax_2 = y_2$ we have

$$A(\lambda x_1 + (1-\lambda)x_2) = \lambda y_1 + (1-\lambda)y_2,$$

and so

$$v(\lambda y_1 + (1-\lambda)y_2) \leq f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2).$$

(The last inequality because $f$ is convex.)
Since this holds for any $x_1, x_2$ such that $Ax_1 = y_1, Ax_2 = y_2$, we can take inf in the right hand side and conclude that

$$v(\lambda y_1 + (1-\lambda)y_2) \leq \lambda v(y_1) + (1-\lambda)v(y_2).$$

(c) We need to show that for any $y \in Y$ the following inequality holds

$$\langle \bar{\phi}, y - \bar{y}\rangle \leq v(y) - v(\bar{y}).$$

Indeed, weak duality implies that for any $y \in Y$

$$\begin{array}{rcl} v(y) & \geq & \sup\{\langle y, \phi\rangle - f^*(A^*\phi) : \phi \in Y\} \\ & \geq & \langle y, \bar{\phi}\rangle - f^*(A^*\bar{\phi}) \\ & = & \langle \bar{\phi}, y - \bar{y}\rangle + \langle \bar{y}, \bar{\phi}\rangle - f^*(A^*\bar{\phi}) \\ & = & \langle \bar{\phi}, y - \bar{y}\rangle + v(\bar{y}). \end{array}$$

Thus

$$v(y) - v(\bar{y}) \geq \langle \bar{\phi}, y - \bar{y}\rangle$$

as we wanted.

(d) We must have $A\bar{x} = \bar{y}$ and

$$\langle \bar{\phi}, \bar{y} \rangle - f^*(A^*\bar{\phi}) = f(\bar{x}).$$

Thus

$$f(\bar{x}) + f^*(A^*\bar{\phi}) = \langle \bar{\phi}, \bar{y} \rangle = \langle \bar{\phi}, A\bar{x} \rangle = \langle \bar{x}, A^*\bar{\phi} \rangle$$

But by the Fenchel-Young inequality

$$f(\bar{x}) + f^*(A^*\bar{\phi}) \geq \langle \bar{x}, A^*\bar{\phi} \rangle$$

with equality if and only if $A^*\bar{\phi} \in \partial f(\bar{x})$. Since the equality indeed holds, we must have

$$A^*\bar{\phi} \in \partial f(\bar{x}).$$

SOLUTIONS QUAL 2002 PART II

## Solution to question on nonlinear programming

(a) By the existence theorem for the central path, we know that for all $\mu > 0$ the following nonlinear system of equations has a solution $(x(\mu), y(\mu), s(\mu))$.

$$\begin{aligned}
&Ax = b\\
&A^T y + s = c\\
&x_i s_i = \mu,\ i = 1, \dots, n\\
&x, s > 0.
\end{aligned}$$

Since we can choose $\mu > 0$ arbitrarily large, at least one of the sets

$$\{x(\mu) : \mu > 0\} \quad\text{or}\quad \{(y(\mu), s(\mu)) : \mu > 0\}$$

must be unbounded (otherwise the condition $x_i s_i = \mu$ could not hold). Since these sets are contained in

$$\{x : Ax = b, x > 0\} \quad\text{and}\quad \{(y, s) : A^T y + s = c, s > 0\}$$

respectively, it follows that at least one of the latter two must be unbounded.

(b) Introducing new variables $z, t$, we can rewrite (P) as

$$(\mathrm{P}) \qquad \begin{array}{ll}
\min & t\\
\text{s.t.} & Ax - z = b\\
& x \geq 0\\
& \|z\| \leq t.
\end{array}$$

That is,

$$(\mathrm{P}) \qquad \begin{array}{ll}
\min & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \\ t \end{bmatrix}\\
\text{s.t.} & \begin{bmatrix} A & -I & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ t \end{bmatrix} = b\\
& \begin{bmatrix} x \\ z \\ t \end{bmatrix} \in \mathbb{R}^n_+ \times K.
\end{array}$$

where $K$ is the second-order cone in $\mathbb{R}^{m+1}$.

(c) The dual of the last problem is

$$(\mathrm{D}) \qquad \begin{array}{ll}
\max & b^T y\\
\text{s.t.} & \begin{bmatrix} A^T \\ -I \\ 0 \end{bmatrix} y + \begin{bmatrix} s \\ w \\ \tau \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\\
& \begin{bmatrix} s \\ w \\ \tau \end{bmatrix} \in \mathbb{R}^n_+ \times K.
\end{array}$$

Strong duality holds because the Slater condition holds for (P): take $\bar{x} > 0$, $\bar{z} = A\bar{x} - b$, and $\bar{t} > \|\bar{z}\|$. Thus

$$\begin{bmatrix} A & -I & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{z} \\ \bar{t} \end{bmatrix} = b$$

with

$$\begin{bmatrix} \bar{x} \\ \bar{z} \\ \bar{t} \end{bmatrix} \in \operatorname{int}(\mathbb{R}^n_+ \times K).$$

(d) We can rewrite the dual as

$$\begin{array}{lll} & \max & b^T y \\ \text{(D)} & \text{s.t.} & A^T y \leq 0 \\ & & \|y\| \leq 1. \end{array}$$

Let $p^*$ and $d^*$ be the optimal values of (P) and (D) respectively. By strong duality $p^* = d^*$.

Furthermore, observe that $d^* = \frac{1}{\beta}$ and $p^* = \alpha$, so

$$\alpha = \frac{1}{\beta}, \text{ i.e., } \alpha\beta = 1.$$

# Questions for the Qualifier of January 2004

## Wednesday, Jan 7 2004

- **ACO Students - 4 hours:** You must answer Questions 1,2,3, and one of 4,5,6 or 7.
- **OM Students - 2 hours for this part:** You must answer Questions 1 and 2 only.
- **OR Students - 4 hours:** You must answer Questions 1,2,3, and one of 4,5 or 6.
- All questions are open-notes open-book.
- Please begin each question on a new page. Write only on one side of every page. Always remember to put your identification number (NOT your name) in the top right-hand corner of every page you turn in.

1. (Linear Programming)

You wish to dig a pit to extract minerals from the ground. You can excavate at $n$ different levels. At each level $j$ you will remove a cylinder of earth with diameter $x_j$, where $0 \le x_j \le 1$. No cylinder can have a larger diameter than those above it ($x_1 \ge x_2 \ge \cdots \ge x_n$). The net value of excavating level $j$ is $v_j x_j$, where possibly $v_j < 0$. The objective to decide how much to dig at each level so as to maximize net value.

The problem can be written as a linear programming problem

$$
\begin{array}{lll}
\max & \displaystyle\sum_{j=1}^{n} v_j x_j & \\
\text{subject to} & -x_j + x_{j+1} \le 0, \quad j = 1, \ldots, n-1 & (\lambda_j) \\
 & 0 \le x_j \le 1, \quad j = 1, \ldots, n & (\mu_j)
\end{array}
$$

(a) State a simple linear-time algorithm that finds an optimal solution. Don't prove optimality at this point, but note that there is always an optimal integral solution.

(b) State the dual problem, using the dual variables $\lambda_j$ and $\mu_j$.

(c) (This is the main question.) State an optimal dual solution and show that it is feasible. Hint: Let $\mu_j = 0$ for $j = 2, \ldots, n$.

(d) Use weak duality to show that your linear-time algorithm obtains an optimal solution.

$\sum \mu_j$

$-\lambda_1 + \mu_1 \ge v_1$

$\lambda_{j-1} - \lambda_j + \mu_j \ge v_j \quad \forall j = 2, \ldots, n-1$

$\lambda_{n-1} + \mu_n \ge v_n$

2. (Integer Programming)

Let $P$ denote the polytope of $R^3$ obtained as the convex hull of the points $(0,0,0)$, $(2,0,0)$, $(0,2,0)$ and $(\frac{1}{2},\frac{1}{2},h)$ with $h > 0$. Consider the mixed integer linear program

$$\max\ y$$
$$(x_1, x_2, y) \in P$$
$$x_1 \in Z, x_2 \in Z, y \in R$$

and let $P_I$ denote the convex hull of its feasible solutions, namely $P_I = conv\{P \cap \{(x_1, x_2, y) \in Z^2 \times R\}\}$.

(a) What is the dimension of $P_I$?

(b) Describe $P_I$ by a system of linear inequalities.

(c) Consider the disjunction $x_1 \leq 0$ or $x_1 \geq 1$. What are the extreme points of $conv\{(P \cap \{(x_1, x_2, y) : x_1 \leq 0\}) \cup (P \cap \{(x_1, x_2, y) : x_1 \geq 1\})\}$?

(d) For $(\pi_0, \pi_1, \pi_2) \in Z^3$, consider the disjunction $\pi_1 x_1 + \pi_2 x_2 \leq \pi_0$ or $\pi_1 x_1 + \pi_2 x_2 \geq \pi_0 + 1$. Show that the point $(\frac{1}{2}, \frac{1}{2}, \frac{h}{10})$ belongs to $P_\pi = conv\{(P \cap \{(x_1, x_2, y) : \pi_1 x_1 + \pi_2 x_2 \leq \pi_0\}) \cup (P \cap \{(x_1, x_2, y) : \pi_1 x_1 + \pi_2 x_2 \geq \pi_0 + 1\})\}$. The *split closure* of $P$, denoted by $P^1$, is the intersection of the polytopes $P_\pi$ over all $\pi = (\pi_0, \pi_1, \pi_2) \in Z^3$. Show that $P^1$ contains the points $(0,0,0)$, $(2,0,0)$, $(0,2,0)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{h}{10})$.

(e) For any integer $k \geq 2$, let $P^k$ denote the split closure of $P^{k-1}$. Show that $P^k$ contains the point $(\frac{1}{2}, \frac{1}{2}, t)$ for some $t > 0$. Conclude that $P^k \neq P_I$.

(f) More generally, for any mixed integer linear program, one can define the $k^{th}$ elementary closure $P^k$ and the convex hull $P_I$ of the feasible solutions.

For a pure integer linear program, is it true that there exists a finite $k$ such that $P^k = P_I$? Support your answer by citing a theorem or by giving a counter example.

For a mixed 0,1 linear program, is it true that there exists a finite $k$ such that $P^k = P_I$? Support your answer by citing a theorem or by giving a counter example.

3. (Advanced Integer Programming)

Consider a 0-1 program

$$\max\{cx : Ax \leq b, x \in \{0,1\}^n\}$$

where part of the constraint set is of the form $e_{Q_i}x = 1$, $i = 1, \ldots, q$, with $\cup_{i=1}^k Q_i = N = \{1, \ldots, n\}$, and $Q_i \cap Q_j = \emptyset$ for all $i, j \in N$.

We wish to generate lift-and-project cuts from disjunctions of the form

$$\begin{pmatrix} Ax & \leq & b \\ e_{Q_{i1}}x & = & 1 \end{pmatrix} \vee \begin{pmatrix} Ax & \leq & b \\ e_{Q_{i2}}x & = & 1 \end{pmatrix}, \qquad (1)$$

for some $i \in \{1, \ldots, q\}$, where $Q_{i1} \cup Q_{i2} = Q_i$, $Q_{i1} \cap Q_{i2} = \emptyset$.

(a) Formulate the cut generating linear program with the standard objective of cutting off the LP optimum $\overline{x}$ by a maximum amount. Use a normalization that guarantees a finite optimum.

(b) What is the connection between cuts obtainable from a disjunction (1), and basic solutions of the cut generating linear program? Define a disjunctive rank in terms of the family of disjunctions of type (1). In terms of your definition, what is the disjunctive rank of $P$, the LP polyhedron associated with the above problem?

5. (Convex Polytopes)

Let $x \in \{0, 1\}^3$, $y \in \{0, 1\}^2$, and let $S$ be the set of pairs $x, y$ satisfying the following condition:

"If $x_i = 1$ for at least 2 components of $x$, then $y_i = 1$ for at least 1 component of $y$."

Give a linear characterization of the convex hull of $S$.

[Hint: Restate the condition as a pair of linear inequalities of which the first implies the second, then restate the implication as a disjunction.]

6. (Advanced Linear Programming)

In your solution to any question you may use any of the previous questions even if you did not solve them.

Consider the primal-dual pair of linear programs

$$\text{(P)}\quad \begin{array}{ll} \min & c^{\mathrm{T}}x \\ & Ax = b \\ & x \geq 0. \end{array} \qquad\qquad \text{(D)}\quad \begin{array}{ll} \max & b^{\mathrm{T}}y \\ & A^{\mathrm{T}}y + s = c \\ & s \geq 0, \end{array}$$

where $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$.

(a) (2pts) Use LP duality to show that $\bar{x}$, $(\bar{y}, \bar{s})$ are optimal solutions to (P) and (D) respectively if and only if $(\bar{x}, \bar{y}, \bar{s})$ solves

$$\begin{aligned} A^{\mathrm{T}}y + s &= c \\ Ax &= b \\ XSe &= 0 \\ x, s &\geq 0, \end{aligned}$$

where $X = \mathrm{Diag}(x)$, $S = \mathrm{Diag}(s)$, and $e = \begin{bmatrix}1 & \dots & 1\end{bmatrix}^{\mathrm{T}}$.

(b) (3pts) Assume $A$ is full row-rank, $\{x : Ax = b, x > 0\} \neq \emptyset$, and $\{(y, s) : A^{\mathrm{T}}y + s = c, s > 0\} \neq \emptyset$. Let $\mu > 0$ be fixed. Prove that $x(\mu)$ and $(y(\mu), s(\mu))$ minimize and maximize respectively

$$(\mathrm{P}_\mu)\quad \begin{array}{ll} \min & c^{\mathrm{T}}x - \mu \sum_{j=1}^n \ln(x_j) \\ & Ax = b \\ & x > 0 \end{array} \qquad\qquad (\mathrm{D}_\mu)\quad \begin{array}{ll} \max & b^{\mathrm{T}}y + \mu \sum_{j=1}^n \ln(s_j) \\ & A^{\mathrm{T}}y + s = c \\ & s > 0 \end{array}$$

if and only if $(x(\mu), y(\mu), s(\mu))$ solves

$$\begin{aligned} A^{\mathrm{T}}y + s &= c \\ Ax &= b \\ XSe &= \mu e \\ x, s &> 0. \end{aligned}$$

(c) (2pts) Assume $A, b, c$ are such that the conditions in (b) hold. Let $\bar{x}$ and $(\bar{y}, \bar{s})$ be optimal solutions to (P) and (D) respectively, and $x(\mu)$ and $(y(\mu), s(\mu))$ be optimal solutions to $(\mathrm{P}_\mu)$ and $(\mathrm{D}_\mu)$ respectively. Prove that

$$x(\mu)^{\mathrm{T}}\bar{s} + s(\mu)^{\mathrm{T}}\bar{x} = n\mu.$$

(d) (3pts) Assume $A, b, c$ are such that the conditions in (b) hold. Furthermore, assume that both (P) and (D) have unique non-degenerate optimal solutions. For $\mu > 0$ let $B(\mu) \subseteq \{1, \ldots, n\}$ be the set of indices of the $m$ largest components of $x(\mu)$. Prove that there exists $\delta > 0$ such that if $\mu < \delta$, then $B(\mu)$ is the optimal basis of (P), (D).

# Questions for the Qualifier of January 2004

## Thursday, Jan 8 2004

- **ACO Students - 3 hours:** You must answer Questions 1,2 and 6.
- **OM Students - 4 hours:** You must answer Questions 1,2,3 and 5.
- **OR Students - 5 hours:** You must answer Questions 1,2,3,4 and 5.
- All questions except 3 are open-notes open-book. For Question 3, you may only use material from a 3 inch by 5 inch notecard filled both sides.
- Please begin each question on a new page. Write only on one side of every page. Always remember to put your identification number (NOT your name) in the top right-hand corner of every page you turn in.

1. (Graph Theory)

   (a) State the Perfect Graph Theorem and the Strong Perfect Graph Theorem, and show that the latter implies the former.

   (b) An interval graph $G$ is the intersection graph of some family $\mathcal{F}$ of intervals on the real line, i.e. $G$ has a vertex for every interval in $\mathcal{F}$, and an edge for every pair of intervals whose intersection is nonempty. Use the Strong Perfect Graph Theorem to prove that interval graphs are perfect.

2. (Networks & Matchings) In a maximum flow problem in a directed network $G$, call $(i, j)$ the most important (least important) arc of $G$ if deleting $(i, j)$ from $G$ results in the largest (the smallest) decrease in the maximum flow.

Prove or disprove (by counterexample) each of the following statements:

(a) A most important arc is one with a maximum capacity $u_{ij}$.

(b) A most important arc is one with a maximum arc flow $x_{ij}$.

(c) A most important arc is one with a maximum flow $x_{ij}$ among the arcs of some minimum cut.

(d) An arc that does not belong to some minimum cut cannot be a most important arc.

(e) A network may have several most important arcs.

(f) Any arc $(i, j)$ with $x_{ij} = 0$ in some maximum flow is a least important arc.

(g) A least important arc is an arc $(i, j)$ with a minimum value of $x_{ij}$ in a maximum flow.

(h) No arc of a minimum cut can be a least important arc.

6. (Discrete Mathematics)

Let $n < N$ be positive integers. We say that $f : [N] \to [n]$ *separates* a set $A \subset [N]$ if the restriction of $f$ to $A$ is an injection (i.e. $f$ separates $A$ if $|f(A)| = |A|$).

(a) Use the probabilistic method to show that if

$$t > \frac{\log \binom{N}{k}}{\log n^k - \log \left(n^k - k!\binom{n}{k}\right)}$$

then there exists a collection of functions $f_1, \ldots, f_t : [N] \to [n]$ such that for all $A \in \binom{N}{k}$ there exists $i \in \{1, \ldots t\}$ such that $f_i$ separates $A$.

(b) Let $k, n$ be fixed constants. Use the Lovász Local Lemma to prove that for $N$ sufficiently large there exists

$$s < \frac{\log \binom{N}{k}}{\log n^k - \log \left(n^k - k!\binom{n}{k}\right)}$$

and a collection of functions $g_1, \ldots, g_s : [N] \to [n]$ such that for all $A \in \binom{N}{k}$ there exists $i \in \{1, \ldots s\}$ such that $g_i$ separates $A$.

# ANSWER–Part 1 Question 1 (Linear Programming)

1. To find an optimal solution, compute the partial sums $S_k = \sum_{j=1}^{k} v_j$ and pick a $k$ that maximizes $S_k$. Let $x_j = 1$ for $j = 1, \ldots, k$ and $x_j = 0$ for $j = k+1, \ldots, n$.

2. The dual is

$$
\begin{array}{lll}
\min & \sum_{i=1}^{n} \mu_i & \\
\text{subject to} & -\lambda_1 + \mu_1 \geq v_1 & (a) \\
& \lambda_{i-1} - \lambda_i + \mu_i \geq v_i, \quad i = 2, \ldots, n-1 & (b) \\
& \lambda_{n-1} + \mu_n \geq v_n & (c) \\
& \lambda_i \geq 0, \quad i = 1, \ldots, n-1 & (d) \\
& \mu_i \geq 0, \quad i = 1, \ldots, n & (e)
\end{array}
$$

3. Let the optimal primal solution be as given above. An optimal dual solution is

$$
\lambda_i = \begin{cases} v_{i+1} + \cdots + v_k & \text{for } i = 1, \ldots, k \\ -(v_{k+1} + \cdots + v_i) & \text{for } i = k+1, \ldots, n-1 \end{cases}
$$

$$
\mu_i = \begin{cases} v_1 + \cdots + v_k & \text{for } i = 1 \\ 0 & \text{for } i = 2, \ldots, n \end{cases}
$$

We check feasibility for each dual constraint:

(a) This becomes $-(v_2 + \cdots + v_k) + (v_1 + \cdots + v_k) \geq v_1$, which is satisfied.

(b) for $i = 2, \ldots, k$. This becomes $(v_i + \cdots + v_k) - (v_{i+1} + \cdots + v_k) \geq v_i$, which is satisfied.

(b) for $i = k+1, \ldots, n-1$. This becomes

$$-(v_{k+1} + \cdots + v_{i-1}) + (v_{k+1} + \cdots + v_i) \geq v_i$$

which is satisfied.

(c) This becomes $-(v_{k+1} + \cdots + v_{n-1}) \geq v_n$, or $v_{k+1} + \cdots + v_n \leq 0$. This is satisfied, since otherwise the algorithm would not have set $x_{k+1} = \cdots = x_n = 0$.

(d) for $i = 1, \ldots, k$. This becomes $v_{i+1} + \cdots + v_k \geq 0$, which is satisfied because otherwise the algorithm would have set $x_k = 0$.

(d) for $i = k+1, \ldots, n$. This becomes $v_{k+1} + \cdots + v_i \leq 0$. This is satisfied, since otherwise the algorithm would not have set $x_{k+1} = \cdots = x_i = 0$.

(e) This becomes $v_1 + \cdots + v_k \geq 0$, which is satisfied since otherwise the algorithm would have set $x_k = 0$.

4. Since the above dual feasible solution has the same value $v_1 + \cdots + v_k$ as the solution obtained by the algorithm, both are optimal by weak duality.

## ANSWER–Part 1 Question 2 (Integer Programming)

(a) Dimension of $P_I = 2$.

(b) $x_1 \geq 0$, $x_2 \geq 0$, $x_1 + x_2 \leq 2$, $y = 0$.

(c) $(0,0,0)$, $(0,2,0)$, $(2,0,0)$, $(1, \frac{1}{3}, \frac{2h}{3})$.

(d) No disjunction of the form proposed is violated by all 3 following points at the same time: $(1, \frac{1}{3})$, $(\frac{1}{3}, 1)$, $(\frac{1}{4}, \frac{1}{4})$. Therefore $P_\pi$ always contains at least one of the following points: $(1, \frac{1}{3}, \frac{2h}{3})$, $(\frac{1}{3}, 1, \frac{2h}{3})$, $(\frac{1}{4}, \frac{1}{4}, \frac{h}{2})$. Thus at least one of the tetrahedra formed by one of these 3 points and the points $(0,0,0)$, $(0,2,0)$, $(2,0,0)$ is always contained in $P_\pi$. Since all 3 tetrahedra contain $(\frac{1}{2}, \frac{1}{2}, \frac{h}{10})$, this point belongs to $P_\pi$ and thus to $P^1 = \cap_\pi P_\pi$.

(e) Follows from (d) by induction.

(f) Yes in both cases: By a theorem of Chvátal for pure IP [also Gomory and Schrijver are acceptable answers] and by a theorem of Balas for mixed 0,1 programs.

# ANSWER–Part 1 Question 3 (Advanced Integer Programming)

(a) The cut generating linear program is

$$
\begin{array}{lllllllllll}
\min & \alpha\bar{x} - \beta \\
& \alpha & + & uA & - & u_0 e_{Q_{i1}} & & & & & \geq 0 \\
& \alpha & & & & & + & vA & - & v_0 e_{Q_{i2}} & \geq 0 \\
& \beta & - & ub & + & u_0 & & & & & = 0 \\
& \beta & & & & & - & vb & + & v_0 & = 0 \\
& & & ue & + & u_0 & + & ve & + & v_0 & = 1 \\
& & & & & & & u, v & & & \geq 0
\end{array}
\qquad \text{(CGLP)}
$$

(b) Any valid cut obtainable from the disjunction (1) either corresponds to a basic solution of (CGLP) or is dominated by such a cut.

The disjunctive rank of $P$ in terms of the family of disjunctions of type (1) can be defined as the number of iterations needed to generate $\text{conv}(P \cap \{0,1\}^n)$ recursively, by obtaining at each step of the recursion all cuts from a disjunction (1) corresponding to basic solutions of (CGLP). In terms of this definition, the disjunctive rank of $P$ is $\sum_{i=1}^{k} \log_2 |Q_i|$.

