composition book

BRIAN KELL

21-701: DISCRETE MATH.

FALL 2009

100 sheets

7 1/2 in x 9 3/4 in (19.1 cm x 24.8 cm)

college ruled

Mon
24 Aug
2009

21-701 Discrete Mathematics Wean 5304
Tom Bohman

Modern Discrete Math

- algebraic combinatorics
  see Stanley, "Enumerative Combinatorics"
- additive combinatorics
  see Tao and Vu, "Additive Combinatorics"
- graph theory
- extremal combinatorics } this course
- probabilistic combinatorics } this course

Ramsey Theory

Defn A simple graph

$$G = (V, E) = (V(G), E(G))$$

$V =$ vertex set
$E \subseteq \binom{V}{2}$

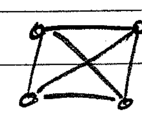
[If X is a set and k is an integer,

$$\binom{X}{k} = \{ Y \subseteq X : |Y| = k \}.$$

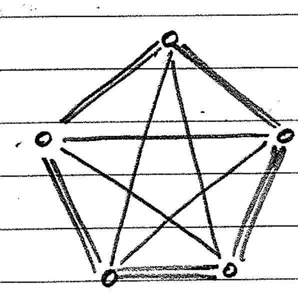
Also, $[n] = \{1, 2, \ldots, n\}$.]

e.g. complete graph K_n

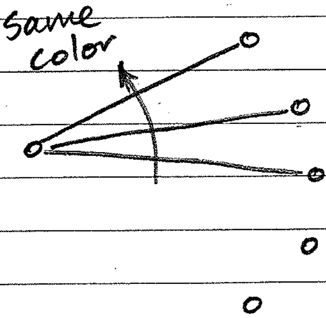
$V = [n]$
$E = \binom{[n]}{2}$



Suppose we color the edges of K_n red and blue.
Is there a monochromatic triangle?



no for $n=5$



yes for $n \geq 6$

(2-color graph) Ramsey theorem:

$\forall\ k, \ell \geq 2 \quad \exists\ n$ such that
if $\binom{[n]}{2} = R \cup B$ then

(i) $\exists A \in \binom{[n]}{k}$ such that $\binom{A}{2} \subseteq R$
(a "red" K_k)

or

(ii) $\exists B \in \binom{[n]}{\ell}$ such that $\binom{A}{2} \subseteq B$.
(a "blue" K_ℓ)

Let $R(k, \ell) =$ minimum such n.
These are the (2-color graph) Ramsey numbers.

e.g. $R(3,3) = 6$.

Proof Induction on $k + \ell$.

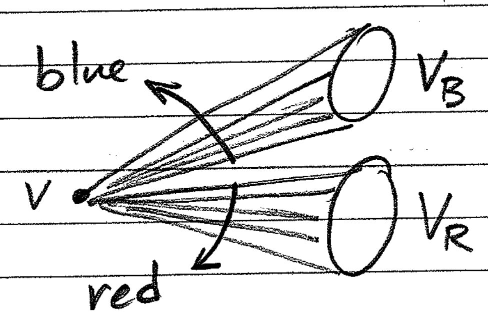
- $R(2, k) = k = R(k, 2)$.
- Claim: For $k, \ell \geq 3$ we have

$$R(k, \ell) \leq R(k-1, \ell) + R(k, \ell-1).$$

It remains to prove the claim.

Pf of Claim: Let $n = R(k-1, \ell) + R(k, \ell-1)$ and consider a coloring $\binom{[n]}{2} = R \cup B$.

Let $v \in [n]$.
Set
$$V_R = \{ u \in [n] : \{u,v\} \in R \}$$
$$V_B = \{ u \in [n] : \{u,v\} \in B \}.$$



Either $|V_B| \geq R(k, \ell-1)$
or $|V_R| \geq R(k-1, \ell)$

[since $R(k-1,\ell) + R(k,\ell-1) = n = 1 + |V_B| + |V_R|$]

If $|V_B| \geq R(k, \ell-1)$ the coloring on V_B has a red K_k or a blue $K_{\ell-1}$. In the latter case attach v to get a blue K_ℓ.

If $|V_R| \geq R(k-1, \ell)$, reverse the roles of R and B in the previous sentence. $\square$

"Full" Ramsey theorem

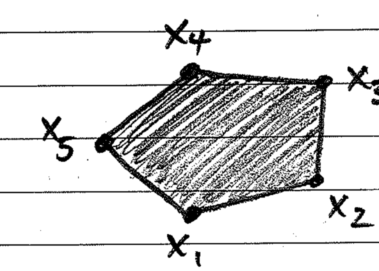
$\forall\ j, k_1, k_2, \ldots, k_m \geq 2$ [m = # colors]
$\exists\ n$ such that if
$$\binom{[n]}{j} = C_1 \cup C_2 \cup \cdots \cup C_m$$
then $\exists\ i$, $1 \leq i \leq m$, and $A \in \binom{[n]}{k_i}$ such that $\binom{A}{j} \subseteq C_i$.

Let $R_j(k_1, k_2, \ldots, k_m)$ be the smallest such n.

A geometric application.

