Chapter 2 review, problem 77.

Evaluate $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3}$.

Solution.

$$\lim_{x \to 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3} = \lim_{x \to 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3} \left(\frac{\sqrt{1 + \tan x} + \sqrt{1 + \sin x}}{\sqrt{1 + \tan x} + \sqrt{1 + \sin x}} \right)$$

$$[\,\text{multiply by conjugate of numerator}\,]$$

$$= \lim_{x \to 0} \frac{(1 + \tan x) - (1 + \sin x)}{x^3 \left(\sqrt{1 + \tan x} + \sqrt{1 + \sin x} \right)}$$

$$[\,\text{let } w = \sqrt{1 + \tan x} + \sqrt{1 + \sin x} \text{ to save space}\,]$$

$$= \lim_{x \to 0} \frac{(1 + \tan x) - (1 + \sin x)}{x^3 w}$$

$$= \lim_{x \to 0} \frac{\tan x - \sin x}{x^3 w}$$

$$= \lim_{x \to 0} \frac{\frac{\sin x}{\cos x} - \frac{\sin x \cos x}{\cos x}}{x^3 w} \qquad [\,\text{rewrite } \tan x, \text{ find common denominator}\,]$$

$$= \lim_{x \to 0} \frac{\left(\frac{\sin x - \sin x \cos x}{\cos x} \right)}{x^3 w}$$

$$= \lim_{x \to 0} \frac{\sin x - \sin x \cos x}{x^3 w \cos x}$$

$$= \lim_{x \to 0} \frac{(\sin x)(1 - \cos x)}{x^3 w \cos x} \qquad [\,\text{factor out } \sin x\,]$$

$$= \lim_{x \to 0} \frac{(\sin x)[2 \sin^2(x/2)]}{x^3 w \cos x}$$

$$[\,\text{from half-angle formula } \sin^2 \theta = (1 - \cos 2\theta)/2, \text{ with } \theta = x/2\,]$$

$$= \lim_{x \to 0} \frac{(\sin x)[2 \sin^2(x/2)]}{4x(x/2)^2 w \cos x} \qquad [\,\text{rewrite denominator}\,]$$

$$= \lim_{x \to 0} \frac{2}{4} \left(\frac{\sin x}{x} \right) \left(\frac{\sin(x/2)}{x/2} \right)^2 \left(\frac{1}{w \cos x} \right) \qquad [\,\text{pull apart}\,]$$

$$= \frac{1}{2} \left(\lim_{x \to 0} \frac{\sin x}{x} \right) \left(\lim_{x \to 0} \frac{\sin(x/2)}{x/2} \right)^2 \left(\lim_{x \to 0} \frac{1}{w \cos x} \right)$$

$$[\,\text{product rule for limits}\,]$$

$$= \frac{1}{2}(1)(1)^2 \lim_{x \to 0} \frac{1}{w \cos x} \qquad \left[\,\text{since } \lim_{x \to 0} \frac{\sin x}{x} = 1\,\right]$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{1}{\left(\sqrt{1 + \tan x} + \sqrt{1 + \sin x} \right) \cos x} \qquad [\,\text{back-substitute } w\,]$$

$$= \frac{1}{2} \left(\frac{1}{\left(\sqrt{1 + \tan 0} + \sqrt{1 + \sin 0} \right) \cos 0} \right) \qquad [\,\text{plug in } x = 0\,]$$

$$= \frac{1}{2} \left(\frac{1}{\left(\sqrt{1 + 0} + \sqrt{1 + 0} \right)(1)} \right)$$

$$= \frac{1}{4}.$$