# ANSWER–Part 1 Question 5 (Convex Polytopes)

The condition

$$x_1 + x_2 + x_3 \geq 2 \quad \Rightarrow \quad y_4 + y_5 \geq 1$$

can be restated, setting $y_i = 1 - x_i$, $i = 1, \ldots, 3$, as

$$y_1 + y_2 + y_3 \geq 2 \quad \vee \quad y_4 + y_5 \geq 1.$$

The convex hull of $S$ is given by

$$\begin{array}{ccccccccc}
y_1 & -y_1^1 & & & -y_1^2 & & & = & 0 \\
\ddots & \ddots & & & \ddots & & & & \vdots \\
y_5 & -y_5^1 & & & -y_5^2 & & & = & 0 \\
 & -y_1^1 & + & z_0^1 & & & & \geq & 0 \\
 & \ddots & & \vdots & & & & & \vdots \\
 & -y_5^1 & + & z_0^1 & & & & \geq & 0 \\
 & y_1^1 + y_2^1 + y_3^1 & - & 2z_0^1 & & & & \geq & 0 \\
 & & & & -y_1^2 & + & z_0^2 & \geq & 0 \\
 & & & & \ddots & & \vdots & & \vdots \\
 & & & & -y_5^2 & + & z_0^2 & \geq & 0 \\
 & & & & y_4^2 + y_5^2 & - & z_0^2 & \geq & 0 \\
 & & & z_0^1 & & + & z_0^2 & = & 1 \\
\end{array}$$

$$y_j^1 \geq 0,\ y_j^2 \geq 0,\ j = 1, \ldots, 5,\ z_0^1 \geq 0,\ z_0^2 \geq 0$$

Projecting this system onto the subspace of $(y_1, \ldots, y_5)$ yields

$$\begin{array}{rcrcrcrcrcl}
y_1 & + & y_2 & + & y_3 & + & 2y_4 & + & 2y_5 & \geq & 2 \\
y_1 & + & y_2 & & & + & y_4 & + & y_5 & \geq & 1 \\
y_1 & & & + & y_3 & + & y_4 & + & y_5 & \geq & 1 \\
 & & y_2 & + & y_3 & + & y_4 & + & y_5 & \geq & 1
\end{array}$$

or, after substituting $x_i = 1 - y_i$ for $i = 1, 2, 3$,

$$\begin{array}{rcrcrcrcrcl}
x_1 & + & x_2 & + & x_3 & - & 2y_4 & - & 2y_5 & \leq & 1 \\
x_1 & + & x_2 & & & - & y_4 & - & y_5 & \leq & 1 \\
x_1 & & & & x_3 & - & y_4 & - & y_5 & \leq & 1 \\
 & & x_2 & + & x_3 & - & y_4 & - & y_5 & \leq & 1.
\end{array}$$

# ANSWER–Part 1 Question 6 (Advanced Linear Programming)

(a) By LP programming duality, we know that $\bar{x}$ and $(\bar{y}, \bar{s})$ are optimal solutions to (P) and (D) respectively if and only if they are both feasible and their optimal values are the same. In other words, if and only if

$$\begin{aligned} A^{\mathrm{T}}y + s &= c \\ Ax &= b \\ c^{\mathrm{T}}x &= b^{\mathrm{T}}y \\ x, s &\geq 0. \end{aligned} \tag{1}$$

But if the first two conditions hold, then $c^{\mathrm{T}}\bar{x} - b^{\mathrm{T}}y = (A^{\mathrm{T}}y + s)^{\mathrm{T}}x - (Ax)^{\mathrm{T}}y = x^{\mathrm{T}}s$. Hence we can replace the third equation in (1) by $x^{\mathrm{T}}s = 0$. But for $x, s \geq 0$ we have $x^{\mathrm{T}}s = 0 \Leftrightarrow x_j s_j = 0,\ j = 1, \ldots, n \Leftrightarrow XSe = 0$. Therefore (1) is equivalent to

$$\begin{aligned} A^{\mathrm{T}}y + s &= c \\ Ax &= b \\ XSe &= 0 \\ x, s &\geq 0. \end{aligned}$$

(b) By the KKT conditions $x(\mu) > 0$ solves $(\mathrm{P}_\mu)$ if and only if $Ax(\mu) = b$ and there exists $\tilde{y}(\mu)$ such that

$$c - \mu X(\mu)^{-1}e = A^{\mathrm{T}}\tilde{y}(\mu).$$

Similarly, $(y(\mu), s(\mu))$ with $s(\mu) > 0$ solves $(\mathrm{D}_\mu)$ if and only if $A^{\mathrm{T}}y(\mu) + s(\mu) = c$ and there exists $\tilde{x}(\mu)$ such that

$$\begin{bmatrix} b \\ \mu S(\mu)^{-1}e \end{bmatrix} = \begin{bmatrix} A\tilde{x}(\mu) \\ \tilde{x}(\mu) \end{bmatrix}.$$

Hence it follows that $\tilde{x}(\mu)$ also solves $(\mathrm{P}_\mu)$. But, because $c^{\mathrm{T}}x - \mu\sum_{j=1}^{n}\ln(x_j)$ is strictly convex, the solution $x(\mu)$ is unique. Therefore $\tilde{x}(\mu) = x(\mu) = \mu S(\mu)^{-1}e$, and consequently $A^{\mathrm{T}}\tilde{y}(\mu) = A^{\mathrm{T}}y(\mu)$. Since $A$ is full row-rank, we must also have $\tilde{y}(\mu) = y(\mu)$. Therefore the KKT conditions for *both* $(\mathrm{P}_\mu)$ and $(\mathrm{D}_\mu)$ are $x(\mu), s(\mu) > 0$, $Ax(\mu) = b$, $A^{\mathrm{T}}y(\mu) + s(\mu) = c$, and $x(\mu) = \mu S(\mu)^{-1}e$. These can be rewritten as

$$\begin{aligned} A^{\mathrm{T}}y + s &= c \\ Ax &= b \\ XSe &= \mu e \\ x, s &> 0. \end{aligned}$$

(c) By (a) and (b) we have

$$\begin{aligned}
x(\mu)^{\mathrm{T}}\overline{s} + s(\mu)^{\mathrm{T}}\overline{x} &= x(\mu)^{\mathrm{T}}(c - A^{\mathrm{T}}\overline{y}) + (c - A^{\mathrm{T}}y(\mu))^{\mathrm{T}}\overline{x} \\
&= x(\mu)^{\mathrm{T}}c - b^{\mathrm{T}}\overline{y} + c^{\mathrm{T}}\overline{x} - b^{\mathrm{T}}y(\mu) \\
&= x(\mu)^{\mathrm{T}}(A^{\mathrm{T}}y(\mu) + s(\mu)) - (Ax(\mu))^{\mathrm{T}}y(\mu) \\
&= x(\mu)^{\mathrm{T}}s(\mu) \\
&= n\mu.
\end{aligned}$$

(d) Let $B$ be the optimal basis of (P), (D) and let $N = \{1, \dots, n\} \setminus B$. Since the optimal solutions to (P) and (D) are not degenerate, we must have $\overline{x}_B > 0$ and $\overline{s}_N > 0$. Let $\epsilon > 0$ be such that all components of $\overline{x}_B$ and $\overline{s}_N$ are bounded below by $\epsilon$. From (c) it follows that

$$x(\mu)_j \leq \frac{n\mu}{\overline{s}_j} \leq \frac{n\mu}{\epsilon},\ j \in N,$$

and

$$s(\mu)_j \leq \frac{n\mu}{\overline{x}_j} \leq \frac{n\mu}{\epsilon},\ j \in B.$$

Since $X(\mu)S(\mu)e = \mu e$, we also have

$$s(\mu)_j \geq \frac{\epsilon}{n},\ j \in N,$$

and

$$x(\mu)_j \geq \frac{\epsilon}{n},\ j \in B.$$

Thus for $\mu < \epsilon^2/n^2$ we have

$$x(\mu)_j < \frac{\epsilon}{n},\ j \in N,$$

and

$$x(\mu)_j \geq \frac{\epsilon}{n},\ j \in B.$$

Therefore, as long as $\mu < \epsilon^2/n^2$, the largest $m$ components of $x(\mu)$ are precisely those whose indices are in $B$. In other words, $B(\mu) = B$ for $\mu < \epsilon^2/n^2$.

# ANSWER–Part 2 Question 1 (Graph Theory)

(a) Given an undirected graph $G = (V, E)$, denote

- $\overline{G} = (V, \overline{E})$, the complement graph of $G$
- $\mathcal{G}$ = collection of induced subgraphs of $G$
- $\alpha(G)$ = the size of a maximum stable set in $G$
- $\theta(G)$ = the size of a minimum clique cover of $G$
- $\omega(G)$ = the size of a maximum clique in $G$
- $\gamma(G)$ = the size of a minimum coloring of $G$.

**Definition.** $G$ is $\gamma$-perfect if $\gamma(G') = \omega(G')$ for all $G' \in \mathcal{G}$.
$G$ is $\alpha$-perfect if $\alpha(G') = \theta(G')$ for all $G' \in \mathcal{G}$

**Perfect Graph Theorem.**
• $G$ is $\alpha$-perfect if and only if it is $\gamma$-perfect.
Equivalently: $G$ is perfect if and only if $\overline{G}$ is perfect.

**Strong Perfect Graph Theorem.**
• $G$ is perfect if and only if it has no odd holes and no odd antiholes.
Equivalently: $G$ is perfect if and only if neither $G$ nor $\overline{G}$ have odd holes.

(b) Interval graphs have no holes (even or odd). Indeed, if the intervals are $[i_1, j_1], \ldots, [i_m, j_m]$, a hole of length $k$ would imply that $[i_\ell, j_\ell], [i_{\ell+1}, j_{\ell+1}]$ intersect for $\ell = 1, \ldots, k-1$ and $[i_k, j_k]$ intersects $[i_1, j_1]$, without intersecting $[i_2, j_2], \ldots, [i_{k-2}, j_{k-2}]$ which is impossible. We claim that this implies that interval graphs have no odd antiholes either, hence according to the Strong Perfect Graph Theorem they are perfect.

**Proof of the claim.** Every odd antihole on $n$ nodes ($n$ odd), with the missing edges being $(1, 1+(n-1)/2), (2, 2+(n-1)/2), \ldots, (n, n+(n-1)/2)$ (with addition taken modulo $n$), has a 4-hole, namely: $(1, 2, 1+(n-1)/2, 3+(n-1)/2)$. Indeed, none of the 4 edges $(1,2), (2, 1+(n-1)/2), (1+(n-1)/2, 3+(n-1)/2)$ and $(3+(n-1)/2, 1)$ are listed as missing; on the other hand, both $(1, 1+(n-1)/2)$ and $(2, 3+(n-1)/2)$ are missing (the latter is the same as $(3+(n-1)/2, 3+(n-1)/2+(n-1)/2) = (3+(n-1)2, 2)$ (modulo $n$).

# ANSWER–Part 2 Question 2 (Networks & Matchings)

(a) False.

(b) False.

(c) False.

(d) False.

(e) True.

(f) True.

(g) False.

(h) False.

# ANSWER–Part 2 Question 6 (Discrete Mathematics)

1. Let $f_1, \ldots, f_t$ be functions chosen uniformly and independently at random from the collection of all $n^N$ functions from $N$ to $n$. For $A \in \binom{[N]}{k}$ let $\mathcal{E}_A$ be the event that no function in the collection $f_1, \ldots, f_t$ separates $A$. We have

$$Pr(\mathcal{E}_A) = \left(1 - \frac{k!\binom{n}{k}}{n^k}\right)^t \qquad \text{函数值} \quad A \to [n]$$

Applying the union bound we have

$$Pr\left(\bigcup_{A\in\binom{[N]}{k}} \mathcal{E}_A\right) \le \sum_{A\in\binom{[N]}{k}} Pr(\mathcal{E}_A) = \binom{N}{k}\left(1 - \frac{k!\binom{n}{k}}{n^k}\right)^t.$$

If this quantity is less than 1 (which is equivalent to the condition on $t$ stated in the problem) then there exists a collection of functions $f_1, \ldots, f_t$ such that for every $A \in \binom{[N]}{k}$ there exists a function in the collection that separates $A$.

2. Let $g_1, \ldots, g_s$ be functions chosen uniformly and independently at random from the collection of all $n^N$ functions from $N$ to $n$. (Note that in this probability space $g_1(1), \ldots, g_1(N)$ are chosen uniformly and independently at random from $[n]$.) For $A \in \binom{[N]}{k}$ let $\mathcal{E}_A$ be the event that no function in the collection $g_1, \ldots, g_s$ separates $A$. We define a dependency graph on this collection of events by setting $\mathcal{E}_A \sim \mathcal{E}_B$ iff $A \cap B \neq \emptyset$. Note that the event $\mathcal{E}_A$ is mutually independent of the collection of events $\{\mathcal{E}_B : \mathcal{E}_A \not\sim \mathcal{E}_B\}$ since these events are determined by the values of the functions on $[N] \setminus A$. The degree in this dependency graph is

$$d = \binom{N}{k} - \binom{N-k}{k}.$$

Applying the Lovász Local Lemma we have

$$4\left(1 - \frac{k!\binom{n}{k}}{n^k}\right)^s \left(\binom{N}{k} - \binom{N-k}{k}\right) < 1 \quad \Rightarrow \quad Pr\left(\bigcup_{A\in\binom{[N]}{k}} \mathcal{E}_A\right) < 1$$

So,

$$s > \frac{2 + \log\left(\binom{N}{k} - \binom{N-k}{k}\right)}{\log n^k - \log\left(n^k - k!\binom{n}{k}\right)} \quad \Rightarrow \quad Pr\left(\bigcup_{A\in\binom{[N]}{k}} \mathcal{E}_A\right) < 1$$

Since $\binom{N}{k} \sim \binom{N-k}{k}$ as $N$ goes to infinity, we have the desired result.

# Questions for the Qualifier of January 2006

## Wednesday January 11, 2006

- **ACO Students - 3 hours:** You must answer Questions 1, 2, and 3.
- **OM Students - 4 hours:** You must answer Questions 1, 2, 4, and 5.
- **OR Students - 5 hours:** You must answer Questions 1, 2, 3, 4, and 5.
- All questions are open-notes, open-book.
- Please begin each question on a new page. Write only on one side of every page. Always remember to put your identification number (NOT your name) in the top right-hand corner of every page you turn in.

1. (Linear Programming)

Let $A$ be an $m \times n$ matrix and $b \in \mathbf{R}^m$.

(a) Using duality, give a necessary and sufficient condition for the system $Ax = b$ to have a solution $x \in \mathbf{R}^n$ with $x \geq 0$. Justify your answer.

(b) Let $v_1, \ldots, v_n$ be given nonzero vectors in $\mathbf{R}^m$. Using duality, give a set of necessary and sufficient conditions on the $v_i$ such that there exist numbers $\lambda_1, \ldots, \lambda_n$, all strictly positive, with $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$. Justify your answer.

(c) Give the simplest geometrical interpretation (in term of hyperplane(s) and/or vector(s)) of the set of conditions you found in b).

2. (Integer Programming)

The pigeonhole principle states that the problem

(P) Place $(n+1)$ pigeons into $n$ holes so that no two pigeons share a hole

has no solution.

(a) Formulate (P) as an integer programming problem with two kinds of constraints:

(a1) those expressing the condition that every pigeon must get into a hole;

(a2) those expressing the condition, through a separate constraint for each pair of pigeons, that at most one member of the pair can get into a given hole.

Show that there is no integer solution satisfying (a1) and (a2), but the linear program with constraints (a1) and (a2) is feasible.

(b) Give a procedure for using (a2) to generate a hierarchy of cutting plane families with binary coefficients such that

(b1) the cutting planes at level 1 of the hierarchy have three coefficients equal to 1 and dominate (make redundant) the inequalities (a2);

(b2) the cutting planes at level $k-2$ of the hierarchy have $k$ coefficients equal to 1 and dominate all cutting planes at level $h < k-2$;

(b3) the cutting planes at level $n-1$ (the top level), together with the constraint (a1), constitute an infeasible system.

3. (Advanced Integer Programming)

Let $P$ be the problem of finding a maximum-weight simple directed cycle in a complete digraph with arc-weights of arbitrary sign.

(a) Give two integer programming formulations of $P$, the first one using arc and node variables, the second one using only arc variables. Prove the correctness of your formulations.

*Hint: In both formulations, in order to eliminate multiple cycles, you have to come up with a family of inequalities satisfied by every simple cycle, but containing for any union $U$ of simple dicycles at least one member violated by $U$.*

(b) Derive the second formulation as a projection of the first one on the space of the arc variables.

(c) Show the connection of the first formulation to the asymmetric TSP. Do you see a way to derive valid inequalities for $P$ from valid inequalities for the asymmetric TSP?

# Questions for the Qualifier of January 2006

## Thursday January 12, 2006

- ACO Students - **4 hours:** You must answer Questions 1, 2, 6 and one of 3, 4, or 5.
- OM Students - **2 hours:** You must answer Questions 1, and 2.
- OR Students - **4 hours:** You must answer Questions 1, 2, 3 and one of 4 or 5.
- All questions are open-notes, open-book.
- Please begin each question on a new page. Write only on one side of every page. Always remember to put your identification number (NOT your name) in the top right-hand corner of every page you turn in.

1. (Graph Theory)

Let $G$ be a connected simple graph with $n$ vertices (no loops or multiple edges). The *tree graph* of $G$ is the graph whose vertices are the spanning trees $T_1, \ldots, T_k$ of $G$ with $T_i$ and $T_j$ joined if and only if they have exactly $n-2$ edges in common.

(a) Consider the case where $G$ has exactly $n$ edges that induce a cycle, i.e. every vertex of $G$ has degree 2. Describe the tree graph of $G$.

(b) Consider the case where $G = K_n$, the complete graph on $n$ vertices. What is the number of vertices of the tree graph of $G$?

(c) Show that the tree graph of any connected simple graph is connected.

(d) Is the tree graph of a connected simple graph always 2-connected?

a)

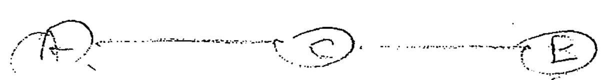
clique

b) Cayley's formula $n^{n-2}$

c) Show that between any two nodes of the tree graph there always is a path.

[illegible]

d) Let A and B be two nodes in the tree graph such that there exists only one path connecting them (they differ by 2 edges)



[illegible]

C [illegible]

2. (Networks & Matchings)

Consider a function $f$ defined for all sequences of numbers such that

(i) $f(x_1, \dots, x_k, x_{k+1}) = f(f(x_1, \dots, x_k), x_{k+1})$

(ii) $f(x_1, \dots, x_k, x_{k+1}) \geq f(x_1, \dots, x_k)$.

Consider a network $N = (V, A)$ with arc weights $w_a$, for all $a \in A$. Let $s, t \in V$. Define the *value* of a path $P$ from $s$ to $v \in V$ to be $f(w_{a_1}, \dots, w_{a_k})$ where $a_1, \dots, a_k$ is the sequence of arcs in the path $P$. The goal is to find a path from $s$ to $t$ with smallest value.

(a) Design a Dijkstra-like algorithm to find a path from $s$ to $t$ with smallest value.

(b) Prove the correctness of your algorithm.

(c) Give examples showing that the algorithm might fail if either (i) or (ii) is omitted.

(d) What is the complexity of a straightforward implementation of your algorithm, i.e. the number of function evaluations?

4. (Convex Polytopes)

The Capacitated Vehicle Routing Problem is the problem of serving customers with capacitated trucks from a central depot. Let $\{1, 2, \ldots, n\}$ be the set of customers with respective demands $d_i \geq 1$ for $i = 1, \ldots, n$. Let $p \geq 2$ be the number of trucks available, each truck having capacity $K \geq 1$. The index 0 is used for indexing the depot.

Let $G$ be the complete graph with vertex set $V = \{0, 1, 2, \ldots, n\}$, each edge $e$ having a cost $c_e$. A *route* in $G$ is an ordered set $\{v_1, \ldots, v_r\}$ of $r \geq 1$ distinct vertices of $V - \{0\}$. The *cost* of a route is the cost of traveling from the depot to $v_1$, plus the cost of traveling the route itself, plus the cost of going back to the depot, i.e. the cost of the edge $(0, v_1)$, plus the sum of the cost of the edges $(v_i, v_{i+1})$ for $i = 1, \ldots, r-1$ plus the cost of the edge $(v_r, 0)$. A feasible solution of the capacitated vehicle routing problem on $G$ is a collection of exactly $p$ routes forming a partition of $V - \{0\}$ and such that the sum of the demand of the customers on each route does not exceed the capacity $K$ of the truck. A solution with minimum sum of the cost of the routes is sought.

(a) Which changes in the following integer linear programming (ILP) formulation of the symmetric traveling salesman problem are required to have an ILP formulation of the Capacitated Vehicle Routing Problem?

$$\text{(STSP)} \qquad \min \sum_{e \in E} c_e x_e \tag{1}$$

subject to

$$x(\delta(v)) = 2 \quad \text{for} \quad v \in V \tag{2}$$
$$x(\delta(S)) \geq 2 \quad \text{for} \quad 3 \leq |S| \leq |V|/2 \tag{3}$$
$$0 \leq x_e \leq 1 \quad \text{for} \quad e \in E \tag{4}$$
$$x_e \text{ integer} \quad \text{for} \quad e \in E \tag{5}$$

*Hint: There is a change in all lines except (1) and (5).*

For the remainder of the problem, let $Q(n)$ be the linear programming relaxation of the formulation you found in point (a) when used for the case $n \geq 3, p = 2, K = n - 1$ and with $d_i = 1$ for $i = 1, \ldots, n$.

(b) What is the dimension of $Q(n)$?

(c) What is the dimension of the convex hull of the integer points contained in $Q(n)$? Give the best upper and lower bounds that you can find.

5. (Advanced Linear Programming)

Consider the primal-dual pair of linear programs

$$
\text{(P)} \quad \begin{array}{ll} \min & c^{\mathrm{T}}x \\ & Ax = b \\ & x \geq 0, \end{array}
\qquad\qquad
\text{(D)} \quad \begin{array}{ll} \max & b^{\mathrm{T}}y \\ & A^{\mathrm{T}}y + s = c \\ & s \geq 0, \end{array}
$$

where $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$.

(a) (2pts) Use LP duality to show that $\bar{x}$, $(\bar{y}, \bar{s})$ are optimal solutions to (P) and (D) respectively if and only if $(\bar{x}, \bar{y}, \bar{s})$ solves

$$
\begin{aligned}
A^{\mathrm{T}}y + s &= c \\
Ax &= b \\
XSe &= 0 \\
x, s &\geq 0,
\end{aligned}
$$

where $X = \mathrm{Diag}(x)$, $S = \mathrm{Diag}(s)$, and $e = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^{\mathrm{T}}$.

(b) (4pts) Assume $A$ is full row-rank, $\{x : Ax = b, x > 0\} \neq \emptyset$, and $\{(y, s) : A^{\mathrm{T}}y + s = c, s > 0\} \neq \emptyset$. Assume $v \in \mathbf{R}^n$ such that $v > 0$ is given.

(i) (2pts) Prove that there exists a unique solution to the system of equations

$$
\begin{aligned}
A^{\mathrm{T}}y + s &= c \\
Ax &= b \\
XSe &= v \\
x, s &> 0.
\end{aligned}
\tag{1}
$$

(ii) (2pts) Prove that for some appropriate convex function $f : \mathbf{R}^n_{++} \to \mathbf{R}$ (depending on $v$) the point $(x_v, y_v, s_v)$ is a solution to (1) if and only if $x_v$ and $(y_v, s_v)$ are the solutions to

$$
(\mathrm{P}_v) \quad \begin{array}{ll} \min & c^{\mathrm{T}}x + f(x) \\ & Ax = b \\ & x > 0 \end{array}
\qquad \text{and} \qquad
(\mathrm{D}_v) \quad \begin{array}{ll} \max & b^{T}y - f(s) \\ & A^{T}y + s = c \\ & s > 0 \end{array}
$$

respectively.

(c) (4pts) Suppose you have a point $(x^0, y^0, s^0)$ such that $x^0, s^0 > 0$ and $Ax^0 = b$, $A^{\mathrm{T}}y^0 + s^0 = c$.

Use part (b) to propose a suitable modification of a feasible interior-point path following algorithm whose starting point is $(x^0, y^0, s^0)$ and that generates approximate solutions to (1) for values of $v$ that converge to 0. Specifically, define:

(i) (2pts) an appropriate "modified central path"

(ii) (1pt) an appropriate "modified neighborhood of the central path"

(iii) (1pt) the steps that should constitute each main iteration of the algorithm.

6. (Discrete Mathematics)

(a) Recall that $G_{n,p}$ is the random graph on $n$ vertices where each edge appears independently with probability $p$. Let $\mathcal{E}$ be the event that $G_{n,p}$ contains a (not necessarily induced) copy of the cycle on 4 vertices.

(i) Prove that if $\lim_{n\to\infty} np = 0$ then $\lim_{n\to\infty} Pr(\mathcal{E}) = 0$.

(ii) Prove that if $\lim_{n\to\infty} \frac{1}{np} = 0$ then $\lim_{n\to\infty} Pr(\mathcal{E}) = 1$.

(b) Prove that every graph $G$ with $e$ edges and maximum degree $\Delta = \Delta(G)$ has a matching of size at least $e/(2\Delta)$.

~~given a graph we can always remove at most half of the edges in order to make it bipartite. (use procedure that terminates in a finite number of steps: assign $v_i$ to $S$ or to $V \setminus S$ randomly, then if a node has more neighbors in one partition, put it in the other) At most we remove half of the edges.~~

# ANSWER–Part 1 Question 1 (Linear Programming)

(a) The dual of (P): $\min\{0^T x \mid Ax = b, x \geq 0\}$ is (D): $\max\{y^T b \mid y^T A \leq 0\}$. Observe that (D) is always feasible. Hence, (P) is feasible if and only if (D) is bounded. Since the linear system in (D) is homogeneous, (D) is bounded if and only if $\{y \mid y^T A \leq 0, y^T b > 0\}$ has no solution.

(b) Let $A$ be the matrix whose columns are the vectors $v_1, \ldots, v_n$. The dual of (P): $\max\{t \mid A\lambda = 0, t - \lambda_i \leq 0 \, \forall i = 1, \ldots, n\}$ is (D): $\min\{0 \mid u^T A - v^T I = 0, v^T 1 = 1, v \geq 0\}$. Observe that (P) is always feasible and has a solution with $t > 0$ if and only if (D) is infeasible. This is equivalent to say that $u^T A \geq 0, u^T A \neq 0$ has no solution.

(c) The above condition is satisfied if and only if there is no hyperplane $H = \{x \in \mathbf{R}^n \mid u^T x = 0\}$ such that all vectors in $\{v_1, \ldots, v_n\}$ lie on the same side of $H$ with at least one vector not lying in $H$ itself.

# ANSWER–Part 1 Question 2 (Integer Programming)

(a)

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i = 1, \ldots, n+1 \tag{1}$$

$$x_{ij} + x_{kj} \leq 1 \quad \forall i, k = 1, \ldots, n+1; \forall j = 1, \ldots, n \tag{2}$$

$$x_{ij} \in \{0, 1\}^n \quad \forall i = 1, \ldots, n+1; \forall j = 1, \ldots, n \tag{3}$$

The variables form a matrix $X = (x_{ij})$ with (n+1) rows and $n$ columns. Constraints (1) imply that $x_{ij} = 1$ for exactly one $j$ in each row $i$, a total of $n+1$ variables. On the other hand, constraints (2) imply that $x_{ij} = 1$ for exactly one $j$ in each column $j$, a total of $n$ variables, a contradiction.

On the other hand, $x_{ij} = \frac{1}{n}$ for all $i, j$ is a feasible solution to the linear system (1), (2).

(b) For every triple $i, k, \ell$ of row indices and for $j = 1, \ldots, n$, we have from (2):

$$x_{ij} + x_{kj} \leq 1$$
$$x_{ij} + x_{\ell j} \leq 1$$
$$x_{kj} + x_{\ell j} \leq 1$$

Multiplying by 0.5 each of these inequalities, adding them up an rounding down the resulting inequality yields: $x_{ij} + x_{kj} + x_{\ell j} \leq 1$ for all $j = 1, \ldots, n$. This family of cuts form the first level of the hierarchy. At level $k-2$, we have the inequalities

$$x_{i_1 j} + x_{i_2 j} + \ldots + x_{i_k j} + 0x_{i_{k+1} j} \leq 1$$
$$x_{i_1 j} + x_{i_2 j} + \ldots + 0x_{i_k j} + x_{i_{k+1} j} \leq 1$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$0x_{i_1 j} + x_{i_2 j} + \ldots + x_{i_k j} + x_{i_{k+1} j} \leq 1$$

Multiplying each inequality by $\frac{1}{k}$, adding them and rounding down yields $x_{i_1 j} + x_{i_2 j} + \ldots + x_{i_k j} + x_{i_{k+1} j} \leq 1$.

Finally, at level $n-1$ (top level), the obtained inequalities are:

$$\sum_{i=1}^{n+1} x_{ij} \leq 1 \text{ for } j = 1, \ldots, n$$

and, together with (1) they form an infeasible system.

# ANSWER–Part 1 Question 3 (Advanced Integer Programming)

(a) First formulation:

$$
\begin{array}{llll}
\max & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}x_{ij} & & (1)\\
\text{s.t.} & x(i,N) = y_i & \forall i \in N & (2)\\
& x(N,j) = y_j & \forall j \in N & (3)\\
& x(S,S) - y(S-\{k\}) + y_\ell \le 1 & \forall S \subset N, k \in S, \ell \in N-S & (4)\\
& x_{ij}, y_j \in \{0,1\} & \forall i,j \in N & (5)
\end{array}
$$

Second formulation:

$$
\begin{array}{llll}
\max & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}x_{ij} & & (6)\\
\text{s.t.} & x(i,N) \le 1 & \forall i \in N & (7)\\
& x(N,i) - x(i,N) = 0 & \forall i \in N & (8)\\
& x(k,N) + x(\ell,N) - x(S,N-S) \le 1 & \forall S \subset N, k \in S, \ell \in N-S & (9)\\
& x_{ij} \in \{0,1\} & \forall i,j \in N & (10)
\end{array}
$$

Constraints (7), (8) correspond to (2), (3): Both sets of constraints are satisfied by dicycles and union of dicycles. Constraints (4) and (9) are satisfied by every dicycle; but if $U$ is a union containing disjoint dicycles $C_1, C_2$, then the inequality of (4) or (9) corresponding to $S = \{\text{nodes of } C_1\}$, any $k \in S$ and $\ell$ a node of $C_2$ is violated by $U$.

(b) Add (2) to $y_i \le 1$ to obtain (7). Substitute $y$ into (4) to obtain (9).

(c) When we set $y_i = 1$ for all $i$, the first formulation is a formulation for the ATSP where the constraints (4) are subtour elimination constraints. It suggests that any valid inequality for the ATSP can be lifted into a valid inequality for $P$.

ANSWER–Part 2 Question 1 (Graph Theory)

ANSWER–Part 2 Question 2 (Networks & Matchings)

# ANSWER–Part 2 Question 4 (Convex Polytopes)

(a)

$$
\begin{array}{llll}
\min & \sum_{e\in E} c_e x_e & & (1)\\
\text{subject to} & & & \\
& x(\delta(v)) = \begin{cases} 2p & \text{if } v = 0 \\ 2 & \text{otherwise} \end{cases} & \text{for } v \in V & (2)\\
& x(\delta(S)) \geq 2\lceil \frac{d(S)}{K} \rceil & \text{for } \emptyset \neq S \subseteq V - \{0\} & (3)\\
& 0 \leq x_e \leq \begin{cases} 2 & \text{if } e \text{ incident to } 0 \\ 1 & \text{otherwise} \end{cases} & \text{for } e \in E & (4)\\
& x_e \text{ integer} & \text{for } e \in E & (5)
\end{array}
$$

(b) Equalities (2) are all linearly independent: This is clear for the $n$ equalities with right hand side 2, since each involves an edge incident to 0 that is not present in the others. The last equality is not linearly dependent with these, as it prevents a single cycle on $V - \{0\}$.

For each inequality (3), either $|S| = 1$ or $|S| = n$ and it is implied by (2), or $1 < |S| < n$ and there exists a feasible integer solution satisfying it strictly.

For each inequality (4), there exists a feasible integer solution satisfying it strictly.

It follows that the equality space of $Q(n)$ is defined by the $n+1$ equalities (2). The dimension of $Q(n)$ is thus $\frac{(n+1)n}{2} - (n+1) = \frac{n^2-n-2}{2}$.

(c) The upper bound is the dimension of $Q(n)$ computed above. Suppose that one other equality $ax = b$ with $a \neq 0$ is linearly independent from the equalities (2) at that the integer hull is contained in $ax = b$. Using multiples of the equalities (2), one can assume that $a_e = 0$ for all $e$ incident to 0, that $a_e \geq 0$ for all $e \in E$ and that at least one edge $\bar{e}$ not adjacent to 0 has $a_{\bar{e}} = 0$.

Since the integer hull is contained in $ax = b$, all feasible solutions satisfy it. Consider a solution $\bar{x}$ using edge $\bar{e}$. Observe that $\bar{x}$ induces two paths $P_1, P_2$ in $G - \{0\}$. Let $e_1, e_2$ be two edges such that $P_1 \cup P_2 \cup \{e_1, e_2\}$ is a cycle. Observe that $(P_1 \cup P_2 - e) \cup e_1$ corresponds to a feasible solution $\bar{x}'$ after adding four edges adjacent to 0. Since $a\bar{x} = a\bar{x}'$, we have $a_e = a_{e_1} = 0$. A similar argument shows that all edges in $P_1 \cup P_2$ have $a_e = 0$, implying $a\bar{x} = 0$ and thus $a = 0$, a contradiction.

# ANSWER–Part 2 Question 5 (Advanced Linear Programming)

(a) By LP programming duality, we know that $\bar{x}$ and $(\bar{y}, \bar{s})$ are optimal solutions to (P) and (D) respectively if and only if they are both feasible and their optimal values are the same. In other words, if and only if

$$\begin{aligned} A^{\mathrm{T}}y + s &= c \\ Ax &= b \\ c^{\mathrm{T}}x &= b^{\mathrm{T}}y \\ x, s &\geq 0. \end{aligned} \tag{1}$$

But if the first two conditions hold, then $c^{\mathrm{T}}\bar{x} - b^{\mathrm{T}}y = (A^{\mathrm{T}}y + s)^{\mathrm{T}}x - (Ax)^{\mathrm{T}}y = x^{\mathrm{T}}s$. Hence we can replace the third equation in (1) by $x^{\mathrm{T}}s = 0$. But for $x, s \geq 0$ we have $x^{\mathrm{T}}s = 0 \Leftrightarrow x_j s_j = 0,\ j = 1, \ldots, n \Leftrightarrow XSe = 0$. Therefore (1) is equivalent to

$$\begin{aligned} A^{\mathrm{T}}y + s &= c \\ Ax &= b \\ XSe &= 0 \\ x, s &\geq 0. \end{aligned}$$

(b) Do parts (i) and (ii) together: consider the function

$$f(x) := -\sum_{j=1}^{n} v_j \log x_j.$$

This function is strictly convex on $\mathbf{R}^n_{++}$ because $\nabla^2 f(x) = VX^{-2} \succ 0$ for all $x \in \mathbf{R}^n_{++}$. By the KKT conditions, $x_v > 0$ solves $(\mathrm{P}_v)$ if and only if $Ax_v = b$ and there exists $\tilde{y}_v$ such that

$$c - VX^{-1}e = A^{\mathrm{T}}\tilde{y}_v.$$

Similarly, $(y_v, s_v)$ with $s_v > 0$ solves $(\mathrm{D}_v)$ if and only if $A^{\mathrm{T}}y_v + s_v = c$ and there exists $\tilde{x}_v$ such that

$$\begin{bmatrix} b \\ VS_v^{-1}e \end{bmatrix} = \begin{bmatrix} A\tilde{x}_v \\ \tilde{x}_v \end{bmatrix}.$$

Hence it follows that $\tilde{x}_v$ also solves $(\mathrm{P}_v)$. But, because $c^{\mathrm{T}}x + f(x)$ is strictly convex, the solution $x_v$ is unique. Therefore $\tilde{x}_v = x_v = VS_v^{-1}e$, and consequently $A^{\mathrm{T}}\tilde{y}_v = A^{\mathrm{T}}y_v$. Since $A$ is full row-rank, we must also have $\tilde{y}_v = y_v$. Therefore the KKT conditions for *both* $(\mathrm{P}_v)$ and $(\mathrm{D}_v)$ are $x_v, s_v > 0$, $Ax_v = b$, $A^{\mathrm{T}}y_v + s_v = c$, and $x_v = VS_v^{-1}e$. These can be rewritten as

$$\begin{aligned} A^{\mathrm{T}}y + s &= c \\ Ax &= b \\ XSe &= v \\ x, s &> 0. \end{aligned} \tag{2}$$

(c) (i) Take $v^0 := X^0 S^0 e$ and define the modified central path as

$$\{(x(\mu), y(\mu), s(\mu)) : (x(\mu), y(\mu), s(\mu)) \text{ solves (2) } \text{ for } v = \mu v^0, \text{ for } \mu > 0\}.$$

(ii) Define the modified neighborhood of the central path as

$$\mathrm{N}(\theta) := \{(x, y, s) : A^{\mathrm{T}} y + s = c, Ax = b, \ x, s > 0, \|XSe - \mu(x, s) v^0\| \leq \theta \mu(x, s)\},$$

where

$$\mu(x, s) := \frac{e^{\mathrm{T}} X S (V^0)^{-1} e}{n}.$$

(iii) Given $(x, y, s) \in \mathrm{N}(\theta)$ put $\mu^+ := \sigma \mu(x, s)$ for some appropriate $\sigma \in (0, 1)$ and solve

$$\begin{aligned} A^{\mathrm{T}} \Delta y + \Delta s &= 0 \\ A \Delta x &= 0 \\ S \Delta x + S \Delta x &= \mu^+ v^0 - XSe \end{aligned}$$

then set

$$(x^+, y^+, s^+) := (x, y, s) + (\Delta x, \Delta y, \Delta s).$$

# ANSWER–Part 2 Question 6 (Discrete Mathematics)

(a) Let the random variable $Y$ be the number of copies of $C_4$ in $G_{n,p}$. We have

$$E[Y] = \frac{(n)_4}{8} \cdot p^4 \sim \frac{(np)^4}{8}.$$

(i) Since $Pr(Y \geq 1) \leq E[Y]$, we have $np \to 0$ implies $Pr(\mathcal{E}) = Pr(Y \neq 0) \to 0$.

(ii) Now $E[Y]$ goes to infinity. To establish that $Y > 0$ with high probability we use the second moment method. Let $\mathcal{A}$ be the collection of all sets $A$ of four edges in $\binom{[n]}{2}$ that form a copy of $C_4$ (i.e. potential copies of $C_4$ in our random graph). For each $A \in \mathcal{A}$ let $Y_A$ be the indicator random variable that is 1 if all the edges in $A$ appear in $G_{n,p}$ and is zero otherwise. Noting that $Y_A$ and $Y_B$ are independent if $A$ and $B$ have no edges in common, we have

$$\begin{aligned} Var[Y] &= \sum_{A,B \in \mathcal{A}: A \neq B} E[Y_A Y_B] - E[Y_A]E[Y_B] + \sum_{A \in \mathcal{A}} E[Y_A^2] - (E[Y_A])^2 \\ &= \sum_{A \in \mathcal{A}} 4 \cdot 2\binom{n-2}{2}(p^7 - p^8) + 4\binom{n-3}{1}(p^6 - p^8) + \sum_{A \in \mathcal{A}} (p^4 - p^8) \\ &\leq \frac{n^6 p^7}{2} + \frac{n^5 p^6}{2} + \frac{n^4 p^4}{8}. \end{aligned}$$

By an application of Chebyshev's inequality we have

$$Pr(Y = 0) \leq Pr(|Y - E[Y]| \geq E[Y]) \leq \frac{Var[Y]}{(E[Y])^2}.$$

Since $np$ tends to infinity as $n$ tends to infinity, we have that $\lim_{n \to \infty} Pr(Y = 0) = 0$.

(b) We first note that $G$ has a bipartite subgraph $H$ with at least $e/2$ edges. To see this, consider a bi-partition of the vertex set taken uniformly at random (note that we can generate such a partition by flipping an independent fair coin for each vertex to determine its location in the bi-partition). Let $X$ be the number of edges that cross this bi-partition. Each edge crosses with probability 1/2. Therefore $E[X] = e/2$. Since $Pr(X \geq E[X])$ is non-zero, the desired bipartite subgraph $H$ exists.

Now, let $X$ be a minimum vertex cover of $H$. We have

$$\frac{e}{2} \leq \sum_{v \in X} d_H(v) \leq \Delta |X| = \Delta \tau(H)$$

where $\tau(H)$ is the vertex cover number. As the König's Theorem stipulates that the vertex cover number and the matching number of a bipartite graph are equal, there is a matching in $H$ (and therefore in $G$) with at least $e/(2\Delta)$ edges.

# Questions for the Qualifier of January 2007

## Wednesday January 10, 2007

- **ACO Students - 4 hours:** You must answer Questions 1, 2, 3 and 4.
- **OM Students - 2 hours:** You must answer Questions 1, and 2.
- All questions are open-notes, open-book.
- Please begin each question on a new page. Write only on one side of every page. Always remember to put your identification number (NOT your name) in the top right-hand corner of every page you turn in.

1. (Graph Theory)

Given a data set $N$ consisting of $n$ points and an index of dissimilarity $w_{ij} \geq 0$ for every pair $i, j \in N$, $i \neq j$, we want to partition $N$ into two nonempty subsets, $N_1$ and $N_2$, so as to maximize the dissimilarity between the two subsets. Solve this problem for the following two objectives:

(a) maximize the minimum dissimilarity between any two points $i, j$ with $i \in N_1$ and $j \in N_2$

(b) minimize the maximum dissimilarity between any two points $i, j$ with either $\{i, j\} \subseteq N_1$ or $\{i, j\} \subseteq N_2$.

Justify your procedures by proving their correctness.

2. (Networks & Matchings)

Let $G = (V, E)$ be an undirected nonbipartite graph.

Basic feasible solutions to the system

$$\begin{aligned} &x \geq 0 \\ &x(\delta(i)) \leq 1, \quad \forall i \in V \end{aligned}$$

are called fractional matchings of *f-matchings*. You are asked to look at conditions under which $G$ has a maximum $f$-matching that is integer. Consider the following:

An odd cycle with vertex set $S$ is called *separable* if $G$ has a maximum matching that "separates" $S$ is the sense that $M \cap \delta(S) = \emptyset$.

Is there any connection between the properties (i) $G$ has a maximum $f$-matching that is integer, and (ii) $G$ has no separable odd cycle?

Does (i) imply (ii)? Does (ii) imply (i)? Why?

3. (Convex Polytopes)

Consider the polyhedra

$$
\begin{aligned}
P_1 &:= \{x \in \mathbb{R}^n : Ax \le b\} \\
P_2 &:= \{y \in \mathbb{R}^n : y = uA \text{ for some } u \ge 0, ub \le 1\}
\end{aligned}
$$

(a) What is the relationship between $P_1$ and $P_2$?

(b) What property of $A$ and $b$ is needed for $P_2$ to be the antiblocker of $P_1$?

(c) Let $\emptyset = P_1 \subseteq K := \{x \in \mathbb{R}^n : 0 \le x \le 1\}$. The outer polar of $P_1$ with respect to the coordinate system centered at $\frac{1}{2}e$ (where $e = (1, \dots, 1)$), is

$$P_1^0(\tfrac{1}{4}n) := \{y \in \mathbb{R}^n : (x - \tfrac{1}{2}e)(y - \tfrac{1}{2}e) \le \tfrac{1}{4}n,\ \forall x \in P_1\}.$$

The outer polar $K^0(\frac{1}{4}n)$ of $K$ is defined in the same way. Show that

(c 1) $P_1 \subseteq K \subseteq K^0(\frac{1}{4}n) \subseteq P_1^0(\frac{1}{4}n)$

(c 2) $P_1 \cap bd(P_1^0(\frac{1}{4}n)) = P_1 \cap \text{vert} K$

4. (Discrete Mathematics)

(a) Generalized Tic-Tac-Toe.

The game is played on a hypergraph $H = (V, E)$. Two players alternate claiming points of $V$, player one going first. Once claimed, a point cannot be claimed by another player. The first player to claim all the points of some $A \in E$ wins the game. If neither player achieves this the game ends in a draw.

In the following example of $[5]^2$ tic-tac-toe (the winning sets are the horizontal, vertical and diagonal lines), the second player to move has a **draw force pairing strategy** to prevent the first player from winning. Namely, at each round, if player one takes a square marked $i$, then player two responds by taking the other square marked $i$, if it is still available, and otherwise chooses some other unclaimed point.

| | | | | |
|---|---|---|---|---|
| 1 | 6 | 10 | 7 | 7 |
| 11 | 3 | 3 | 2 | 12 |
| 9 | 6 | * | 5 | 9 |
| 11 | 2 | 4 | 4 | 12 |
| 8 | 8 | 10 | 5 | 1 |

Suppose $H = (V, E)$ is $k$-uniform (every $A \in E$ has $|A| = k$) and $d$-regular (every $x \in V$ is contained in exactly $d$ members of $E$). Show that if $k \geq 2d$ then the second player has a draw force pairing strategy.

(b) Let $G = (V, E)$ be a graph. We consider two random experiments.

Let $H$ be a subgraph of $G$ chosen at random by independently picking each edges $e \in E$ to be an edge of $H$ with probability $1/2$. Let $p$ be the probability that $H$ is connected and spanning.

Let $f : E \to \{\text{Red}, \text{Blue}\}$ be a 2-coloring of the edge set of $E$ chosen uniformly at random; that is, suppose that for each $e \in E$ we independently set $Pr(f(e) = \text{Red}) = Pr(f(e) = \text{Blue}) = 1/2$. Let $G_B$ be the subgraph of $G$ given by the Blue edges (formally, $G_B = (V, \{e \in E : f(e) = \text{Blue}\})$), and let $G_R$ be the subgraph given by the Red edges. We define $q$ to be the probability that $G_R$ and $G_B$ are both connected and spanning.

Prove or disprove: $q \leq p^2$.

# Questions for the Qualifier of January 2007

## Thursday January 11, 2007

- **ACO Students - 3 hours:** You must answer Questions 1, 2, and 3.
- **OM Students - 3 hours:** You must answer Questions 1, 2, 4.
- All questions are open-notes, open-book.
- Please begin each question on a new page. Write only on one side of every page. Always remember to put your identification number (NOT your name) in the top right-hand corner of every page you turn in.

1. (Linear Programming)

The goal of this question is to prove the following result:

Assume $P \subseteq \mathbf{R}^n$ is a non-empty polyhedron, and $a, b \in \mathbf{R}^n$, $\alpha, \beta \in \mathbf{R}$ are given. Then the following two statements are equivalent:

**Statement A:**

*For all $x \in P$ we have $a^{\mathrm{T}}x \leq \alpha$ or $b^{\mathrm{T}}x \leq \beta$.*

**Statement B:**

*There exists $\lambda \in [0, 1]$ such that for all $x \in P$ we have $(\lambda a + (1-\lambda)b)^{\mathrm{T}}x \leq \lambda\alpha + (1-\lambda)\beta$.*

(a) (2pts) Prove that Statement B implies Statement A.

(b) (1pt) Assume $P \cap \{x \in \mathbf{R}^n : a^{\mathrm{T}}x > \alpha\} = \emptyset$. Prove that under this condition Statement A implies Statement B.

(c) (2pts) Assume $P \cap \{x \in \mathbf{R}^n : a^{\mathrm{T}}x > \alpha\} \neq \emptyset$. Prove that if Statement A holds, then the following statement holds as well:

**Statement C:** *For all $x \in P \cap \{x \in \mathbf{R}^n : a^{\mathrm{T}}x \geq \alpha\}$ we have $b^{\mathrm{T}}x \leq \beta$.*

(d) (2pts) Let $A \in \mathbf{R}^{m \times n}$, $c \in \mathbf{R}^m$ be such that $Q := \{x \in \mathbf{R}^n : Ax \leq c\} \neq \emptyset$. Prove that the following two statements are equivalent:

**Statement D:** *For all $x \in Q$ we have $b^{\mathrm{T}}x \leq \beta$.*

**Statement E:** *There exists $y \geq 0$ such that $A^{\mathrm{T}}y = b$ and $c^{\mathrm{T}}y \leq \beta$.*

(e) (3pts) Assume $P \cap \{x \in \mathbf{R}^n : a^{\mathrm{T}}x > \alpha\} \neq \emptyset$. Use (c) and (d) to prove that under this condition Statement A implies Statement B.

2. (Integer Programming)

Consider a MIP with constraints $Ax \geq b$, $x \geq 0$, $x_j \in \{0,1\}$, $j \in N_1 \subseteq N$, and let

$$x_k = \bar{a}_{k0} - \sum_{j \in J} \bar{a}_{kj} x_j \qquad (1)$$

be a row of the simplex tableau associated with nonbasic set $J$.

(a) Derive the simple disjunctive cut from $x_k \leq 0 \vee x_k \geq 1$ applied to (1). Can this always be done? If not, why? In what sense is this an intersection cut?

(b) Show how to strengthen this cut by using the integrality of $x_j$, $j \in J \cap N_1$. Why is this strengthening valid? What is the relationship of the resulting cut to the mixed integer Gomory cut from source row (1)?

(c) What is the deepest lift-and-project cut from $x_k \leq 0 \vee x_k \geq 1$, relative to a basic solution $\bar{x}$ to the LP relaxation? derive the cut generating linear program $(\text{CGLP})_k$. How does the cut obtained by solving $(\text{CGLP})_k$ compare with the cuts of (a) and (b)?

3. (Advanced Integer Programming)

Given a set of $n$ items available for processing on a machine one at a time without repetition, and a setup cost of $c_{ij} \geq 0$ for processing item $j$ right after item $i$ for all $i, j$, select at least $p$ items, where $p$ is a given integer satisfying $1 \leq p \leq n$, and order them into a processing sequence of minimum total setup cost.

(a) Formulate the problem as an integer program

(a 1) in variables corresponding to items and variables corresponding to pairs of items

(a 2) in variables corresponding to pairs of items only.

Show that formulation (a 2) is a projection of formulation (a 1)

(b) Outline a procedure for generating facets of the polytope associated with both formulations from facets of the asymmetric traveling salesman polytope

(c) Illustrate the procedure on facets of the ATS polytope corresponding to odd CAT inequalities involving 4 or 5 nodes.

# ANSWER–Part 1 Question 1 (Graph Theory)

(a) The problem can be stated as

$$\max_{(N_1,N_2)\in P} \min_{i\in N_1, j\in N_2} w_{ij}$$

where $P$ is the set of all partitions $(N_1, N_2)$ of $N$ into two nonempty subsets.

Solution:

- Find a minimum-weight spanning tree $T^*$ in the graph $G = (N, E)$ with edge weights $w_e$, $e \in E$
- If $e_0 = \arg\min \max_{e\in T^*} w_e$, the solution is the cutset of $G$ induced by $e_0$ and $T^*$.

*Proof.* Every cutset has an edge in $T^*$, and every edge of $T^*$ is a minimum-weight edge of the cutset defined by it and $T^*$. Since $e_0$ is maximum-weight among the edges of $T^*$, the cutset defined by $e_0$ and $T^*$ maximizes the minimum weight of any edge with one end in $N_1$ and the other in $N_2$.

□

(b) This problem can be stated as

$$\min_{(N_1,N_2)\in P} \max \left\{ \max_{\{i,j\}\in N_1} w_{ij}, \max_{\{i,j\}\in N_2} w_{ij} \right\}$$

where $P$ is as in (a).

Solution:

Find a maximum-weight spanning tree $T$ in $G$. Then the bipartition $(N_1, N_2)$ of $N$ induced by $T$ (where all the edges of $T$ join vertices in $N_1$ to those in $N_2$) is the solution.

*Proof.* Let

$$(i_*, j_*) = \arg\max \left\{ \max_{\{i,j\}\in N_1} w_{ij}, \max_{\{i,j\}\in N_2} w_{ij} \right\}$$

for the constructed partition $(N_1, N_2)$. Let $C^*$ be the cycle created by adding $(i_*, j_*)$ to $T$. Then $w_{ij} \geq w_{i_*,j_*}$ for all $(i,j) \in C^*$, since $T$ is maximum-weight.

Since by choice $\{i_*, j_*\} \in N_1$ or $\{i_*, j_*\} \in N_2$, the path joining $i_*$ to $j_*$ in $T$ has an even number of edges; hence $C^*$ is an odd-length cycle. But then every bipartition $(N_1', N_2')$ of $N$ must have at least one edge of $C^*$ either in $G[N_1]$ or in $G[N_2]$, hence it cannot be better than $(N_1, N_2)$.

□

# ANSWER–Part 1 Question 2 (Networks & Matchings)

(i) implies (ii).

*Proof.* We prove the counterpositive. Let $M$ be a maximum matching that separates an odd cycle $C$ with vertex set $S$. Then removing the $\lfloor |S|/2 \rfloor$ edges of $M$ contained in $C$ and assigning a value of $\frac{1}{2}$ to all $|S|$ edges of $C$ yields an $f$-matching of value $|M| + \frac{1}{2}$. □

(ii) implies (i).

*Proof.* Again we prove the counterpositive. Suppose $\hat{x}$ is a maximum $f$-matching of value strictly larger than a maximum matching. We will then show that $G$ has a separable odd cycle. Assume wlog that $\hat{x}$ has fractional components for a minimum number of odd cycles $C_i$ with vertex sets $S_i$, $i = 1, \ldots, q$, and let $S = \bigcup_{i=1}^{q} S_i$. The edges $e$ corresponding to $\hat{x}_e = 1$ define a maximum matching $M'$ in $G[V \setminus S]$. For $i = 1, \ldots, q$, let $M_i$ be a maximum matching in $C_i$. Then $M := M' \cup C_1 \cup \ldots \cup C_q$ is a matching that separates $C_i$, $i = 1, \ldots, q$.

It remains to be shown that $M$ is maximum. Suppose not; then there exists an $M$-augmenting path $P$. Wlog we may assume that $P$ contains no edge of any $C_i$ (otherwise $M_i$ can be changed so that the first vertex of $S_i$ that $P$ meets is exposed, hence $P$ stops). Also, since $M'$ is maximum in $G(V \setminus S)$, $P$ has at least one end in $S$. But then augmenting along $P$ produces a matching $M^*$ with $|M^*| = |M| + 1$. From $M^*$ one can then construct an $f$-matching of value at least equal to that of $\hat{x}$ and with fractional components for at least one less odd cycle, contrary to the assumption on $\hat{x}$. □

# ANSWER–Part 1 Question 3 (Convex Polytopes)

(a) If $P_1 \neq \emptyset$, $P_2$ is the polar $P_1^0$ of $P_1$.

*Proof.* Let $P_1 \neq \emptyset$ and $y \in P_2$. Then there exists $u \geq 0$ such that $uA = y$, $ub \leq 1$. Multiplying $y = uA$ by $x$ yields $yx = uAx \leq ub \leq 1$, i.e. $y \in P_1^0$. □

(b) If $P_1 \subseteq \mathbb{R}^n_+$ and in the definition of $P_2$ we require $y \in \mathbb{R}^n_+$, then $P_2$ is the antiblocker of $P_1$.

(c 1) $P_1 \subseteq K$ by assumption. Let $x^* \in K$, i.e. $0 \leq x^* \leq 1$; then $x^* \in K^0(\frac{1}{4}n)$ if and only if

$$\max_{x \in K}\{(x^* - \tfrac{1}{2}e)(x - \tfrac{1}{2}e)\} \leq \tfrac{1}{4}n.$$

The maximum of the left hand side is attained for

$$\overline{x}_j = \begin{cases} 1 & j \in N^+ := \{j \in N : x_j^* > \frac{1}{2}\} \\ 0 & j \in N^0 := \{j \in N : x_j^* \leq \frac{1}{2}\}, \end{cases}$$

and

$$\begin{aligned} (x^* - \tfrac{1}{2}e)(\overline{x} - \tfrac{1}{2}e) &= \tfrac{1}{2}\textstyle\sum_{j \in N^+}(x_j^* - \frac{1}{2}) + \frac{1}{2}\sum_{j \in N^0}(\frac{1}{2} - x_j^*) \\ &\leq \tfrac{1}{4}n. \end{aligned}$$

Hence $K \subseteq K^0(\frac{1}{4}n)$.

Finally, $K^0(\frac{1}{4}n) \subseteq P_1^0(\frac{1}{4}n)$ follows from $P_1 \subseteq K$ and the inclusion-reversing property of polarity.

(c 2) Let $x \in P_1 \cap \text{bd}(P_1^0(\frac{1}{4}n))$. Then $x \in K$ and $xy = \frac{1}{4}n$ for some $y \in P_1 \subseteq K$. But then $x_i = y_i$ and $|x_i| = |y_i| = \frac{1}{2}$ for $i = 1, \ldots, n$, i.e. $x \in$ vert $K$.

## ANSWER–Part 1 Question 4 (Discrete Mathematics)

(a) A draw forcing pairing strategy exists if and only if there is collection of pairwise disjoint sets $\{A_e : e \in E\}$ such that $A_e \subseteq e$ and $|A_e| = 2$ for each $e \in E$. When player one picks an element of $A_e$, player two picks the other element of $A_e$ unless he has already picked it (in which case he picks any other unclaimed point). Player one will never totally occupy any $A_e$ and hence never any $e \in E$.

Form a bipartite graph $G$ with bipartition $X \cup Y$ where

$$X = \{a_e : e \in E\} \cup \{b_e : e \in E\}$$

and $Y = V$. Note that $|X| = 2|E|$. We place an edge between $a_e$ and $v$ in this graph if $v \in e$. We also place an edge between $b_e$ and $v$ if $v \in e$. We claim that $G$ has an $X$-perfect matching $M$. This gives the collection of sets $A_e$ needed for the draw forcing pairing strategy: For each $e \in E$ let $A_e$ be the vertices in $V$ that are matched with $a_e$ and $b_e$ in $M$.

To prove that the $X$-perfect matching $M$ exists, we apply Hall's Theorem. Let $X' \subseteq X$ and define $N(X')$ to be the set of neighbors of $X'$ in $G$. Let $m$ be the number of edges from $X'$ to $N(X')$. We have $|X'|k = m \leq (2d)|N(X')|$. (Since $H$ is $k$ uniform, each point in $X$ has degree $k$ and since $H$ is $d$-regular each point in $v$ has degree $2d$.) Since $k \geq 2d$ this implies $|X'| \leq |N(X')|$.

(b) In the first experiment each subset of $E$ is equally likely to be the edge set of $H$. Let $\mathcal{F} \subseteq 2^E$ be the collection of edge sets that give connected, spanning subgraphs. We have

$$p = \frac{|\mathcal{F}|}{2^{|E|}}.$$

On the other hand, in the second experiment both graphs are connected and spanning if and only if

$$\{e : f(e) = \text{Red}\}, \{e : f(e) = \text{Blue}\} \in \mathcal{F} \quad \Leftrightarrow \quad \{e : f(e) = \text{Red}\} \in \mathcal{F} \cap \{\overline{X} : X \in \mathcal{F}\}.$$

Therefore,

$$q = \frac{|\mathcal{F} \cap \{\overline{X} : X \in \mathcal{F}\}|}{2^{|E|}}.$$

Note that $\mathcal{F}$ is upwardly closed (i.e. a filter) while $\{\overline{X} : X \in \mathcal{F}\}$ is downwardly closed (i.e. an ideal). Therefore, Kleitman's Lemma implies

$$2^{|E|}|\mathcal{F} \cap \{\overline{X} : X \in \mathcal{F}\}| \leq |\mathcal{F}| \cdot |\{\overline{X} : X \in \mathcal{F}\}| = |\mathcal{F}|^2.$$

Dividing by $2^{2|E|}$ gives $q \leq p^2$.

# ANSWER–Part 2 Question 1 (Linear Programming)

(a) (2pts) Prove that Statement B implies Statement A.

Let $x \in P$ be given. Then

$$(\lambda a + (1-\lambda)b)^{\mathrm{T}} x \leq \lambda\alpha + (1-\lambda)\beta,$$

i.e.,

$$\lambda(a^{\mathrm{T}}x - \alpha) + (1-\lambda)(b^{\mathrm{T}}x - \beta) \leq 0.$$

Since $\lambda \in [0,1]$ we have $\min\{u,v\} \leq \lambda u + (1-\lambda)v$ for all $u, v \in \mathbf{R}$. Thus

$$\min\{a^{\mathrm{T}}x - \alpha, b^{\mathrm{T}}x - \beta\} \leq \lambda(a^{\mathrm{T}}x - \alpha) + (1-\lambda)(b^{\mathrm{T}}x - \beta) \leq 0.$$

Therefore, we have $a^{\mathrm{T}}x \leq \alpha$ or $b^{\mathrm{T}}x \leq \beta$. Since this holds for any $x \in P$ we conclude that Statement A holds.

(b) (1pt) Assume $P \cap \{x \in \mathbf{R}^n : a^{\mathrm{T}}x > \alpha\} = \emptyset$. Prove that under this condition Statement A implies Statement B.

From this condition it follows that $a^{\mathrm{T}}x \leq \alpha$ for all $x \in P$. Thus Statement B holds for $\lambda = 1$.

(c) (2pts) Assume $P \cap \{x \in \mathbf{R}^n : a^{\mathrm{T}}x > \alpha\} \neq \emptyset$. Prove that if Statement A holds, then the following statement holds as well:

**Statement C:** *For all $x \in P \cap \{x \in \mathbf{R}^n : a^{\mathrm{T}}x \geq \alpha\}$ we have $b^{\mathrm{T}}x \leq \beta$.*

We shall prove the counterpositive, that is, we shall show that if Statement C does not hold, then Statement A does not hold either: Assume that there is some $\overline{x} \in P \cap \{x \in \mathbf{R}^n : a^{\mathrm{T}}x \geq \alpha\}$ such that $b^{\mathrm{T}}\overline{x} > \beta$. Take $\hat{x} \in P \cap \{x \in \mathbf{R}^n : a^{\mathrm{T}}x > \alpha\}$. Then for $\lambda > 0$ sufficiently small we have $\lambda\hat{x} + (1-\lambda)\overline{x} \in P \cap \{x \in \mathbf{R}^n : a^{\mathrm{T}}x > \alpha\}$ and $b^{\mathrm{T}}(\lambda\hat{x} + (1-\lambda)\overline{x}) > \beta$. Thus we have a point $\lambda\hat{x} + (1-\lambda)\overline{x} \in P$ with $a^{\mathrm{T}}(\lambda\hat{x} + (1-\lambda)\overline{x}) > \alpha$ and $b^{\mathrm{T}}(\lambda\hat{x} + (1-\lambda)\overline{x}) > \beta$. Hence Statement A does not hold.

(d) (2pts) Let $A \in \mathbf{R}^{m\times n}$, $c \in \mathbf{R}^m$ be such that $Q := \{x \in R^n : Ax \leq c\} \neq \emptyset$. Prove that the following two statements are equivalent:

**Statement D:** *For all $x \in Q$ we have $b^{\mathrm{T}}x \leq \beta$.*

**Statement E:** *There exists $y \geq 0$ such that $A^{\mathrm{T}}y = b$ and $c^{\mathrm{T}}y \leq \beta$.*

First, assume Statement E holds. Then for all $x \in Q$ we have $b^{\mathrm{T}}x = y^{\mathrm{T}}Ax \leq y^{\mathrm{T}}c \leq \beta$. The first inequality holds because $Ax \leq c$ and $y \geq 0$. Thus Statement D holds.

On the other hand, assume Statement D holds. Then the optimal value of the following linear program is at most $\beta$:

$$\begin{array}{rl} \max & b^{\mathrm{T}}x \\ \text{such that} & Ax \leq c \end{array}$$

In particular, this problem is feasible and bounded. Thus by strong duality, the dual

$$\begin{array}{rl} \min & c^{\mathrm{T}}y \\ \text{such that} & A^{\mathrm{T}}y = b \\ & y \geq 0 \end{array}$$

is feasible and has optimal value less than or equal to $\beta$. Hence there exists $y \geq 0$ such that

$$A^{\mathrm{T}}y = b,\ c^{\mathrm{T}}y \leq \beta.$$

(e) (3pts) Assume $P \cap \{x \in \mathbf{R}^n : a^{\mathrm{T}}x > \alpha\} \neq \emptyset$. Use (c) to prove that under this condition Statement A implies Statement B.

Assume $P = \{x : \tilde{A}x \leq \tilde{c}\}$ for some $\tilde{A} \in \mathbf{R}^{m\times n}$, $\tilde{c} \in \mathbf{R}^m$. Assume Statement A holds. Then by part (c) it follows that Statement C holds, which is the same as Statement D in part (d) for $A = \begin{bmatrix} \tilde{A} \\ -a^{\mathrm{T}} \end{bmatrix}$, $c := \begin{bmatrix} \tilde{c} \\ -\alpha \end{bmatrix}$. Observe that for this choice of $A, c$ we have $\{x : Ax \leq c\} = P \cap \{x \in \mathbf{R}^n : a^{\mathrm{T}}x \geq \alpha\} \supseteq P \cap \{x \in \mathbf{R}^n : a^{\mathrm{T}}x > \alpha\} \neq \emptyset$. Therefore, by part (d) it follows that there exist $y, t \geq 0$ such that

$$\tilde{A}^{\mathrm{T}}y - ta = b \ \text{ and } \ \tilde{c}^{\mathrm{T}}y - t\alpha \leq \beta.$$

Thus for $\overline{y} := \frac{t}{1+t}y \geq 0$ and $\lambda := \frac{t}{1+t} \in [0,1]$ we have

$$\tilde{A}^{\mathrm{T}}\overline{y} = \lambda a + (1-\lambda)b \ \text{ and } \ \tilde{c}^{\mathrm{T}}\overline{y} \leq \lambda\alpha + (1-\lambda)\beta.$$

Thus Statement E in part (d) holds for $A = \tilde{A}, c = \tilde{c}$. Observe that for this choice of $A, c$ we have $\{x : Ax \leq c\} = P \supseteq P \cap \{x \in \mathbf{R}^n : a^{\mathrm{T}}x > \alpha\} \neq \emptyset$. Hence by the equivalence between Statement D and Statement E in part (d), it follows that for all $x \in P$ we have $(\lambda a + (1-\lambda)b)^{\mathrm{T}}x \geq \lambda\alpha + (1-\lambda)\beta$, which is precisely Statement B.

# ANSWER–Part 2 Question 2 (Integer Programming)

(a) The simple disjunctive cut from $x_k \le 0 \ \vee \ x_k \ge 1$ applied to (1) is $\alpha x \ge 1$, with

$$\alpha_j = \max\{\frac{\bar{a}_{kj}}{\bar{a}_{k0}}, \frac{-\bar{a}_{kj}}{1-\bar{a}_{k0}}\}, j \in J.$$

This is an intersection cut from the convex set $S := \{x \in \mathbb{R}^n : 0 \le x_k \le 1\}$ because the cut-hyperplane is defined by the $n$ intersection points of the extreme rays of the LP cone with the boundary of $S$.

(b) The cut $\alpha x \ge 1$ can be strengthened by replacing its coefficients $\alpha_j$ with

$$\bar{\alpha}_j = \begin{cases} \min\left\{\frac{\bar{a}_{kj}-\lfloor \bar{a}_{kj}\rfloor}{\bar{a}_{k0}}, \frac{-\bar{a}_{kj}+\lceil \bar{a}_{kj}\rceil}{1-\bar{a}_{k0}}\right\} & j \in J_1 := J \cap N_1 \\ \max\left\{\frac{\bar{a}_{kj}}{\bar{a}_{k0}}, \frac{-\bar{a}_{kj}}{1-\bar{a}_{k0}}\right\} & j \in J \setminus J_1 \end{cases}$$

This strengthened cut is the same as the mixed integer Gomory cut from source row (1).

(c) The lift and project cut from $x_k \le 0 \ \vee \ x_k \ge 1$ is obtained by describing the convex hull $C$ of the disjunctive set $\begin{pmatrix} Ax & \ge & b \\ 0 \le x & \le & 1 \\ x_k & \le & 0 \end{pmatrix} \vee \begin{pmatrix} Ax & \ge & b \\ 0 \le x & \le & 1 \\ x_k & \ge & 1 \end{pmatrix}$ as

$$\begin{array}{rlllllll} C = \{x : x - y & & & - & z & & \le & 0 \\ Ay & - & by_0 & & & & \ge & 0 \\ -y_k & & & & & & \ge & 0 \\ & & & & Az & - \ bz_0 & \ge & 0 \\ & & & & z_k & - \ z_0 & \ge & 0 \\ & & y_0 & & + & z_0 & = & 1 \\ & & & & & y_0, z_0 & \ge & 0\} \end{array}$$

and projecting $C$ onto the $x$-space, using the projection cone

$$\begin{array}{rllllllll} W := & \{(\alpha, \beta, u, u_0, v, v_0) : \\ & \alpha & - & uA & + & u_0 e_k & & & = 0 \\ & \alpha & & & & - \ vA & - & v_0 e_k & = 0 \\ & -\beta & + & ub & & & & & \ge 0 \\ & -\beta & & & & + \ vb & + & v_0 & \ge 0 \\ & & & & & & & u, u_0, v, v_0 & \ge 0\}. \end{array} \qquad (1)$$

Introducing a normalization, for instance

$$ue + u_0 + ve + v_0 = 1, \tag{2}$$

the deepest lift-and-project cut is $\overline{\alpha}x \geq \overline{\beta}$, where $\overline{\alpha}, \overline{\beta}$ are components of an optimal solution to

$$\min \overline{x}^T\alpha - \beta \tag{CGLP$_k$}$$

subject to (2), (3).

# ANSWER–Part 2 Question 3 (Advanced Integer Programming)

Add a node 0 with $c_{0j} = c_{j0} = 0$ for all $j$.

(a 1) Variables:

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is a pair of the sequence} \\ 0 & \text{else} \end{cases}$$

$$y_i = \begin{cases} 0 & \text{if item } i \text{ is selected} \\ 1 & \text{else} \end{cases}$$

$N := \{0, 1, \ldots, n\}$

$$\begin{array}{rcll} & \min cx & & \\ x(i,N) + y_i & = & 1 & i \in N \\ x(N,j) + y_j & = & 1 & j \in N \\ y_0 & = & 0 & \\ y(N) & \leq & n - p & \\ x(S,S) + y(S \setminus \{k\}) - y_\ell & \leq & |S| - 1 & \text{for all } S \subset N, \\ & & & 2 \leq |S| \leq p - 1, \\ & & & \text{and } k \in S,\ \ell \in N \setminus S \\ & x_{ij} \in \{0,1\}, y_j \in \{0,1\}, \forall i,j & & \end{array} \tag{1}$$

(a 2) Variables: $x_{ij}$ as above

$$\begin{array}{rcll} & \min cx & & \\ x(i,N) & \leq & 1 & i \in N - \{0\} \\ x(N,i) - x(i,N) & = & 0 & i \in N \\ x(0,N) & = & 1 & \\ x(N,N) & \geq & p & \\ x(k,N) + x(\ell,N) - x(S, N \setminus S) & \leq & 1 & S \subset N, 2 \leq |S| \leq p-1, \\ & & & k \in S, \ell \in N \setminus S \\ & x_{ij} \in \{0,1\}, \forall i,j & & \end{array} \tag{2}$$

**Proposition** The polytope $P_1$ defined by the constraints (2) with $x_{ij} \in \{0,1\}$ replaced by $x_{ij} \geq 0$ is the projection on the $x$-space of the polytope $P_2$ defined by the constraints of (1), with $x_{ij} \in \{0,1\}$, $y_{ij} \in \{0,1\}$ replaced by $x_{ij} \geq 0$, $0 \leq y_j \leq 1$ for all $i, j$.

*Proof.* Substituting for each $i \in N$ the expression for $y_i$ obtained from the first set of equations of (1) into the corresponding equation of the second set, we obtain the $n$ homogeneous equations of (2). Substituting the same expressions for $y_i$, $i \in N$, into $y(N) \leq n - p$, we obtain the inequality $x(N,N) \geq p$. Finally, substituting the same expressions for $y_i$, $i \in N$, into the last set of inequalities of (1), we obtain the last set of inequalities of (2).

□

(b) Setting $y_i = 0$, $i \in N$, in (1) yields the ATS polytope.

Let $\alpha x \leq \alpha_0$ be a facet defining inequality for the ATS polytope. Then there exist lifting coefficients $\beta_i$ for $y_i$, $i \in N$, such that $\alpha x + \beta y \leq \alpha_0$ is a facet defining inequality for $P_1$. Eliminating the $y_i$ from such a lifted facet defining inequality (by substituting from the first set of equations of (1)) yields a facet defining inequality for $P_2$.

(c) The odd CAT inequality for the ATS polytope

$$x_{12} + x_{13} + x_{32} + x_{34} + x_{43} \leq 2$$

can be lifted to

$$x_{12} + x_{13} + x_{32} + x_{34} + x_{43} + y_3 \leq 2,$$

which is facet defining for $P_1$. Substituting $y_3 = 1 - x(3, N)$ yields

$$x_{12} + x_{13} + x_{43} - x(3, N \setminus \{2, 4\}) \leq 1$$

which is facet defining for $P_2$.

# Questions for the Qualifier of January 2008

## Monday January 7, 2008

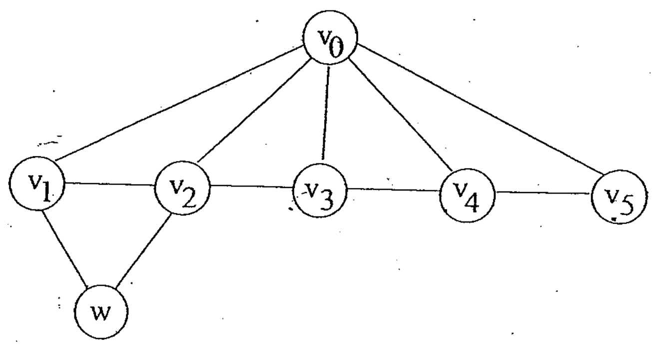
- **ACO Students - 4 hours:** You must answer Questions 1, 2, 3 and 4.
- All questions are open-notes, open-book.
- Please begin each question on a new page. Write only on one side of every page. Always remember to put your identification number (NOT your name) in the top right-hand corner of every page you turn in.

1. (Graph Theory)

Let $G = (V, E)$ be a simple graph and let $v_0 \in V$. Consider the following operations starting with $G = (V, E)$ and yielding graph $G(v_0) = (V'', E'')$:

1. Let $N[v_0] = \{v_1, \ldots, v_p\}$ be the neighbors of $v_0$ in $G$.
2. Let $G' = (V', E')$ be the graph obtained from $G$ by deleting the vertices in $v_0 \cup N[v_0]$.
3. Let $V'' := V' \cup \{v_{ij} \mid 1 \leq i < j \leq p \text{ and } (v_i, v_j) \notin E\}$.
4. Let $E'' = E' \cup \{(v_{ij}, v_{k\ell}) \mid i \neq k \text{ or } (v_j, v_\ell) \in E\} \cup \{(v_{ij}, v) \mid v \in V' \text{ with } (v_i, v) \in E \text{ or } (v_j, v) \in E\}$

(a) Construct the graph $G(v_0)$ for the graph $G$ below.




(b) Let $\alpha(G)$ be the size a maximum stable set in $G$ and $\alpha(G(v_0))$ be the size a maximum stable set in $G(v_0)$. Prove that $\alpha(G) - 1 = \alpha(G(v_0))$.

(c) Does this transformation yield a good algorithm for computing the stability number of a graph? Explain.

2. (Networks & Matchings)

For a graph $G = (V, E)$ and $S \subseteq V$, let $O(G-S)$ denote the number of odd components of $G - S$.

(a) Prove or disprove: Let $G = (V, E)$ be a tree. Then $G$ has a perfect matching if and only if, for all $S \subseteq V$ with $|S| \leq 1$, we have $O(G - S) \leq |S|$.

(b) Prove or disprove: Let $G = (V, E)$ be a connected graph with no cycle of length four or more. Then $G$ has a perfect matching if and only if, for all $S \subseteq V$ with $|S| \leq 1$, we have $O(G - S) \leq |S|$.

3. (Advanced Linear Programming)

Consider an equality-constrained linear programming problem

$$\min\{c^T x \mid Ax = b, x \geq 0\} ,$$

where $A$ is an $m \times n$ matrix. Path-following methods solve the problem by iteratively solving the primal-dual system $F(x, \lambda, s) = 0$, $(x, s) \geq 0$ with a modified Newton method, where

$$F(x,\lambda,s) = \begin{bmatrix} A^T\lambda + s - c \\ Ax - b \\ XSe \end{bmatrix} .$$

If $(x, \lambda, s)$ is the current iterate, the next iterate is $(x, \lambda, s) + \alpha(\Delta x, \Delta\lambda, \Delta s)$, where $(\Delta x, \Delta\lambda, \Delta s)$ solves

$$J(x,\lambda,s) \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -XSe + \sigma\mu e \end{bmatrix} . \tag{1}$$

The goal is to reduce $\mu$ to zero.

We wish to modify the path-following method to solve an inequality-constrained problem $\min\{c^T x \mid Ax \geq b, x \geq 0\}$ by searching $(x, \lambda, s, t)$-space. Here $t$ is a vector of surplus variables in the constraints, which become $Ax - t = b$.

(a) The primal-dual system is now $F(x, \lambda, s, t) = 0, (x, \lambda, s, t) \geq 0$. What is $F(x, \lambda, s, t)$?

(b) The system (1) is now

$$J(x,\lambda,s,t) \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \\ \Delta t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -XSe + \sigma\mu e \\ -\Lambda Te + \tau\mu e \end{bmatrix} \tag{2}$$

where $\mu = (x^T s + \lambda^T t)/(m + n)$. Note that there are two centering parameters $\sigma, \tau$. What is $J(x, \lambda, s, t)$?

(c) Show that $\Delta x^T \Delta s + \Delta\lambda^T \Delta t = 0$.

(d) Derive an expression (solely in terms of $\mu$, $\sigma$, $\tau$, $m$, $n$, and $\alpha$) for the amount by which $\mu$ is reduced after taking the modified Newton step.

4. (Discrete Mathematics)

Let $\binom{X}{k} = \{Y \subseteq X : |Y| = k\}$ denote the set of all $k$-subsets of $X$.

(a) Prove the following claim. Let $X$ be an infinite set and $k$ be a positive integer. Let $c : \binom{X}{2} \to \mathbb{N}$ be a coloring of $\binom{X}{2}$ such that

$$\Big|\{\, c(\{x, y\}) : y \in X \setminus \{x\} \,\}\Big| \leq k, \quad \text{for every } x \in X, \tag{1}$$

that is, every element $x \in X$ sees at most $k$ different colors. Then there is an infinite set $Y \subseteq X$ such that $\binom{Y}{2}$ is monochromatic.

(b) Does the statement of (a) remain valid if we weaken Assumption (1) by requiring only that every element of $X$ sees finitely many colors, that is, there is a function $k : X \to \mathbb{N}$ such that

$$\Big|\{\, c(\{x, y\}) : y \in X \setminus \{x\} \,\}\Big| \leq k(x), \quad \text{for every } x \in X?$$

(c) Let $R(k, k, k)$ be the 3-color Ramsey number, that is, the smallest integer $n$ such that every coloring of $\binom{\{1,\ldots,n\}}{2}$ with 3 colors yields a $k$-set $Y \subseteq \{1, \ldots, n\}$ with $\binom{Y}{2}$ monochromatic. Prove that $R(k, k, k) > 3^{k/2}$ for all sufficiently large $k$.

# Questions for the Qualifier of January 2008

## Tuesday January 8, 2008

- **ACO Students - 3 hours:** You must answer Questions 1, 2, and 3.
- **OM Students - 4 hours:** You must answer Questions 1, 2, 4, and 5.
- All questions are open-notes, open-book, except question 4 (Probability and Stochastic Processes). That question is a closed book question and no reference of any kind can be used while answering that question.
- Please begin each question on a new page. Write only on one side of every page. Always remember to put your identification number (NOT your name) in the top right-hand corner of every page you turn in.

1. (Linear Programming)

Consider the linear program

$$\max\{cx : Ax \leq b,\ 0 \leq x \leq u\} \tag{LP}$$

where the $m \times n$ matrix $A$, the $m$-vector $b$ and the $n$-vectors $c$, $u$, are given, with $u > 0$ and $A$ of full row rank, and such that (LP) has an optimal solution.

(a) Show that any basic feasible solution to LP in which all components of the slack vector $s := b - Ax$ are basic, satisfies the condition

$$x(u - x) = 0 \tag{1}$$

(b) Consider the following heuristic for finding a "good" solution that satisfies (1):

0. Start with an optimal solution $x^*$ and the associated simplex tableau.
1. Choose a nonbasic slack variable $s_j$ whose reduced cost is closest to 0. If there is none, stop: the current solution satisfies (1).
2. Introduce $s_j$ into the basis through a pivot that maintains primal feasibility. Return to 1.

Is this procedure guaranteed to find a solution satisfying (1)? What difficulties do you expect to encounter? Any ideas for overcoming them?

2. (Integer Programming)

Give a procedure for restating an arbitrary 0-1 program (of unknown feasibility) as one in the same variables but with a constraint set having only coefficients equal to 0, 1 or -1 on the lefthand side.

First prove that such equivalent formulations always exist, then show how to find one. Evidently, you are expected to prove that the formulation your procedure finds is equivalent to the original. Is your formulation unique? If not, what distinguishes the formulation your procedure finds from those found by other procedures?

Finally, illustrate your procedure on the following instance:

$$\begin{array}{rcrcrcrcl} 2x_1 & - & x_2 & - & 5x_3 & + & 6x_4 & = & 1 \\ -x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 2 \end{array}$$

3. (Advanced Integer Programming)

Consider the Asymmetric Traveling Salesman Problem (ATSP) where in addition every tour must satisfy some constraints of the form $i \prec j$, meaning that given a fixed "home" city 1, in any feasible tour city $i$ must be visited before city $j$. Let $B := \{(i,j) \in A : i \prec j\}$. Note that the relation $\prec$ is transitive, i.e., if $(i,k) \in B$ and $(k,j) \in B$, then $(i,j) \in B$. For any set $S \in V \setminus \{1\}$, let

$$\pi(S) := \{i \in V \setminus \{1\} : (i,j) \in B \text{ for some } j \in S\}$$

and

$$\sigma(S) := \{j \in V \setminus \{1\} : (i,j) \in B \text{ for some } i \in S\}$$

denote the predecessors and successors, respectively, of $S$. Thus $\pi(V \setminus \{1\})$ is the set of all predecessor nodes and $\sigma(V \setminus \{1\})$ is the set of all successor nodes of $G$.

Formulate the above described problem $\mathcal{P}$ on the complete digraph $G = (V, A)$ as an integer program, by starting with a formulation of the standard ATSP and adding additional constraints derived from the following considerations:

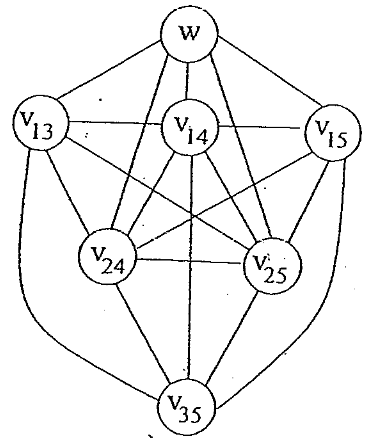
1. Certain arcs cannot be part of any feasible tour. Identify the set $F$ of these forbidden arcs, i.e. show that no arc of $F$, but every arc of $A \setminus F$, can be contained in a feasible tour.
2. Derive a set of valid inequalities from the condition that if $(i,j) \in B$, then no elementary path $P(j,i)$ from $j$ to $i$ can form a segment of a feasible tour (i.e., can be contained in its entirety in a feasible tour). Show that appending all the "path inequalities" to the standard formulation of the ATSP yields a valid formulation of $\mathcal{P}$.
3. Show that a more concise formulation results from replacing the path inequalities for a given pair $j, i$ by the stronger inequalities that put an upper bound for any set $Q \in V \setminus \{1, i, j\}$, on the total number of arcs in $(j, Q) \cup (Q, Q) \cup (Q, i)$ that can belong to a feasible tour. What is the upper bound? Why are these inequalities valid? In what sense are they stronger than the path inequalities for the same pair $j, i$?
4. The cutset inequalities $x(S, \overline{S}) \geq 1$ for the ATSP can be strengthened for $\mathcal{P}$ in

several ways. Which of the following inequalities is valid for $\mathcal{P}$ and why?

$$
\begin{aligned}
x(S \setminus \pi(S), \overline{S}) &\geq 1 \\
x(S, \overline{S} \setminus \pi(S)) &\geq 1 \\
x(S \setminus \pi(S), \overline{S} \setminus \pi(S)) &\geq 1.
\end{aligned}
$$

# ANSWER–Part 1 Question 1 (Graph Theory)

(a) The resulting graph $G(v_0)$ is




(b) We first prove that $\alpha(G) - 1 \leq \alpha(G(v_0))$. Let $S$ be a maximum stable set in $G$. If $S' := S \cap (v_0 \cup N[v_0])$ has cardinality $\leq 1$, then as $S'' := S - S'$ is stable in $G''$, we are done. Otherwise, $S' = \{v_{i_1}, \ldots, v_{i_k}\}$ contains at least two nodes from $N[v_0]$. Assuming that $i_1 < \ldots < i_k$, the $k-1$ nodes $W := \{v_{i_1 i_2}, \ldots, v_{i_1 i_k}\}$ are stable in $G''$ and $(S - S') \cup W$ is a stable set in $G''$ with cardinality $\alpha(G) - 1$.

We now prove that $\alpha(G) - 1 \geq \alpha(G(v_0))$. Let $S''$ be a maximum stable set in $G''$. If $S''$ does not contain a vertex $v_{ij}$, then $S'' \cup v_0$ is stable in $G$ and we are done. Otherwise, the vertices $v_{ij}$ in $S''$ must all share the same first index, by step 4, say $\{v_{ij_1}, \ldots, v_{ij_t}\}$. Then $S'' \cup \{v_i, v_{j_1}, \ldots, v_{j_t}\}$ is stable in $G$.

(c) No, this does not give a good algorithm, as the number of vertices may explode. A reduction of 1 in the size of the stable set might require a graph with a number of vertices that is squared.

# ANSWER–Part 1 Question 2 (Networks & Matchings)

It is clear that if $O(G-S) > |S|$ for some $S \subseteq V$ with $|S| \leq 1$, then $G$ has no perfect matching, as implied by Tutte's condition. It remains to prove the converse.

Assume that $O(G-S') \leq |S'|$ for all $S' \subseteq V$ with $|S'| \leq 1$. This implies in particular that $|V|$ is even. Assume for a contradiction that $G$ does not have a perfect matching. By Tutte's condition, there exists $S \subseteq V$ with $O(G-S) > |S|$. Consider the smallest such $S$ and note that we have $|S| \geq 2$.

(a) Let $v_0$ be any vertex of $V$. Direct all edges in the tree away from $v_0$ to obtain an arborescence rooted at $v_0$. Let $s \in S$ such that no descendant of $s$ in the arborescence is in $S$. The components of $G - s$ are either *downstream components* (i.e. components containing only descendants of $s$) or the *upstream component* (i.e. the component containing $v_0$).

Let $U := S - s$. Since $O(G-s) = 1$ either one of the downstream components is odd or the upstream component is odd. In the former case we have $O(G-U) = O(S)-1 > |U|$, since the union of all downstream components together with $s$ is an even component merged with a unique component contained in the upstream component. In the latter case, we have $O(G-U) \geq O(S)-1 > |U|$ since the union of all downstream components together with $s$ is an odd component merged with a unique component contained in the upstream component. In both cases, we get $O(G-U) > |U|$, a contradiction with the minimality of $|S|$.

(b) We can assume without loss of generality that $G$ has no parallel edges and no loops. Then any cycle in $G$ has length 3. Observe that two distinct cycles in $G$ can share a single vertex at most, since otherwise $G$ would have a cycle of length 4 or more. Remove exactly one edge from each cycle in $G$ to obtain a tree $G'$. Let $v_0$ be any vertex of $G'$ and direct the edges of $G'$ away from $v_0$ to get an arborescence rooted at $v_0$. Direct the edges of $G - G'$ so that the resulting orientation of $G$ is without directed cycles.

Note that any node $w \in V$ has at most two entering arcs $(v_1, w)$ and $(v_2, w)$ in the oriented $G$. Indeed, $w$ has at most one entering arc $(v_1, w)$ from $G'$ and any other entering arc $(v_2, w)$ must be part of a cycle $C$ of length 3 in $G$. If $C$ does not contain $(v_1, w)$, then it must use two arcs $(w, v_3)$ and $(v_2, v_3)$ of $G'$, a contradiction since $v_3$ would have two entering arcs in $G'$.

If $w$ has two entering arcs $(v_1, w)$ and $(v_2, w)$ in the oriented $G$, then $(v_1, v_2) \in E$. It follows that for any $s \subseteq V$, $G - s$ has at most one upstream component. The proof of point (a) above applies.

# ANSWER–Part 1 Question 3 (Advanced Linear Programming)

(a)

$$F(x, \lambda, s, t) = \begin{bmatrix} A^T\lambda + s - c \\ Ax - t - b \\ XSe \\ \Lambda Te \end{bmatrix}.$$

(b) The system (2) becomes

$$\begin{bmatrix} 0 & A^T & I & 0 \\ A & 0 & 0 & -I \\ S & 0 & X & 0 \\ 0 & T & 0 & \Lambda \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \\ \Delta t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -XSe + \sigma\mu e \\ -\Lambda Te + \tau\mu e \end{bmatrix} \tag{1}$$

(c) Multiplying the first row of (1) by $\Delta x^T$, we have

$$\Delta x^T A^T \Delta\lambda + \Delta x^T \Delta s = 0$$

But the second row of (1) implies $\Delta x^T A^T = \Delta t^T$, and from the above we have $\Delta x^T \Delta s + \Delta\lambda^T \Delta t = 0$.

(d) The reduction in $\mu$ is

$$\begin{aligned} &\mu - \frac{(x + \alpha\Delta x)^T(s + \alpha\Delta s) + (\lambda + \alpha\Delta\lambda)^T(t + \alpha\Delta t)}{n + m} \\ &= \mu - \frac{x^T s + \lambda^T t + \alpha x^T\Delta s + \alpha s^T\Delta x + \alpha\lambda^T\Delta t + \alpha t^T\Delta\lambda}{n + m} \\ &= -\alpha\frac{x^T\Delta s + s^T\Delta x + \lambda^T\Delta t + t^T\Delta\lambda}{n + m} \end{aligned} \tag{2}$$

where the first equation is due to (c) and the second to the definition of $\mu$. Now, adding $e^T$ times the third row of (1) to $e^T$ times the fourth row, we get

$$\begin{aligned} e^T S\Delta x + e^T T\Delta\lambda + e^T X\Delta s + e^T\Lambda\Delta t = \\ -e^T XSe - e^T\Lambda Te + \sigma\mu e^T e + \tau\mu e^T e \end{aligned}$$

or

$$\begin{aligned} s^T\Delta x + t^T\Delta\lambda + x^T\Delta s + \lambda^T\Delta t &= -x^T s - \lambda^T t + \mu(n\sigma + m\tau) \\ &= -(m + n)\mu + \mu(n\sigma + m\tau) \\ &= (n(\sigma - 1) + m(\tau - 1))\mu \end{aligned}$$

Substituting this into (2), the reduction in $\mu$ is

$$\frac{n(1 - \sigma) + m(1 - \tau)}{n + m}\alpha\mu$$

If $\sigma = \tau$, the reduction is $(1 - \sigma)\alpha\mu$ as in the standard algorithm.

# ANSWER–Part 1 Question 4 (Discrete Mathematics)

(a) Choose an arbitrary $x_1 \in X$ and let $Y_0 = X$. Having a vertex $x_i \in X$ and an infinite set $Y_{i-1} \ni x_i$, we inductively define $x_{i+1}$ and $Y_i$ as follows. The pairs connecting $x_i$ to $Y' := Y_{i-1} \setminus \{x_i\}$ are colored in finitely many colors, so one of the colors appears infinitely offen. Let $Y_i \subseteq Y'$ be the corresponding (infinite) set of neighbors and let $x_{i+1}$ be an arbitrary vertex of $Y_i$.

Since each set $Y_i$ is infinite, this process does not terminate, producing an infinite sequence of nested sets $Y_0 \supseteq Y_1 \supseteq \dots$ and vertices $x_i \in Y_{i-1} \setminus Y_i$ for $i \in \mathbb{N}$. Also, for any positive integers $i < j$, we have $x_j \in Y_{j-1} \subseteq Y_i$ and the color of $\{x_i, x_j\}$ does not depend on $j$ (as long as $j > i$). Let $c'(x_i)$ be this common color. Let $X' = \{x_1, x_2, \dots\}$.

The function $c'$ assumes at most $k$ different values. Indeed, if $x_{i_1}, \dots, x_{i_{k+1}}$ get different $c'$-values, then for any $j > \max(i_1, \dots, i_{k+1})$, the pairs $\{x_{i_h}, x_j\}$, for $1 \le h \le k+1$, get pairwise different $c$-colors. Thus $x_j$ sees more than $k$ colors under the coloring $c$, contradicting our assumption.

Since $c' : X' \to \mathbb{N}$ assumes only $k$ possible values and $X$ is infinite, there is an infinite set $Y \subseteq \{x_1, x_2, \dots\}$ such that $c'$ assumes the same value, say 1, on all elements of $Y$. Then for any $x_i, x_j \in Y$ with $i < j$ we have $c(\{x_i, x_j\}) = c'(x_i) = 1$. Thus, $Y$ is the required set.

(b) No, the statement is false. A simple counterexample is given by the following coloring: let $X = \mathbb{N}$ and let the color of $\{i, j\}$ with $i < j$ be $i$.

(c) Let $k$ be large, $t = \lfloor 3^{k/2} \rfloor$, and $T = \{1, \dots, t\}$. Consider a random coloring of $\binom{T}{2}$, where each pair of elements of $T$ chooses its color uniformly at random, independently of all other choices. Then the expected number of $k$-sets $Y \subseteq T$ with $\binom{Y}{2}$ monochromatic is

$$\binom{t}{k} 3^{1-\binom{k}{2}} \le t^k 3^{-k^2/2} \times \frac{3^{1+k/2}}{k!} < t^k 3^{-k^2/2} \le 1.$$

Hence, there is a coloring without any such set, which shows that $R(k, k, k) > t$, as required.

ANSWER–Part 2 Question 1 (Linear Programming)

ANSWER–Part 2 Question 2 (Integer Programming)

ANSWER–Part 2 Question 3 (Advanced Integer Programming)

# Questions for the Qualifier of January 2009

## Part I – Monday January 5, 2009

- **ACO Students – 4 hours**: You must answer all questions 1, 2, 3, and one question out of 4 and 5 (you get to choose).
- All questions are open-notes, open-book.
- Please begin each question on a new page. Write only on one side of every page. Always remember to put your identification number (NOT your name) in the top right-hand corner of every page you turn in.

## 1. Graph Theory

Consider a graph $G = (V, E)$ with $|V| = n$ that has $k$ disjoint chordless cycles (holes) of length 4 and no other chordless cycles of length $\geq 4$. For fixed $k$, give a $O(n^2)$ procedure for finding a maximum clique in $G$ and identifying the $k$ 4-holes.

[Hint: Think of how you would proceed in the absence of 4-holes.]

## 2. Networks and Matching

Let $G = (V, E)$ be an undirected connected graph with a clique $K$ such that every odd cycle of $G$ contains a vertex of $K$, and $K$ is minimal with respect to this property. Give a polynomial time algorithm to find a maximum stable set in $G$.

[Hint: How would you proceed if $|K| = 1$?]

if $|K| = 1$ then removing that vertex in $G$ has no odd cycle
$\Rightarrow$ $G'$ is bipartite and finding stable set is easy since
I just need to determine. $S = V - VC$ and $VC =$ König [illegible] adding

In the max stable set I can have at most one node from the clique $K$.

$|K| = 1$ do the following.

Remove $K$ first [illegible]

put $K \in I$ and remove $G[K]$
[illegible]

## 3. Discrete Mathematics

1. Suppose $\mathcal{G} \subseteq 2^{[n]}$ has the property that it does not contain a chain of length $s+1$; to be precise, there do not exist $A_1, A_2, \ldots, A_{s+1} \in \mathcal{G}$ such that

$$A_1 \subset A_2 \subset \cdots \subset A_{s+1}.$$

Show that

$$\sum_{i=0}^{n} \frac{|\{A \in \mathcal{G} : |A| = i\}|}{\binom{n}{i}} \leq s.$$

2. Let $d$ be a positive integer and consider the following random graph model, known as $d$-out. Our vertex set is $[n]$. Each vertex $v$ chooses a list $v_1, v_2, \ldots, v_d$ of neighbors uniformly at random from $[n]$ with replacement. The edge set consists of all pairs of the form $\{v, v_i\}$ where $v \in [n]$ and $i = 1, \ldots, d$. Note that loops, which correspond to $v_i = v$ for some $i$, and multiple edges, which can arise if a vertex appears twice in a list $v_1, \ldots, v_d$ or if $x = y_i$ and $y = x_j$, can appear in the graph. The number of graphs in the probability space is $n^{dn}$, and each graph in the probability space has minimum degree at least $d$ and average degree $2d$.

Let the random variable $X$ give the number of triangles in $d$-out where $d = d(n)$ tends to infinity as $n$ tends to infinity but $d = o\left(n^{1/20}\right)$. (Formally, let a triangle be defined by a set of 3 edges. So there could be several triangles that share the same set of 3 vertices.) Show that

$$\lim_{n \to \infty} Pr(X = 0) = 0.$$

## 4. Advanced Linear Programming

Assume $A \in \mathbb{R}^{m\times n}$ and consider the "matrix game" problem

$$\min_{x\in\Delta_n} \max_{y\in\Delta_m} y^{\mathrm{T}}Ax. \tag{1}$$

Here $\Delta_n, \Delta_m$ denote the standard simplexes in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. In other words, $\Delta_k := \{z \in \mathbb{R}^k : z \geq 0, z_1 + \cdots + z_k = 1\}$ for $k = n, m$.

(a) (2pts) Prove that (1) can be recast as the primal-dual linear programming pair

$$\begin{array}{llll} \min\limits_{x,t} & t & \max\limits_{y,\tau} & \tau \\ & -Ax + te_m \geq 0 & & -A^{\mathrm{T}}y + \tau e_n \leq 0 \\ & e_n^{\mathrm{T}}x = 1 & & e_m^{\mathrm{T}}y = 1 \\ & x \geq 0 & & y \geq 0 \end{array} \tag{2}$$

where $e_n$ and $e_m$ are the vectors of all ones in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

(b) (1pt) Use (a) to show the "min-max/max-min" theorem:

$$\min_{x\in\Delta_n} \max_{y\in\Delta_m} y^{\mathrm{T}}Ax = \max_{y\in\Delta_m} \min_{x\in\Delta_n} y^{\mathrm{T}}Ax.$$

(c) (3pts) Assume $x^0 \in \Delta_n$, $y^0 \in \Delta_m$ are strictly positive, i.e., all of their components are greater than zero. Assume also that $t^0, \tau^0 \in \mathbb{R}$ are such that both $t^0 e_m > Ax^0$ and $\tau^0 e_n < A^{\mathrm{T}}y^0$.

Put $s^0 := t^0 e_m - Ax^0$ and $z^0 := A^{\mathrm{T}}y^0 - \tau^0 e_n$.

Prove that for some suitable weights $u \in \mathbb{R}^n_{++}, v \in \mathbb{R}^m_{++}$ the points $(x^0, t^0, z^0)$ and $(y^0, \tau^0, s^0)$ are the solutions to

$$\begin{array}{ll} \min\limits_{x,t,z} & t - \sum\limits_{j=1}^{n} u_j \log x_j - \sum\limits_{i=1}^{m} v_i \log z_i \\ & -Ax + te_m - z = 0 \\ & e_n^{\mathrm{T}}x = 1 \\ & x > 0 \\ & z > 0 \end{array}$$

and

$$\begin{array}{ll} \max\limits_{y,\tau,s} & \tau + \sum\limits_{j=1}^{n} u_j \log s_j + \sum\limits_{i=1}^{m} v_i \log y_i \\ & -A^{\mathrm{T}}y + \tau e_n + s = 0 \\ & e_m^{\mathrm{T}}y = 1 \\ & y > 0 \\ & s > 0 \end{array}$$

respectively.

(d) (4pts) Use (c) to propose a modification of a feasible interior-point path following algorithm that starts from the pair of points $(x^0, t^0, z^0), (y^0, \tau^0, s^0)$ and generates a sequence pair of points that converge to solutions to the primal-dual linear programming pair (2).

More precisely, proceed as follows:

(i) (2pts) Define an appropriate "modified central path".

(ii) (1pt) Define an appropriate "modified neighborhood of the central path".

(iii) (1pt) Define the steps that should constitute each main iteration of the algorithm.

5. **Convex Polyhedra**

Let $P := \{x \in \mathbb{R}^n : Ax \leq b\} \neq \emptyset$ and define

$$P^0 := \{y \in \mathbb{R}^n : xy \leq 1, \forall x \in P\}, \quad P^\# := \{y \in \mathbb{R}^n : xy \geq 1, \forall x \in P\}.$$

(i) Prove or disprove:

(a) $P^0 = \{y \in \mathbb{R}^n : y = uA \text{ for some } u \geq 0 \text{ such that } ub \leq 1\}$

(b) $P^0 = \{y \in \mathbb{R}^n : y = uA \text{ for some } u \geq 0 \text{ such that } ub \geq 1\}$

(c) $P^\# = \{y \in \mathbb{R}^n : y = uA \text{ for some } u \leq 0 \text{ such that } ub \geq 1\}$.

(ii) For the nonconvex set $P := \bigcup_{i \in Q} P_i$, where $P_i := \{x \in \mathbb{R}^n : A^i x \geq b^i\} \neq \emptyset, i \in Q$, prove or disprove:

(d) $P^\# = \{y \in \mathbb{R}^n : y = u^i A^i \text{ for some } u^i \geq 0,\ i \in Q, \text{ such that } u^i b^i \geq 1\}$.

# Questions for the Qualifier of January 2009

## Part II – Tuesday January 6, 2009

- **ACO Students – 3 hours**: You must answer questions 1, 2, and 3.
- **OM Students – 4 hours**: You must answer questions 1, 2, 4, 5, and 6.
- All questions are open-notes, open-book.
- Please begin each question on a new page. Write only on one side of every page. Always remember to put your identification number (NOT your name) in the top right-hand corner of every page you turn in.

## 1. Linear Programming

Consider a simple blending problem in which the objective is to formulate a minimum-cost mixture of which each ingredient $i$ comprises a fraction $x_i$ that is $p_i$ or greater. The problem can be written

$$\begin{aligned} &\min \sum_i c_i x_i \\ &\sum_i x_i \geq 1 \\ &x_i \geq p_i, \ i = 1, \ldots, n \end{aligned} \tag{1}$$

We suppose $0 < c_1 < c_2 < \cdots < c_n$, each $p_i > 0$, and $\sum_i p_i < 1$.

(a) The problem can be solved on inspection. What is the unique optimal solution?

(b) What is the unique optimal dual solution? Prove that it and your solution in (a) are optimal.

(c) How much can a given $p_i$ be increased without changing the optimal dual solution?

(d) Suppose that the simplex method is applied to (1) and starts with basic solution $x_i = p_i$ for $i = 1, \ldots, n-1$, and $x_n = 1 - \sum_{i<n} p_i$. Derive the reduced costs of all the nonbasic variables in this starting solution, including surplus variables. Show that the optimal solution is obtained in one simplex step, using Dantzig's pivoting rule (i.e., pivot on the most negative reduced cost).

## 2. Integer Programming

Let $S = \left\{ (x, y) \in \mathbb{Z} \times \mathbb{R} : \begin{array}{rcl} y & \geq & x + \frac{2}{3} \\ y & \geq & 0 \end{array} \right\}.$

(a) Show that $y \geq \frac{2}{3}(x+1)$ is a valid inequality for $S$.

(b) Show that $y \geq \frac{2}{3}(x+1)$ is a facet of conv $S$.

(c) Describe conv $S$ by a system of linear inequalities. Describe the recession cone of conv $S$ by a system of linear inequalities.

(d) Let $a_1, \ldots, a_n \in \mathbb{Z}$ and $S = \{(x_1, \ldots, x_n, y) \in \mathbb{Z}^n \times \mathbb{R} : y \geq \frac{2}{3} + \sum_{i=1}^{n} a_i x_i,\ y \geq 0\}$.

Is $y \geq \frac{2}{3}(\sum_{i=1}^{n} a_i x_i + 1)$ a valid inequality for $S$? Justify your answer.

(e) Assume that $\gcd(a_1, \ldots, a_n) = 1$.

Can you describe conv $S$ by a system of three linear inequalities? Justify your answer.

## 3. Advanced Integer Programming

Consider

$$S \quad \begin{cases} x = \bar{x} + \sum_{j \in N} a^j s_j \\ s_j \geq 0 \quad \text{for all } j \in N \\ x \in \mathbb{Z}^m. \end{cases}$$

Here $\bar{x}$ and $a^j$, $j \in N$, are given vectors in $\mathbb{R}^m$.
Suppose $\bar{x} \notin \mathbb{Z}^m$.
Let $C$ be any bounded closed convex set in $\mathbb{R}^m$ such that $\bar{x} \in \operatorname{int} C$ and $\mathbb{Z}^m \cap \operatorname{int} C = \emptyset$.

(a) Let $\alpha_j > 0$ be the largest value such that $\bar{x} + \alpha_j a^j$ belongs to $C$. Show that $\sum_{j \in N} \frac{s_j}{\alpha_j} \geq 1$ is a valid inequality for $S$.

(b) In the remaining questions, let

$$S \quad \begin{cases} x_1 = \frac{1}{2} + s_1 - s_2 \\ x_2 = \frac{1}{2} + s_3 - s_4 \\ s_j \geq 0 \quad i = 1, \ldots, 4 \\ x \in \mathbb{Z}^2. \end{cases}$$

Show that $s_1 + s_2 + s_3 + s_4 \geq 1$ is a valid inequality for $S$ by exhibiting a convex set $C$ as described in (a). Is the set $C$ a maximal convex set with no point of $\mathbb{Z}^2$ in its interior?

(c) Are the inequalities

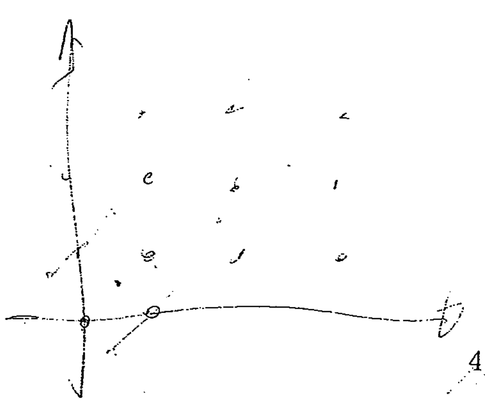
$$2s_1 + 2s_2 \geq 1$$

and

$$2s_3 + 2s_4 \geq 1$$

valid for $S$? Justify your answer.

(d) Is the inequality $s_1 + s_2 + s_3 + s_4 \geq 1$ a facet of $\operatorname{conv} S$? Justify your answer.



## 4. Dynamic Programming

At the beginning of year 0, Julie Ripe has an initial wealth of $W_0 > 0$ dollars. During year $t$, Julie chooses to consume $C_t$ dollars and invests the rest of her wealth in an index fund.

Each dollar invested in the index fund yields $R$ dollars at the beginning of the next year. Assume $R$ is lognormal with $\log R \sim N(0.1, 0.0225)$.

Julie's level of consumption in year $t$ is bounded above by the amount of wealth available that year. If Julie consumes $C_t$ dollars in year $t$, she earns a utility

$$U(C_t) = \log(C_t).$$

Julie's goal is to determine how much to consume each year so that she maximizes her total expected utility of consumption over the years $t = 0, 1, \ldots, T-1$ plus utility of final wealth:

$$\mathbb{E}\left[\sum_{t=0}^{T-1} \log(C_t) + \log(W_T)\right].$$

Here $W_T$ is the amount of wealth left at the beginning of year $T$. Assume Julie will die on year $T$ so her last year of consumption is $T-1$.

(a) (3pts) Formulate this problem as a sequential decision problem, or equivalently as a discrete-time system. Define clearly the stages, states, controls, law of motion, and objective.

(b) (2pts) Determine the value-to-go function at the last two stages, and the corresponding optimal decision rules.

(c) (2pts) Use Bellman equation to characterize, as explicitly as possible, the value-to-go functions and optimal decision rules at all stages.

(d) (3pts) Repeat the above with the following two additional features:

- In addition to the index fund, Julie can simply keep part of her unconsumed wealth in a money market account.

  A dollar invested in the money market account yields $\tilde{R}$ dollars at the beginning of next year, where $\tilde{R}$ is constant.

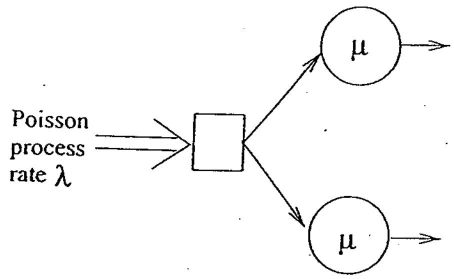
- Julie's goal is to determine how much to consume each year so that she maximizes her total expected discounted utility of consumption

$$\mathbb{E}\left[\sum_{t=0}^{T-1} \rho^t \log(C_t) + \rho^T \log(W_T)\right].$$

  for some discount factor $\rho \in (0, 1)$.

5. Performance Modeling (M/M/2/3)

Below is drawn a 2-server system with a waiting room which can hold only 1 job. Any arrival that sees 3 jobs in the system is dropped. Jobs arrival from outside according to a Poisson process with rate $\lambda = 1$. Whenever a server finished serving a job, it grabs the job from the waiting area, if there is one. Job sizes are Exponentially-distributed with rate $\mu = 1$.




(a) 11 pts Draw a CTMC where the state represents the total number of jobs in the system. Be careful! Many remaining parts depend on this.

(b) 11 pts Suppose that there are exactly 2 jobs in the system. What is the probability that a job arrives before a job completes?

(c) 11 pts Use your CTMC to determine the probability that the system is idle (both servers are idle).

(d) 11 pts What is the throughput of the system?

(e) 11 pts What is $\mathbf{E}[N]$, the expected number of jobs in the system?

(f) 11 pts What is $\mathbf{E}[T]$, the expected response time (for those jobs not dropped)?

(g) 11 pts If you're offered to replace the 2 servers with a single server that is twice as fast, should you go for it? Assume your goal is to minimize $\mathbf{E}[T]$ for those jobs not dropped. You don't have to derive the answer exactly – just be able to explain the intuition in words.

(h) 11 pts We say that the system becomes "idle" when the number of jobs drops to zero. What is the expected time from when the system becomes idle until it next becomes idle again?

(i) 12 pts Consider the process of arrivals to the system that aren't dropped. Is this a Poisson process? Why or why not?

## 6. Stochastic Processes

We are evaluating two potential configurations for a stochastic service system. For both systems, jobs arrive according to an iid interarrival process; job $n$ has interarrival times having distribution $T_n$, with finite mean $\lambda$ and variance $\sigma_T^2$.

For the first system, jobs will be served by a single server at rate 1. Job durations are iid; the duration of job $n$ is drawn according to the distribution $S_n$, with finite mean and variance $\mathrm{E}[S]$ and $\sigma_S^2$. Jobs will be served in this system according to FIFO, with no preemptions.

For the second system each task will be broken up into two parts which are done sequentially; the times to do these two tasks are mutually independent iid sequences: The first part of task $n$ has time $A_n$ then the second part has time $B_n$. The time to do these operations have finite mean and variance $\mathrm{E}[A]$ and $\sigma_A^2$, and $\mathrm{E}[B]$ and $\sigma_B^2$, respectively. A task is completed when both $A$ and $B$ have been completed; for task $n$ this takes a total processing time of $A_n + B_n$. Again a single server will serve jobs at rate 1. In this system, the server will try to do the same operations consecutively, so the server will start working on part $A$ of all jobs present, and will continue to work on jobs of type $A$ as long as any are present. As soon as there are no jobs of type $A$, the server will do all of the type $B$ jobs present. Once these are all done she will return to doing jobs of type $A$. Jobs within each type are completed according to FIFO; there are no preemptions.

For any job $\mathrm{E}[A] + \mathrm{E}[B] = \mathrm{E}[S]$ and $\sigma_A^2 + \sigma_B^2 = \sigma_S^2$. Also, assume $\lambda \mathrm{E}[S] < 1$.

Just to be perfectly clear, let's look at a concrete example. Assume the job is to put wheels on bicycles. I the first system, every time a bicycle appears the server will put both wheels on and then the job is complete. In the second system the server will put the front wheels on as many bicycles as possible. When all of the bikes present have their front wheels on, the server will put back wheels on all of the bikes that have had their front wheels put on. After all of these bikes are completed, the server will return to putting on front wheels. She will always put front wheels on first.

1. Prove that both system 1 and system 2 converge to a proper stationary distribution.
2. For both systems define a *busy period* as the time from when a job arrives to an empty system until the first time the system is empty again. How do the busy periods of the two systems compare? Prove what you are able to.
3. The completion time of a job is the time from when it arrives to when it leaves the system (or for the bicycle example the time from when it arrives until when it has both wheels on). How do the completion times of jobs compare in the two systems? Prove what you are able to.

# Questions for the Qualifier of January 2010

## Part I – Monday January 4, 2010

- **ACO Students – 4 hours**: You must answer all questions 1, 2, 3, and one question out of 4 and 5 (you get to choose).
- All questions are open-notes, open-book.
- Please begin each question on a new page. Write only on one side of every page. Always remember to put your identification number (NOT your name) in the top right-hand corner of every page you turn in.

## 1. Graph Theory

Given an undirected graph $G = (V, E)$ and integer $k \geq 1$, the $k^{th}$ power $G^k$ of $G$ is the graph on vertex-set $V$ and edge-set $\{(u, v) \mid u, v \in V,\ d(u, v) \leq k\}$. Here $d(u, v)$ denotes the shortest path distance between vertices $u$ and $v$ in $G$ (i.e. minimum number of edges in a path between $u$ and $v$).

1. (2 points) Prove or Disprove: If $G$ is 2-edge-connected, then $G$ contains a Hamilton cycle.
2. (2 points) Prove or Disprove: If $G$ is 3-vertex-connected, then $G$ contains a Hamilton cycle.
3. (3 points) Prove or Disprove: If $G$ is connected, then $G^3$ contains a Hamilton cycle.
4. (3 points) Prove or Disprove: If $G$ is connected, then $G^3$ contains a Hamilton path with end-points $u$ and $w$, for every $u, w \in V$.

## 2. Networks and Matchings

Consider a directed graph $N = (V, A)$ with integer arc capacities $c_a$, for $a \in A$. Let $s \in V$ be a source node and $t \in V$ a sink node, where $t \neq s$. Let $n = |V|$ and $m = |E|$. Assume that $c_a \leq n$ for all $a \in A$ and that a maximum flow from $s$ to $t$ has value $mn$.

(a) Give an upper bound (as a function of $m$ and $n$) on the computing time it takes to find a maximum flow using the Ford-Fulkerson augmenting path algorithm.

(b) Construct an example where the computing time it takes to find a maximum flow using the Ford-Fulkerson augmenting path algorithm is as close as possible to the upper bound found in (a).

(c) Give an upper bound (as a function of $m$ and $n$) on the computing time it takes to find a maximum flow using a shortest augmenting path algorithm.

(d) What is the best algorithm that you know to find a maximum flow in the above directed graph, i.e. an algorithm that gives the smallest upper bound on computing time, as a function of $m$ and $n$?

## 3. Discrete Mathematics

Let $w(k)$ be the smallest integer $n$ such that for every 2-coloring of the set $[n] := \{1, \ldots, n\}$ there is a monochromatic arithmetic progression of length $k$. (We allow only progressions whose difference is non-zero.)

i) Let $k \geq 2$ and let $a(n, k)$ be the number of arithmetic progressions of length $k$ that lie entirely inside $[n]$. Prove that if $a(n, k) \cdot 2^{1-k} \leq 1$ (where equality is allowed), then $w(k) \geq n$. Deduce that $w(k) \geq 2^{(1/2 - o(1))k}$ as $k \to \infty$. [2 Points]

ii) Define when an event $A$ is mutually independent of events $B_1, \ldots, B_n$. Give an example of three events $A, B, C$ such that every two are independent of each other while $A$ is not mutually independent of $B$ and $C$. [1 Point]

iii) Carefully state the Lovász Local Lemma (both versions, symmetric and asymmetric). Show how the asymmetric version implies the symmetric one. [4 Points]

iv) Improve the bound of Item i) to $w(k) \geq 2^{(1-o(1))k}$ as $k \to \infty$. [3 Points]

## 4. Advanced Linear Programming

Consider an LP defined over a box:

$$\begin{aligned} &\min\ c^T x \\ &0 \le x \le e \end{aligned} \tag{1}$$

where $c > 0$ and $e$ is a vector of ones. Because the variables can be rescaled, there is no loss of generality in considering the unit box.

(a) Write a closed-form expression for the projection onto $x$-space of the primal-dual central path.

(b) Given a centering parameter $\sigma$ and an interior primal-dual feasible point with $x$-coordinate $x^k$, write the projection $\Delta x$ onto $x$-space of the Newton direction in terms of $c$, $\sigma$, $\mu_k$, and $x^k$. Show that $\Delta x$ can be scaled so that $\Delta x = -c$ when $\sigma = 0$. Also show that for $\sigma > 0$, $\Delta x_i < -c_i$ when $x_i^k > 1/2$ and $\Delta x_i > -c_i$ when $x_i^k < 1/2$. This implies a centering effect when $\sigma > 0$.

## 5. Convex Polyhedra

Given a polyhedron

$$P := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : Ax + By \leq d\},$$

show how to project $P$ onto the subspace of the $x$-variables.

- Does the projection of $P$ have fewer, more, or the same number of facets as $P$?
- What is the connection between the dimension of $P$ and that of its projection?
- When is the projection of a facet of $P$ a facet of the projection of $P$?

# Questions for the Qualifier of January 2010

## Part II – Tuesday January 5, 2010

- **ACO Students – 3 hours**: You must answer questions 1, 2, and 3.
- **OM Students – 4 hours**: You must answer questions 1, 2, 4, 5, and 6.
- All questions are open-notes, open-book.
- Please begin each question on a new page. Write only on one side of every page. Always remember to put your identification number (NOT your name) in the top right-hand corner of every page you turn in.

## 1. Linear Programming

Given the linear program

$$z = \max\{cx : Ax \leq b,\ x \geq 0\}$$

and its dual

$$w = \min\{ub : uA \geq c,\ u \geq 0\},$$

characterize $z(b)$ and $z(c)$ (i.e. the primal optimum as a function of the righthand side, and as a function of the cost vector), as well as $w(b)$ and $w(c)$.

## 2. Integer Programming

Consider a MIP with the opimal solution to its LP relaxation given as

$$x_i = a_{i0} - \sum_{j \in J} a_{ij} x_j \qquad i \in I, \tag{1}$$

where $x_i \geq 0$, $i \in I \cup J$, and $x_i$ integer, $i \in I' \cup J'$. Further, let $0 < a_{h0} < 1$ for $h = i, k \in I'$.

Derive an intersection cut from the set $S$ defined by the inequalities

$$\begin{array}{rcrcrcl} 0 & \leq & x_i & + & x_k & \leq & 2 \\ -1 & \leq & x_i & - & x_k & \leq & 1 \end{array} \tag{2}$$

for some pair $i, k \in I'$.

(a) Show that $S$ is a suitable convex set for this role

(b) Show that the intersection cut from $S$ is equivalent to a disjunctive cut from a 4-term disjunction involving $x_i + x_k$ and $x_i - x_k$.

(c) Derive the cut from the 4-term disjunction

## 3. Advanced Integer Programming

Given a digraph $G$ with arc- and node-weights, formulate the problem of finding a directed cycle of length at least $k$ in $G$, that minimizes the sum of arc- and node-weights. Prove the correctness of your formulation.

Show how facet defining inequalities of the asymmetric traveling salesman polytope can be used to derive corresponding facet defining inequalities for the polytope resulting from your formulation.

2011

## Linear Programming Question

The *multicommodity flow problem* is a minimum cost network flow problem in which several types of flow are routed over a capacitated network. Let $c_{ijk}$ be the unit cost of sending commodity $k$ over arc $(i, j)$, and $s_{ik}$ the net supply of commodity $k$ at node $i$. Several commodities may flow over the same arc, subject to arc capacity $d_{ij}$. If $x_{ijk}$ is the flow of commodity $k$ on arc $(i, j)$, the problem can be stated

$$
\begin{aligned}
&\min \sum_{ijk} c_{ijk} x_{ijk} \\
&\sum_{k} x_{ijk} \le d_{ij}, \quad \text{all } i, j \qquad (i) \\
&\sum_{j} x_{ijk} - \sum_{j} x_{jik} = s_{ik}, \quad \text{all } i, k \qquad (ii) \\
&x_{ijk} \ge 0, \quad \text{all } i, j, k \qquad (iii)
\end{aligned}
\tag{1}
$$

The multicommodity problem lacks the nice properties of a standard network flow problem, but Dantzig-Wolfe decomposition can exploit the network structure. You can define the pricing subproblem so that it decouples into single-commodity flow problems.

(a) Write the restricted master problem that results.[1] You need not consider extreme rays. Why?

(b) Write the pricing subproblem.[1]

(c) The subproblem decouples into an uncapacitated single-commodity flow problem for each $k$. Write this problem. If $z_k$ is its optimal value, what is the optimal value of the subproblem as a whole?

(d) Suppose you solve (2) with the simplex method. How can you use the optimal dual solution of (2) to construct an optimal dual solution of the final restricted master problem, without actually solving the problem by Dantzig-Wolfe?

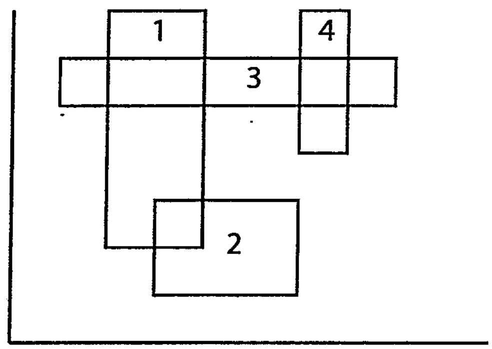
[According to Lawrence Rapp, "(2)" is supposed to be "(1)".]

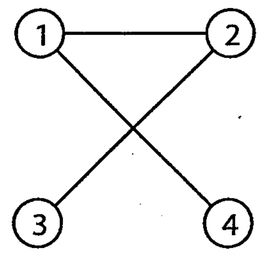
[1] Don't write a general restricted master or subproblem in terms of some coefficient matrix $A$. Write the master and subproblem for this specific problem.

2011

## Graph Theory Question

Let $G(\mathcal{F}) = (V, E)$ be the intersection graph of a family of $k$-dimensional rectangles in $\mathbb{R}^k$ whose sides are parallel to the cordinate system. Here is an example for $k = 2$:




(a) Show that there exist $k$ interval graphs $G_i = (V_i, E_i)$, $i = 1, \ldots, k$, such that $E = \bigcap_{i=1}^{k} E_i$.

(b) Based on (a), show that for fixed $k$, the number of maximal cliques of $G(\mathcal{F})$ is polynomial in the number of rectangles, and outline an algorithm for listing all maximal cliques.

(c) Is $G(\mathcal{F})$ perfect? (prove your answer)

2011

**Networks and Matchings Question**

Let $A$ be the $n \times m$ vertex-edge incidence matrix of an undirected graph $G = (V, E)$; then a maximum matching in $G$ is given by

$$\max\{1x : Ax \leq 1,\ x \in \{0,1\}^m\}, \qquad \text{EM}(G)$$

an integer program whose linear programming relaxation

$$\max\{1x : Ax \leq 1,\ x \geq 0\} \qquad \text{LEM}(G)$$

is known to have fractional basic solutions $x$ with $x_j \in \{0, \frac{1}{2}, 1\}$. It is also known that if $G$ is bipartite then LEM($G$) has integer basic solutions.

If $M$ is a matching and $C$ is an odd cycle with vertex set $S$, we say that $M$ separates $C$ if $M \cap (S, V \setminus S) = \emptyset$. An odd cycle $C$ of $G$ is called *separable* if $G$ has a maximum matching that separates $C$. Prove the following

**Proposition.** *If* LEM($G$) *has an integer optimum then* $G$ *has no separable odd cycle.*

2011

## Dynamic Programming Question

Consider a sales person who sells products in two cities: Pittsburgh and Cleveland. Let $P$ (or $C$) denote the random variable that represents the monthly sales in Pittsburgh (or in Cleveland, respectively). Based on his past experience, the sales person knows that $P$ dominates $C$ in the sense of first order stochastic dominance. Each month he can sell products in only one of the two cities. In the beginning of each month, he may stay in the same city as in the previous month or change the city. If he changes the city, the cost of $s$ ($> \max\{E[P], E[C]\}$) will occur. There is a monthly discount factor $\alpha$ ($\in (0,1)$). The objective of the sales person is to maximize the expected total discounted sales over infinite horizon.

(a) [3 points] Set up a dynamic program for this problem.

(b) [4 points] Let $V_\pi(i)$ denote the expected total discounted sales when the stationary policy $\pi$ is employed and the initial state is $i$. Find $V_\pi(i)$ for all feasible stationary policies $\pi$ and all states $i$. Show that an optimal policy is to stay in the city he starts in when $\alpha$ is sufficiently small.

(c) [3 points] Suppose $\alpha = 0.8$, $s = 4$, $E[P] = 2$, and $E[C] = 1$. Suppose the initial policy of the sales person is to change the city every month. Improve this initial policy by applying the policy improvement method once. (In fact, by repeating this procedure about 30 times in Excel, you can compute the optimal expected total discounted sales within an error of 0.01.)

2011

## Stochastic Processes (Performance Modeling) Question

**Stochastic Processes:** There is a holiday light display that changes colors between blue lights and red lights. People arrive to see the lights according to a Poisson process with rate 1 person every ten minutes.

The lights work as follow:

- People can only look at them one person at a time, so people may have to wait in line to see the lights.
- If there is no one looking at the lights, they turn off. They automatically start up as soon as a new person arrives.
- The blue light show lasts for an exponential amount of time with mean two minutes, and the red for an exponential amount of time with mean three minutes.
- After the light show of one color finishes, a new show starts (if someone is present) with a color randomly chosen with probability one half for red, and one half for blue. So there could be a sequence of shows of the same color lights.
- Likewise, when a new person arrives to an empty system, the color they see is chosen randomly between red and blue with probability one half each.
- Each person stays until they have seen one complete show of each color. So a person may have to watch a number of red shows before they see their first blue show, or a number of blue shows before they see a red one. Only after they see at least one show of each color will they leave. When that person leaves, if there is anyone in the line, they start seeing a new show, again with the color chosen randomly.

Questions:

1. If a customer arrives and the light show is going on (i.e. there is someone already watching, so the customer has to wait in line), can you say anything about the probability that the show currently going on is red? Justify your answer.
2. Can you say anything about the probability that the first show a customer sees is red? Justify your answer.
3. Find the mean time in queue (mean delay) for a customer.
4. What proportion of the mean delay in part 3 is due to (i) the expected time waiting for the person watching the lights (*in service*) when a customer arrives to finish; and what proportion is due to (ii) the expected time waiting for the people *in queue* when the customer arrives to watch the lights themselves.
5. Answer questions 1 - 4 if customers arrive twice as fast and the light shows are twice as fast, so lambda is one customer every five minutes and the mean duration is two minute for blue, and 1.5 minutes for red.
6. Answer questions 1 - 4 if the lights are on for a *deterministic* amount of time, two minutes for blue, and three minutes for red.

2011

## Discrete Mathematics Question

1. Let $\mathcal{F}$ be a collection of subsets of $[n] = \{1, 2, \ldots, n\}$ with the following property: If $A$ and $B$ are distinct sets in $\mathcal{F}$ then the symmetric difference of $A$ and $B$ contains at least $n/3$ elements of $[n]$. Show that such a collection $\mathcal{F}$ exists where $|\mathcal{F}| > e^{cn}$ for some constant $c$.

2. Let $\mathcal{G}$ be a collection of subsets of $[n]$ that is downwardly closed; that is, if $A \in \mathcal{G}$ and $B \subset A$ then $B \in \mathcal{G}$. (Such a collection of subsets of $[n]$ is also known as an ideal.) Prove that the average size of a set in $\mathcal{G}$ is at most $n/2$.

2011

## Integer Programming Question

An $m \times n$ matrix $A$ is *unimodular* if it has rank $m$, it is integral and $det(B) = 0, \pm 1$ for every $m \times m$ submatrix $B$ of $A$.

An $m \times n$ matrix $A$ is *totally unimodular* if every square submatrix of $A$ has a determinant equal to $0, \pm 1$.

Let $I$ denote the $m \times m$ identity matrix.

(a) (3 points) Let $A$ be an $m \times n$ totally unimodular matrix. Prove that $(A, I)$ is also totally unimodular.

(b) (3 points) Let $A$ be an $m \times n$ matrix. Prove that $A$ is totally unimodular if and only if $(A, I)$ is unimodular.

(c) (4 points) Let $A$ be an $m \times n$ totally unimodular matrix of full row rank and let $B$ be an $m \times m$ nonsingular submatrix of $A$. Prove that $B^{-1}A$ is totally unimodular.

2011

## Advanced Integer Programming Question

Consider integer constraints of the form

$$(S)\begin{cases} x_i = b_i + \sum_{j=p+1}^{n} a_{ij}x_j & \text{for } i = 1,\dots,p \\ x_i \in \mathbb{Z}_+ & \text{for } i = 1,\dots,n \end{cases}$$

where the vector $b = \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix} \notin \mathbb{Z}^p$. Assume $a^j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{pj} \end{pmatrix} \neq 0$ for $j = p+1,\dots,n$.

Let $K \subset \mathbb{R}^p$ be a closed convex set containing $b$ but no point of $\mathbb{Z}_+^p$ in its interior. Assume that $K$ is bounded. Let $\alpha_j = \max\{\alpha \in \mathbb{R}_+ : b + \alpha a^j \in K\}$.

(a) (5 points) Prove that $\sum_{j=p+1}^{n} \frac{x_j}{\alpha_j} \geq 1$ is a valid inequality for $(S)$.

(b) (2 points) Assume that $K$ is a polytope of the form

$$K = \{x \in \mathbb{R}^p : \sum_{i=1}^{p} c_i^k (x_i - b_i) \leq 1 \text{ for } k = 1,\dots,t\}.$$

Show that $\frac{1}{\alpha_j} = \max_{k=1,\dots,t} \sum_{i=1}^{p} c_i^k a_{ij}$.

(c) (3 points) Are the inequalities in (a) familiar to you? If so, give an example from the literature. What happens if one removes the assumption that $K$ is bounded? Does (b) still hold in this case?

2011

**Constraint Programming Question**

Define a global constraint

$$\texttt{path}((x_1, \ldots, x_k), G) \tag{2}$$

so that $(x_1, \ldots, x_k)$ satisfies the constraint if and only if $x_1, \ldots, x_k$ are the sequence of vertices on some directed path of the directed acyclic graph $G$. For example, if $G$ is the graph of Fig. 1, then $(x_1, x_2, x_3) = (2, 5, 7)$ satisfies (2), but $(x_1, x_2, x_3) = (1, 2, 4), (2, 1, 3)$ do not satisfy (2).

Consider the constraint set

$$\begin{array}{l} \texttt{path}((x_1, x_2, x_3), G) \\ \texttt{path}((x_2, x_3, x_4), G) \\ \texttt{path}((x_1, x_4, x_5), G) \end{array} \tag{3}$$

where $G$ is the graph of Fig. 1. Let the initial domain of $x_1$ be $\{1\}$ and the initial domain of $x_2, \ldots, x_5$ be $\{1, \ldots, 7\}$.

(a) What are the domains of $x_1, x_2, x_3$ if domain completeness[2] is achieved for the *first* constraint of (3)?

(b) Filter domains to achieve domain consistency for each constraint of (3), and propagate the reduced domains to the next constraint. Cycle through the constraints until a fixed point is reached (no further filtering is possible). What are the resulting domains of $x_1, \ldots, x_5$?

(c) After reducing domains as in (b), what is the maximum $k$ for which (3) is strongly $k$-consistent?

(d) Propagation to a fixed point achieves domain completeness for the entire constraint set (3). However, this is not the case for every constraint set. Provide a counterexample on a small graph $G$ (two path constraints are enough).

[2] The terms *domain completeness*, *domain consistency*, *hyperarc consistency*, and *generalized arc consistency* have the same meaning.

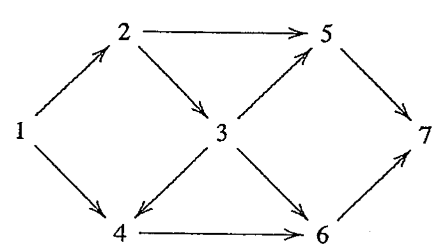



Figure 1: *The graph G.*

2011

## Convex Analysis Question

Let $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be closed convex functions.

**(a)** Let $s \in \mathbb{R}^n$. Prove that $f^*(s) + g^*(-s) \leq 0$ if and only if there exists $r \in \mathbb{R}$ such that

$$f(x) \geq \langle s, x \rangle + r \geq -g(x) \quad \text{for all} \quad x \in \mathbb{R}^n.$$

**(b)** Suppose $\bar{x} \in \mathbb{R}^n$ is such that $f(\bar{x}) = -g(\bar{x}) < \infty$. Prove that

$$\{s \in \mathbb{R}^n : f^*(s) + g^*(-s) \leq 0\} = \partial f(\bar{x}) \cap -\partial g(\bar{x}).$$

2011

## Convex Polyhedra Question

Let $S := \bigcup_{i \in Q} P_i$ where $P_i$, $i \in Q$ are convex polyhedra in $\mathbb{R}^n$, and let $H := \{x \in \mathbb{R}^n : ax = a_0\}$ be a hyperplane such that $S$ is contained in one of the halfspaces defined by $H$, say $S \subseteq H^+$. Let $T_1 := \text{conv}(H \cap S)$, $T_2 := H \cap \text{conv}S$.

Which of the relations $T_1 \subseteq T_2$, $T_1 \supseteq T_2$, $T_1 = T_2$ holds?

2011

## Advanced Linear Programming Question

Given a linear program of the form

$$\begin{array}{rrcl} \min & cx & & \\ & Ax & \geq & b \\ & x & \geq & 0 \end{array} \qquad (P)$$

where $A$ is $m \times n$, and $m = O(2^n)$, under certain conditions (P) can be solved in time polynomial in $n$ and $\log U$, where $U$ is an upper bound on the size of entries of $(A, b)$.

State these conditions, define the separation problem, and discuss the equivalence of separation and optimization. Give an example of a linear program with exponentially many constraints solvable in polynomial time.

## Qualifier Question — Linear Programming

January 2012

Consider the upper-bounded continuous knapsack problem

$$
\begin{array}{lll}
\max\ cx & & \\
ax \leq b_0 & (i) & \\
0 \leq x \leq b & (ii) &
\end{array}
\tag{1}
$$

where $b_0 > 0$ is a scalar, $x = (x_1, \ldots, x_n)$, $a > 0$, $b > 0$, $c \geq 0$, and

$$
\frac{c_1}{a_1} \geq \cdots \geq \frac{c_n}{a_n} \tag{2}
$$

(a) What is an optimal solution of (1)?

(b) Write the dual of (1). What is its optimal solution? Prove that your primal and dual solutions are optimal.

(c) Suppose the problem is to be solved by Dantzig-Wolfe decomposition, where (ii) goes into the pricing subproblem. State the restricted master problem. (Note that the subproblem is bounded).

(d) State the pricing subproblem. What is its optimal solution?

(e) The restricted master need only contain two columns, corresponding to two extreme point solutions of the subproblem, to ensure that its optimal value is optimal in (1). Which two extreme points?

(f) If the restricted master contains the two columns mentioned in (e), what is its optimal dual solution?

# Graph Theory
# Qualifying Exam

January 2012

Your answer to this question will be scanned and sent for grading. Therefore, please write your answer to this question in a separate piece of standard A4 paper legibly so that it is easy to scan.

1. (2 points) Let $G$ be an undirected graph with distinct nodes $a, b$ and $c$. Suppose there are $k$ node-disjoint paths between $a$ and $b$, and there are also $k$ node-disjoint path between $b$ and $c$. Prove or disprove: There exists $k$ node-disjoint paths between $a$ and $c$.

2. Let $A$ and $B$ be two $k$-node cuts separating $a$ and $b$ in a $k$-node connected undirected graph. Let $C = \{v \in A \cup B \mid \text{ there is a path from } a \text{ to } v \text{ containing no other nodes of } A \cup B\}$. Prove the following.

   (a) (2 points) $C$ is a node cut separating $a$ and $b$.

   (b) (2 points) $|C| \geq k$.

   (c) (4 points) $|C| \leq k$.

2012

## Integer Programming Question

Prove that every 0-1 programming problem with a constraint set $P_0$ is equivalent to a 0-1 programming problem with a constraint set $P_1$ in the same variables but with constraints whose lefthand side coefficients are 0, 1 or -1. What can you say about the righthand side coefficients? Is $P_1$ unique?

Derive $P_1$ for the following constraint set $P_0$:

$$\begin{array}{rcrcrcrcr} 2x_1 & - & x_2 & + & 4x_3 & + & 4x_4 & \leq & 4 \\ -2x_1 & + & 2x_2 & - & 4x_3 & - & x_4 & = & -2 \\ & & & & x \in \{0,1\}^4 & & & & \end{array}$$

## Dynamic Programming Question (2012)

Consider a worker who begins each period with a current wage offer $w$ and has two alternative actions. He can work at that wage or he can search for a new wage offer. If he chooses to search, he earns nothing during the current period, and his new wage is drawn from a probability distribution with density $f(\cdot)$, cumulative distribution $F(\cdot)$, and support $[0, \bar{w}]$. Assume search cost is zero. If he chooses to work during the current period, then with probability $1-\theta$ the same wage is available to him next period; but with probability $\theta$ he will lose his job at the beginning of next period and begin next period with a wage of zero. The one-period discount factor is $\beta \in (0,1)$. The worker's goal is to maximize his total discounted expected wage over infinite horizon.

(a) [3 points] Set up a dynamic program for this problem.

(b) [4 points] Show that an optimal policy is to search for a new wage offer if and only if the current offer $w$ is lower than some reservation wage $w^*$, where $w^*$ is a unique value between 0 and $\bar{w}$.

(c) [3 points] Show that $w^*$ satisfies the following equation:

$$[1 + \beta\theta - \beta F(w^*)]w^* = \beta \int_{w^*}^{\bar{w}} w f(w) dw.$$

If you do not have enough time to derive this equation exactly, provide a sketch of the proof by showing how you can obtain this equation.

## 2012 ACO Qualifying Exam: Discrete Mathematics

1. (a) State Ramsey's theorem for graphs.

   (b) Let $t$ be fixed and $n \to \infty$. Prove that any graph $G$ on $n$ vertices has at least $\Omega(n^t)$ *homogeneous* $t$-sets. (A homogeneous $t$-set is a set of $t$ vertices that spans either a clique or an independent set).

2. (a) State Markov's Inequality.

   (b) Let $t \geq 2$ be fixed while $n \to \infty$. By considering the random graph $G_{n,p}$ and using alterations prove that the maximum number of edges in a $K_{t,t}$-free graph with $n$ vertices is at least $\Omega(n^{2-\frac{2}{t+1}})$. Here $K_{t,t}$ is the complete bipartite graph with both parts having $t$ vertices.

3. Let $G$ be the graph on vertex set $\binom{[n]}{3} = \{Y \subseteq [n] : |Y| = 3\}$ in which $A, B$ are connected if and only if $|A \cap B|$ is even. Assuming only the standard facts of linear algebra (i.e., you are not allowed to use any intersection theorem), prove that neither $G$ nor its complement $\overline{G}$ contains $K_{n+1}$.

2012

# Qualifying Exam for Performance Modeling (15-857)

This is a **Closed Book** exam.

YOUR NAME: ______________________________

2012

For each of the 12 expressions below, fill in either (a), (b), (c), (d), (e), (f), or (g). Read the Glossary on the next page to make sure you understand all the expressions.

**Expressions:**

Throughout assume an M/G/1 queue, unless M/M/1 is specified. Assume that the average arrival rate is $\lambda$ and that job sizes are i.i.d. instances of random variable $S$.

1. $\mathbf{E}\{T\}^{M/G/1/FCFS}$ ______
2. $\mathbf{E}\{T\}^{M/G/1/PS}$ ______
3. $\mathbf{E}\{T\}^{M/G/1/LCFS}$ ______
4. $\mathbf{E}\{T\}^{M/G/1/PLCFS}$ ______
5. $\mathbf{E}\{T\}^{M/M/1/FCFS}$ ______
6. $\mathbf{E}\{T\}^{M/M/1/PS}$ ______
7. $\mathbf{E}\{T\}^{M/M/1/FB}$ ______
8. $\rho$ ______
9. $\mathbf{E}\{B\}^{M/G/1/FCFS}$ ______
10. $\mathbf{E}\{B\}^{M/M/1/FCFS}$ ______
11. $\mathbf{E}\{S_e\}$ ______
12. $\mathbf{E}\{W\}^{M/G/1/FCFS}$ ______

**Formulas:**

(a) $\lambda\mathbf{E}\{S\}$

(b) $\frac{\mathbf{E}\{S^2\}}{2\mathbf{E}\{S\}}$

(c) $\frac{\mathbf{E}\{S\}}{1-\rho}$

(d) $\frac{\rho}{1-\rho}\mathbf{E}\{S_e\}$

(e) $\frac{\rho}{1-\rho}\mathbf{E}\{S_e\}+\mathbf{E}\{S\}$

(f) $\frac{\mathbf{E}\{S_e\}}{1-\rho}$

(g) None of the above

2012

Glossary:

$\rho$ = load = fraction of time server is busy
$T$ = Response time
$B$ = Busy period duration
$\lambda$ = Average arrival rate
$S$ = Service requirement for jobs
$S_e$ = Excess of $S$
$W$ = Work seen by an arrival into the queue
FCFS = First-Come-First-Served scheduling
PS = Processor-Sharing scheduling
LCFS = Non-preemptive Last-Come-First-Served scheduling
PLCFS = Preemptive-Last-Come-First-Served scheduling
FB = Foreground-Background (a.k.a, Least-Attained-Service) scheduling

2012

**Advanced Integer Programming Question**

Let $\mathcal{F}_n$ be the family of all $n$-component permutation vectors, i.e. vectors whose components are permutations of the integers from 1 to $n$. For instance, $\mathcal{F}_3 = \{(1,2,3),\ (1,3,2),\ (2,1,3),\ (2,3,1),\ (3,1,2),\ (3,2,1)\}$. You want to characterize $\mathcal{F}_n$ as the set of extreme points of a polytope. Let $N := \{1,\ldots,n\}$, and consider the family of inequalities

$$\sum_{i\in S} x_i \leq n|S| - \frac{1}{2}|S|(|S|-1) \qquad (1)_S$$

for all $S \subseteq N$.

(a) Show that every $x \in \mathcal{F}_n$ satisfies the inequalities $(1)_S$ for all $S \subseteq N$.

[Hint: interpret the right hand side as the sum of certain components of $x$. Which components?]

(b) Which inequalities of the above system are satisfied at equality by a given $x \in \mathcal{F}_n$? How many such inequalities are there?

(c) Show that every $x \in \mathcal{F}_n$ is an extreme point (vertex) of the polyhedron $P$ defined by the above system together with the inequality

$$\sum_{i\in N} x_i \geq \frac{1}{2}n(n+1).$$

[Hint: use your answer to (b).]

# Networks and Matchings
# Qualifying Exam

January 2012

Your answer to this question will be scanned and sent for grading. Therefore, please write your answer to this question in a separate piece of standard A4 paper legibly so that it is easy to scan.

1. (2 points) True or False: Let $G$ be a bipartite graph with minimum degree $k \in \mathbb{N}$ and the same number of nodes on both sides of the vertex bipartition. Then, $G$ has at least $k$ (edge-)disjoint perfect matchings.

2. (4 points) Let $G = (V, E)$ be a bipartite graph and let $b : V \to \mathbb{Z}_+$. Prove that $G$ has a subgraph $G' = (V, E')$ such that $deg_{G'}(v) = b(v)$ for each $v \in V$ if and only if each $X \subseteq V$ contains at least

$$\frac{1}{2}(\sum_{v \in X} b(v) - \sum_{v \in V \setminus X} b(v))$$

edges.

3. (4 points) Let $G$ be a bipartite graph with the same number of nodes on both sides of the vertex bipartition and let $k \in \mathbb{N}$. Prove that $G$ has $k$ disjoint perfect matchings if and only if each $X \subseteq V$ contains at least $k(|X| - \frac{1}{2}|V|)$ edges.

## Qualifier Question — Constraint Programming

January 2012

Let $I = \{1, 2, \ldots, n\}$ be an index set, and let $S$ be a set of distinct unordered pairs from $I$, i.e., $S \subseteq \{(i, j) \mid i, j \in I, i < j\}$. Given a set of finite-domain variables $X = \{x_1, x_2, \ldots, x_n\}$, we consider the CSP consisting of the following constraints:

$$x_i \neq x_j \quad \forall (i, j) \in S. \tag{1}$$

Figure 1 represents the constraint network of a particular instance CSP $P$ (with $n = 8$). The set $S$ can be deduced from this graph: $(i, j) \in S$ if edge $(x_i, x_j)$ exists in the constraint network, for $i < j$.

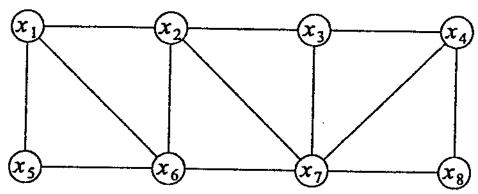



Figure 1: *Constraint network for P.*

For questions (a) and (b), assume the following initial variable domains:

$$D(x_1) = \{1, 2\},\quad D(x_2) = \{1, 2, 3\},\quad D(x_3) = \{1, 2, 3\},\quad D(x_4) = \{1, 2, 3\},$$
$$D(x_5) = \{1, 2\},\quad D(x_6) = \{1, 2, 3\},\quad D(x_7) = \{1, 2, 3\},\quad D(x_8) = \{1, 2, 3\}.$$

**(a)** Make $P$ arc consistent. Report the resulting variable domains.

Next, let $C$ be the set of all maximal cliques in the constraint network (for example, $\{x_1, x_2, x_6\}$ is a maximal clique). We reformulate our CSP as

$$\texttt{alldifferent}(x_i \mid i \in c) \quad \forall c \in C, \tag{2}$$

and we call this CSP $P'$.

**(b)** Make $P'$ arc consistent. Report the resulting variable domains.

**(c)** Sketch the dual graph representation for $P'$.

Lastly, we reformulate our CSP using a single global constraint `some-different`, which is defined as follows:

$$\texttt{some-different}(X, S) := \bigwedge_{(i,j) \in S} x_i \neq x_j, \tag{3}$$

and we call the resulting CSP $P''$.

**(d)** Argue how establishing arc consistency on $P'$ can establish hyper-arc consistency on $P''$ (for arbitrary variable domains), or give a counterexample showing that this is not possible.

2012

# Convex Analysis

**Problem** [10 points]
Let $p(x)$ be a norm on $\mathbf{R}^n$. Recall that it's dual norm is the norm $p_*(x)$ on $\mathbf{R}^n$ given by

$$p_*(x) = \max_y\{\langle x, y\rangle : p(x) \leq 1\}.$$

You may take it for granted that this formula indeed defines a norm. While proving the statements below, you can use any of the parts even if you have not proved it.

**(a)** [5 points]

**(i)** [2 points] Prove that

$$\langle x, y\rangle \leq p_*(y)p(x)$$

holds for all $x, y \in \mathbf{R}^n$ and that for every $y \in \mathbf{R}^n$ there exists $x \in \mathbf{R}^n$ with $p(x) = 1$ such that

$$\langle x, y\rangle = p_*(y) \qquad [= p_*(y)p(x)],$$

**(ii)** [3 points] Let $K = \{(x,t) \in \mathbf{R}^n \times \mathbf{R} : \ t \geq p(x)\}$. You may take it for granted that $K$ is a closed convex and pointed cone with nonempty interior. Prove that the cone dual to $K$ is given by

$$K_* = \{(y, \tau) \in \mathbf{R}^n \times \mathbf{R} : \ \tau \geq p_*(y)\}.$$

**(b)** [3 points] Let $p(x)$ and $q(x)$ be norms on $\mathbf{R}^n$; then $r(x) = \max\{p(x), q(x)\} + \alpha p(x) + \beta q(x)$ with $\alpha, \beta \geq 0$ also is a norm on $\mathbf{R}^n$. (You may take it for granted that $r(x)$ defines a norm.) Prove that $r_*(x) \leq 1$ if and only if there exists a representation

$$x = u + v$$

and $\gamma \in [0,1]$ such that $p_*(u) \leq \alpha + \gamma$ and $q_*(v) \leq \beta + 1 - \gamma$.

**(c)** [2 points] Let $r(\cdot)$ and $\|\cdot\|$ be norms on $\mathbf{R}^n$ and suppose there exists $L > 0$ such that

$$\frac{1}{L}\|\nabla r(y) - \nabla r(x)\|^2 \leq \langle \nabla r(y) - \nabla r(x), y - x\rangle$$

holds for all $x, y \in \mathbf{R}^n$. Show that the conjugate function of $r(\cdot)$ denoted by $r^*(\cdot)$ satisfies the following inequality

$$r^*(s') \geq r^*(s) + \langle s' - s, x\rangle + \frac{1}{2L}\|s' - s\|^2$$

for all $s, s'$ such that $\partial r^*(s) \neq \emptyset$, $\partial r^*(s') \neq \emptyset$ and for all $x \in \partial r^*(s)$.

<u>Hint:</u> First derive an upper bound on $r(\cdot)$ in terms of $r^*(\cdot)$ and then use the definition of conjugate function.

2012

**Nonlinear Programming Question**

Consider the problem

$$\begin{array}{ll} \min & x^2 + y^2 \\ \text{s.t.} & x^2 - (y-1)^3 = 0. \end{array}$$

(a) Solve the problem geometrically.

(b) Prove that there does not exist any point that satisfies the Karush-Kuhn-Tucker optimality conditions.

(c) Find all the points that satisfy the Fritz-John optimality conditions.

(d) One may be tempted to solve the above optimization problem by substituting $x^2 = (y-1)^3$ in the objective, thereby reducing it to the unconstrained problem

$$\min \ y^2 + (y-1)^3.$$

However, something goes wrong with this approach. What is it, and how can it be corrected?

# Convex Analysis

Let $f : Q \to \mathbb{R}$ be a strongly convex function with convexity parameter $\mu > 0$, where $Q \subseteq \mathbb{R}^n$ is a bounded, closed, convex set, which contains the origin.

Let $\widehat{Q} = \operatorname{conv}(Q, -Q)$ and consider the function $\widehat{f}$ given by

$$\widehat{f}(x) = \min_{\alpha,u,v}\{f(u) + f(v) : x = u - v, u \in \alpha Q, v \in (1-\alpha)Q, 0 \leq \alpha \leq 1\}$$

for all $x \in \widehat{Q}$ and $\widehat{f}(x) = +\infty$ for all $x \notin \widehat{Q}$.

(a) [6 points] Show that $\widehat{f}$ is strongly convex with parameter $\frac{1}{2}\mu$ on its domain, $\widehat{Q}$.

(b) [2 points] Show that if a function $g : C \to \mathbb{R}$ on a convex set $C$ is strongly convex with parameter $\mu > 0$ then the following holds:

$$\langle s_y - s_x, y - x\rangle \geq \mu\|y - x\|^2$$

for all $x, y \in C$, $s_x \in \partial g(x)$ and $s_y \in \partial g(y)$.

(c) [2 points] Consider the conjugate function $\widehat{f}^*(\cdot)$ of $\widehat{f}(\cdot)$ defined as

$$\widehat{f}^*(s) = \sup_x\{\langle s, x\rangle - \widehat{f}(x) : x \in \widehat{Q}\}.$$

Let $x, y \in \widehat{Q}$, $s_x \in \partial\widehat{f}(x)$ and $s_y \in \partial\widehat{f}(y)$ be such that $\widehat{f}^*$ is differentiable at $s_x$ and $s_y$. Show that

$$\|\nabla\widehat{f}^*(s_y) - \nabla\widehat{f}^*(s_x)\| \geq \frac{1}{\mu}\|s_y - s_x\|$$

holds.

Hint: You may use the previous parts even if you haven't proved them.

2013

## Discrete Mathematics

1. Let $\mathcal{H}$ be a finite 3-uniform hypergraph on vertex set $V$ and let $\mathcal{C}$ be a set of colors. For each $v \in V$ a prescribed set of colors $C_v \subseteq \mathcal{C}$ is given. A vertex coloring $f : V \to \mathcal{C}$ is **proper with respect to $(C_v)_{v\in V}$** if every vertex $v$ gets a color from its prescribed set and no edge of the hypergraph is monochromatic. (To be precise, a coloring is proper with respect to the prescribed sets if $f(v) \in C_v$ for all $v \in V$ and for all $e \in \mathcal{H}$ there are $u, v \in e$ such that $f(u) \neq f(v)$.)

   Now, suppose the hypergraph $\mathcal{H}$ and the sets $(C_v)_{v\in V}$ have the property that for every vertex $v$ there are at most $d$ edges $e$ that contain a vertex $u$ such that $C_u \cap C_v \neq \emptyset$. Prove that there is a constant $C$ (which does not depend on $d$ or $|V|$) such that if $|C_v| \geq C\sqrt{d}$ for all $v \in V$ then there is a coloring that is proper with respect to $(C_v)_{v\in V}$.

2. (a) State the definition of the 2-color van der Waerden number $W(k)$.

   (b) Let $N = W(t^2)$. Let $f$ be a coloring of $\{1, \dots, N\}$ with 2 colors. Prove that there exists a non-trivial $t$-term arithmetic progression

   $$\{a + i \cdot d : i = 0, \dots, t-1\}$$

   which together with its difference $d$ is monochromatic. I.e. $f(d) = f(a + i \cdot d)$ for $i = 0, \dots, t-1$, and $d$ is non-zero.

2013

## Graph Theory

Six professors, $A$, $B$, $C$, $D$, $E$, $F$, have been to the library on the day the rare tractate was stolen. Each entered once, stayed for some time, and then left. If two were in the library at the same time, then at least one of them saw the other. Detectives who questioned the professors gathered the following testimony:

- $A$ said he saw $B$ and $C$ in the library
- $B$ said he saw $A$ and $D$
- $C$ said he saw $B$ and $E$
- $D$ said he saw $E$ and $C$
- $E$ said he saw $D$ and $F$
- $F$ said he saw $A$ and $D$

One of the professors lied when he claimed to have seen one of his colleagues. Who was it?

## Integer Programming

Let $n \geq 2$ be an integer and let $S \subseteq \{0,1\}^n$ have the property that, for every $x \in \{0,1\}^n \setminus S$, the set $S$ contains the $n$ vertices of the cube $[0,1]^n$ that are adjacent to $x$.

(i) Show that the following algorithm solves the problem $\min\{cx : x \in S\}$ :

Let $\overline{x}$ be defined by $\overline{x}_j = \begin{cases} 1 & \text{if } c_j < 0, \\ 0 & \text{otherwise.} \end{cases}$

If $\overline{x} \in S$, return $\overline{x}$.

Otherwise, let $k$ be an index such that $|c_k| = \min_j |c_j|$.

Let $\tilde{x}$ be defined by $\tilde{x}_j = \begin{cases} \overline{x}_j & \text{for } j \neq k \\ 1 - \overline{x}_k & \text{for } j = k. \end{cases}$

Return $\tilde{x}$.

(ii) Prove that any formulation of the problem $\min\{cx : x \in S\}$ as an integer program

$$\min\{cx : Ax \leq b, x \in \{0,1\}^n\}$$

has a number of constraints at least as large as $2^n - |S|$.

(iii) Let $S$ be the set of $0,1$ vectors in $\mathbb{R}^n$ that have an odd number of 1s. Show that $\text{conv}(S)$ has at least $2^{n-1}$ facets.

2013

# Advanced Integer Programming

Let $A \in \mathbb{Q}^{m \times n}$ be a matrix, $b \in \mathbb{Q}^m$ a column vector, $c \in \mathbb{Q}^n$ a row vector, and $d \in \mathbb{Q}$ a scalar. Let

$$
\begin{aligned}
&P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\},\\
&Q = P \cap \{x \in \mathbb{R}^n : cx \leq d\},\\
&R = P \cap \{x \in \mathbb{R}^n : x_1 \cdot x_2 = 0\} \text{ where } x_1, x_2 \text{ are the first two components of vector } x.
\end{aligned}
$$

(i) Let $u \in \mathbb{R}^m$ and $\nu \in \mathbb{R}^n_+$ be row vectors and $u_0 \in \mathbb{R}_+$ a scalar. Show that $\alpha x \leq \beta$ is a valid inequality for $Q$ for any

$$
\begin{aligned}
\alpha &= uA - \nu + u_0 c\\
\beta &= ub + u_0 d.
\end{aligned}
$$

(ii) Conversely, is every valid inequality for $Q$ of the form $\alpha x \leq \beta$ where $\alpha$ and $\beta$ are defined as in (i)?

(iii) Let $Q_i = P \cap \{x \in \mathbb{R}^n : x_i \leq 0\}$. Applying (i) to $Q_1$ and $Q_2$, give sufficient conditions for an inequality $\alpha x \leq \beta$ to be valid for $R$.

(iv) Let $\nu \in \mathbb{R}^n_+$ and $u_0 \in \mathbb{R}_+$ such that there exists $w \in \mathbb{R}^m$, $\mu \in \mathbb{R}^n_+$, $v_0 \in \mathbb{R}_+$ satisfying

$$
\begin{aligned}
wA &= \nu - \mu - u_0 e^1 + v_0 e^2\\
wb &= 0
\end{aligned}
$$

where $e^1$ and $e^2$ are the first two unit vectors ($\nu, w, \mu, e^1, e^2$ are all row vectors). Show that the inequality $\sum_{j=1}^n \nu_j x_j - u_0 x_1 \geq 0$ si a valid inequality for $R$.

(v) Assume that we have a tableau for problem $Ax = b, x \geq 0$ where $x_1$ and $x_2$ are both basic and strictly positive:

$$
\begin{aligned}
x_1 + \sum_{j \in N} \bar{a}_{1j} x_j &= \bar{b}_1\\
x_2 + \sum_{j \in N} \bar{a}_{2j} x_j &= \bar{b}_2,
\end{aligned}
$$

where $N$ indexes the nonbasic variables and $\bar{b}_1 > 0, \bar{b}_2 > 0$. Prove that

$$
-\frac{x_1}{\bar{b}_1} + \sum_{j \in N} \left( \frac{\bar{a}_{2j}}{\bar{b}_2} - \frac{\bar{a}_{1j}}{\bar{b}_1} \right)^+ x_j \geq 0 \qquad \text{is a valid cut for } R.
$$

2013

## Linear Programming

(a) A unit simplex is $\{x \in \mathbb{R}^n : ex \le 1, x \ge 0\}$ where $e = (1, \ldots, 1)$. Consider the problem of maximizing $cx$ over a unit simplex, where $c_1 > \cdots > c_n > 0$. State the optimal primal and dual solution.

(b) Write the optimal tableau for the problem in (a). Add the constraint $x_1 \le 0$ and perform one iteration of the dual simplex method to restore feasibility.

(c) Suppose that $\max\{cx + dy : Ax + By \le b\}$ is to be solved by Benders decomposition, with $x$ in the master problem. Show that the projection of the feasible set onto $x$ is described by the set of all possible Benders cuts arising from infeasible subproblems.

(d) Use (c) to show that the projection of a unit simplex onto any subspace is a unit simplex. State explicitly the extreme ray solutions that yield the relevant Benders cuts. *Be sure to treat nonnegativity constraints as part of the system $Ax + By \le b$ and use all of them in the subproblem.*

2013

## Networks and Matchings

Let $G$ be a connected bipartite graph with bipartition $(A, B)$, each of whose edges is contained in a perfect matching. Show that $|A| = |B|$ and for each nonempty proper subset $X$ of $A$, $|N(X)| > |X|$. (Here $N(X)$ is the set of vertices adjacent to some vertex in $X$).

2014

## Advanced Integer Programming Question

The asymmetric assignment (AA) problem is the assignment problem defined on a digraph $G = (N, A)$ with the additional restriction that cycles of length 2 are forbidden. The monotone relaxation of the AA problem, in which the assignment equations are replaced by $\leq$ inequalities, is the asymmetric partial assignment (APA) problem. What is the dimension of the APA and the AA polytopes? What are some of the facets of the APA polytope? (Hint: use the intersection graph of the coefficient matrix of the system defining the APA polytope to formulate an equivalent vertex packing problem.)

Networks and Matchings 2014

Assume that you have a maximum flow algorithm at your disposal for networks with a source $s$, a sink $t$, and a positive capacity $u_{ij}$ for each arc $ij$.

Consider a network $G = (N, A)$ in which there is a positive lower bound $l_{ij}$ on the flow of each arc $ij \in A$ as well as the upper bound $u_{ij}$. A flow is *feasible* in the network $G$ if it satisfies the lower and upper bound constraints and conservation of flow at all nodes other than $s$ and $t$.

(a) (3 points) Give an algorithm to find a feasible flow in the network $G$, or to show that none exists.

(b) (4 points) Rather than find a feasible flow that sends the most amount of flow from $s$ to $t$, the objective is to find a feasible flow in the network $G$ that sends the *least* amount of flow from $s$ to $t$. Give an algorithm to solve this problem.

(c) (3 points) Define the *capacity* of an $st$-cut in the network $G$ as the sum of the lower bounds on the forward arcs of the cut minus the sum of the upper bounds on the backward arcs. Is it correct to say that the minimum value of a flow from $s$ to $t$ equals the maximum capacity of an $st$-cut?

Convex Polyhedra Question

# 2014

Given a full-dimensional $V$-polyhedron in $\mathbb{R}^n$

$$P := \operatorname{conv} V + \operatorname{cone} W,$$

where $V = (v^1, \ldots, v^p)$ and $W = (w^1, \ldots, w^q)$, describe a method for finding the equivalent $H$-polyhedron

$$P := \{x \in \mathbb{R}^n : Ax \leq b\}$$

$$x - \begin{bmatrix} v^1 v^2 \cdots v^p \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{bmatrix} - \begin{bmatrix} w^1 \cdots w^q \end{bmatrix} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_q \end{bmatrix} = 0$$

$$\mathbb{1} \lambda = 1$$

$$\lambda_i, \mu_i \geq 0$$

$$\begin{array}{llllll}
I\vec{x} & - & V\vec{\lambda} & - W\vec{\mu} & \leq 0 & a^- \\
-I & + & V & + W & \geq 0 & a^+ \\
0 & & \vec{\mathbb{1}}^T & 0 & = +1 & b^+ \\
0 & & -\vec{\mathbb{1}}^T & 0 & = 1 & b^- \\
0 & & -I & 0 & \leq 0 & c \\
0 & & 0 & -I & \leq 0 & d
\end{array}$$

$$\underbrace{\hspace{6em}}_{A}$$

# 47-862 Constraint Programming

## Qualifier Exam Question, January 2014

Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of Boolean variables. Let $b$ be a fixed Boolean indicator parameter representing 'odd' ($b$ = TRUE) or 'even' ($b$ = FALSE). We define the constraint `Parity`$(X, b)$ as:

`Parity`$(X,$ TRUE$)$ is true when an *odd* number of variables in $X$ is true,
`Parity`$(X,$ FALSE$)$ is true when an *even* number of variables in $X$ is true.

1. Design an efficient (polynomial-time) filtering algorithm for a single constraint `Parity`$(X, b)$, establishing domain consistency. Provide the algorithm, provide its worst-case time complexity, and show that it establishes domain consistency.

2. We next define the global constraint `ParitySystem` that represents the conjunction of arbitrary `Parity` constraints on subsets of $X$. Let $S_i \subseteq X$, and let $b_i$ a fixed Boolean parameter, for $i = 1, \dots, k$:

$$\texttt{ParitySystem}(\{(S_i, b_i)\}_{i=1}^{k}) = \bigwedge_{i=1}^{k} \texttt{Parity}(S_i, b_i)$$

   Design an efficient filtering algorithm for the `ParitySystem` constraint, establishing domain consistency, or show that no such algorithm exists.

## Dynamic Programming Question (2014)

**(Question 1)** Consider a machine that can be in either of two states. good or bad. At the beginning of each day. the machine produces an item that is either defective or nondefective. The probability of a defective item is $P_1$ when in the good state. and $P_2$ when in the bad state. Based on the past data. $P_2 > P_1$. Once in the bad state. the machine remains in this state until it is replaced. However, if the machine is in the good state at the beginning of one day. then with probability $\gamma$ it will be in the bad state at the beginning of the next day. A decision as to whether or not to replace the machine must be made each day after observing the item produced. Let $R$ be the cost of replacing the machine and let $c$ be the cost incurred whenever a defective item is produced. Assume that at time zero there is a known probability that the machine is in the bad state. The one-period discount factor is $\alpha \in (0,1)$.

(a) [3 points] Set up a dynamic program for this problem.
(b) [3 points] Derive an optimal policy.

**(Question 2)** Consider random variables $X$ and $Y$. Cumulative probability distributions for $X$ and $Y$ are $F$ and $G$. respectively. Define the second-order stochastic dominance as follows: $X$ dominates $Y$ in the sense of second-order stochastic dominance ($X \geq_{SOSD} Y$) if and only if $\int_{-\infty}^{a} F(x)dx \leq \int_{-\infty}^{a} G(y)dy$ for all $a$.
(c) [1 point] State whether the following statement is true or false:

$$X \geq_{SOSD} Y \Rightarrow Var[X] \leq Var[Y].$$

(d) [1 point] State whether the following statement is true or false:

$$Var[X] \leq Var[Y] \text{ and } E[X] = E[Y] \Rightarrow X \geq_{SOSD} Y.$$

(e) [2 points] Prove or disprove with a counter-example your answer in (c) **or** (d). You will get 2 points if you choose either (c) or (d). and answer it correctly. If you choose to answer both (c) and (d). each correct answer will get 1 point.

Integer Programming Question 2014

Consider an integer program with the constraint set

(1) $Ax \geq b$, $x \in \mathbb{Z}^n_+$

(2) at least one of the variables $x_i, x_j$ must be positive

(a) Derive some valid cuts from condition (2) and show that they cut off some points that satisfy the standard representation of (2) as $x_i + x_j \geq 1$.

(b) Somebody claims to have derived a cut from (2) that strictly dominates yours. How do you check whether it is valid?

Lift-and-Project.

it must be true for $Ax \geq b$, $x_i \geq 1$ and $Ax \geq b$, $x_j \geq 1$

$$\begin{array}{ccccc|c|l}
x & y_1 & y_2 & y_0^1 & y_0^2 & \geq & \\
I & -I & -I & 0 & 0 & 0 & z_1 \\
-I & I & I & 0 & 0 & 0 & z_2 \\
 & A & 0 & -b & 0 & 0 & z_3 \\
 & & A & 0 & -b & 0 & z_4 \\
 & e_i & & 1 & 0 & 0 & z_5 \\
 & & e_j & 0 & 1 & 0 & z_6 \\
 & & & 1 & 1 & 1 & z_7 \\
 & & & -1 & -1 & -1 & z_8 \\
 & & & 1 & 0 & 0 & z_9 \\
 & & & 0 & 1 & 0 & z_{10}
\end{array}$$

b A

$vA = 0$

$-z_1 + z_2 + z_3 A + (z_5)_i = 0$

$-z_1 + z_2 + z_4 A + (z_6)_j = 0$

$-b z_3 + z_5 + z_7 - z_8 + z_9 = 0$

$-b z_4 + z_6 + z_7 - z_8 + z_{10} = 0$

# Linear Programming (January 2014)

convex

**Problem** [10 points]

(a) **[3 pts]** Let $f : \mathbf{R} \to \mathbf{R}$ be a convex function and define

$$P_f := \{(x, y) \in \mathbf{R}^2 : x + ty \geq f(t)\ \forall t \in [-1, 1]\}$$

Show that $P_f$ is an unbounded polyhedral set. What is its recession cone? What are its extreme points? we can always go in the direction $(1,0)$

(b) **[2.5 pts]** Let $f_i : \mathbf{R} \to \mathbf{R}$ for $i = 1, \ldots, 2014$ be convex functions and for each $i$, define $P_{f_i}$ as above. Consider the optimization problem given by

$$\min_{x,y,z}\{\alpha x + \beta y + \gamma z : \ (x, y) \in P_{f_i},\ i = 1, \ldots, 1007,\ (x, z) \in P_{f_i},\ i = 1008, \ldots, 2014\}. \quad (P)$$

Suppose that the objective value of this problem is zero. Clearly if we drop one or more of the 2014 constraints of form $(x, y) \in P_{f_i}$ or $(x, z) \in P_{f_i}$ from this problem, the resulting objective value can only remain the same or decrease. Can we drop any of these constraints corresponding to some $1 \leq i \leq 2014$ without changing the optimal objective value? If so, what is the maximum number of constraints we can safely drop while keeping the objective value of this problem still the same?

(c) **[4.5 pts]** Consider $0 = s_0 \leq s_1 \leq \ldots \leq s_{n-1} \leq s_n = 1$ and let $S = \{s_0, \pm s_1, \ldots, \pm s_n\}$. Suppose that $\xi$ is a discrete random variable taking values from $S$ and has mean 0, i.e., $\mathbf{E}[\xi] = 0$. Give an explicit analytical expression[1] for the maximum possible value of $\mathbf{E}\left[\max\left\{2|\xi|, 2^{2\xi} - \xi\right\}\right]$.

<u>Hint:</u> You may use the previous parts even if you haven't proved them.

[1]Here I am expecting either an explicit expression involving $s_i$'s or a numerical value.

# Graph Theory
# Qualifying Exam

January 2014

This portion of the exam is open-book open-notes, so you may consult any notes or books you bring with you. However, you may not use the internet or any communication device in solving the problem.

Your answer to this question will be scanned and sent for grading. Therefore, please write your answer to this question in a separate piece of standard A4 paper legibly so that it is easy to scan.

**Question:** (10 points) Let $G = (V, E)$ be an undirected graph where $m = |E|$. We will consider functions $d : V \to \mathbb{Z}$ that assigns integer values to nodes. For any $X \subseteq V$, define $d(X) = \sum_{v \in X} d(v)$. For any subset $X \subseteq V$, denote by $e(X)$ the number of edges in $E$ with at least one endpoint in $X$, and by $i(X)$, the number of edges with both endpoints in $X$.

1. (1 point) Suppose the edges of $G$ are oriented (directed in one of two ways) so that $d(v)$ represents the in-degree (number of incoming arcs) of node $v$ in the orientation. Then show the following.
   (A) $d(V) = m$.
   (B) For any subset of nodes $X \subseteq V$, $e(X) \geq d(X)$.
   (C) For any subset of nodes $X \subseteq V$, $i(X) \leq d(X)$.
2. (2 points) Suppose a function $d$ obeys both (A) and (B) above. Show that it also obeys (C).
3. (2 points) Show that the set function $e(X)$ is submodular, i.e. for any subsets $S, T \subseteq V$, we have $e(S) + e(T) \geq e(S \cap T) + e(S \cup T)$.
4. (2 points) Given a function $d$ that obeys (B) above, define a subset of nodes $X \subseteq V$ to be *tight* if it satisfies $e(X) = d(X)$, i.e. it is tight for the condition (B). Show that if $X, Y \subseteq V$ are both tight, so are $X \cap Y$ and $X \cup Y$.
5. (3 points) Given a function $d$ that obeys both (A) and (B), show that it is the degree sequence of an orientation of $G$.