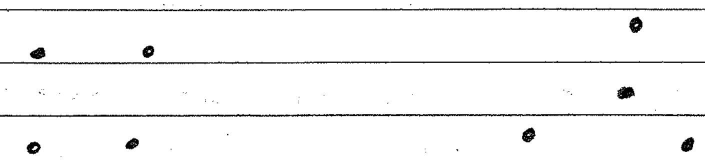
__Defn__ For $X = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}^m$, the convex hull of X is
$$\operatorname{conv}(X) = \{ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n : 0 \leq \lambda_i \text{ and } \sum_{i=1}^{n} \lambda_i = 1 \}.$$

__e.g.__



__Defn__ X is in __convex position__ if $\nexists\ i$ such that
$$x_i \in \operatorname{conv}(X \setminus \{x_i\}).$$

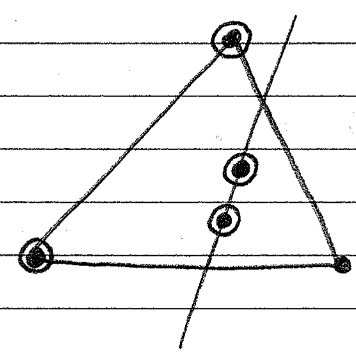
e.g.



in convex position   not in convex position

Claim: Of 5 points in the plane, no 3 of which lie on a line, there are 4 that lie in convex position.

Pf



$\square$

Erdős–Szekeres Theorem

For any $n \geq 3$ there is an integer N such that among any N points in the plane, (no 3 on a line) there are n in convex position.

Proof Let $N = R_4(\underset{\text{red}}{5}, \underset{\text{blue}}{n})$.

Consider N points in the plane, no 3 on a line. Given a set X of 4 of these points we say

X is red if X is not in convex position;
X is blue if X is in convex position.

By Ramsey's theorem there is

① a set Y of points such that $|Y| = 5$ and all 4-element subsets of Y are red (i.e., not in convex position),

or

② a set Z of points such that $|Z| = n$ and all 4-element subsets of Z are blue (i.e., in convex position).

By the claim, ① is not possible. So we have ②.

Now consider a set Z of n points, every 4 of which are in convex position. AFSOC that Z is not in convex position. There is some $x \in Z$ such that

$$x \in \operatorname{conv}(Z \setminus \{x\}).$$

Consider a triangulation of $\operatorname{conv}(Z \setminus \{x\})$. Then x lies in the interior of one of the triangles. This triangle and x is a 4-element subset of Z not in convex position. ↯

$\square$

On Ramsey numbers

Asymptotic questions

How do $R(k,k)$, $R(3,k)$, and $R(\underbrace{3,3,\ldots,3}_{k \text{ times}})$ behave as $k\to\infty$?

$\underline{R(k,k)}$ [diagonal Ramsey numbers]

(Exercise) $\longrightarrow$ $\boxed{EX}$ $R(k,\ell) \le R(k-1,\ell) + R(k,\ell-1)$

$$\Longrightarrow R(k,\ell) \le \binom{k+\ell-2}{k-1}$$

It follows that

$$R(k,k) \le \binom{2k-2}{k-1} \sim \frac{c}{\sqrt{k}} 2^{2k}$$

for some constant c.

[$f \sim g$ means $\lim f/g = 1$]

Prop (Erdős 1947)

$$R(k,k) > \frac{k\,2^{k/2}}{e\sqrt{2}}.$$

Wed 26 Aug 2009

Asymptotic questions

[Recall: $R(k,\ell)$ is the minimum n such that any red/blue coloring of the edges of K_n has a red K_k or a blue K_ℓ.]

How do $R(k,k)$, $R(3,k)$, and $R(\underbrace{3,3,\ldots,3}_{k})$ behave as $k\to\infty$?

$\underline{R(k,k)}$

Last time we showed

$$R(k,k) \le \binom{2k-2}{k-1} \sim \frac{c}{\sqrt{k}} 2^{2k} = \Theta\left(\frac{2^{2k}}{\sqrt{k}}\right)$$

[$f = \Theta(g)$ means $\exists$ constants c_1 and c_2 such that $c_1 f \le g \le c_2 f$.]

Proposition (Erdős 1947)

$$R(k,k) > \frac{k\,2^{k/2}}{e\sqrt{2}}.$$

Pf We randomly and independently color each edge of K_n.

$$\Pr(e \text{ is red}) = \Pr(e \text{ is blue}) = \frac{1}{2} \quad \forall e \in \binom{[n]}{2}.$$

Let $\mathcal{E}$ be the event that there is no monochromatic K_k.

Note that $\Pr(\mathcal{E}) > 0 \Rightarrow R(k,k) > n$.

For $A \in \binom{[n]}{k}$ let $\mathcal{B}_A$ be the event that $\binom{A}{2}$ is monochromatic.

$$\Pr(\mathcal{B}_A) = 2 \cdot \left(\tfrac{1}{2}\right)^{\binom{k}{2}}$$

Note that

[union of events in a probability space]

$$\bigvee_{A \in \binom{[n]}{k}} \mathcal{B}_A = \overline{\mathcal{E}}.$$

[complement of event $\mathcal{E}$]

Furthermore,

[union bound, or Boole's inequality]

$$\Pr\Big(\bigvee_{A \in \binom{[n]}{k}} \mathcal{B}_A\Big) \le \sum_{A \in \binom{[n]}{k}} \Pr(\mathcal{B}_A)$$

$$= \binom{n}{k} 2^{1-\binom{k}{2}}.$$

So, if $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$

then $\Pr(\mathcal{E}) > 0$

so $R(k,k) > n$.

$\boxed{\text{EX}^*}$ $\binom{n}{k} \le \left(\frac{ne}{k}\right)^k$. [related to Stirling's formula]

So $\underbrace{\left(\frac{ne}{k}\right)^k 2^{1-\binom{k}{2}}}_{2\left(\frac{ne}{k \cdot 2^{(k-1)/2}}\right)^k} < 1 \Rightarrow R(k,k) > n.$

Using $n = \frac{k \cdot 2^{k/2}}{2e}$, this is $2\left(\frac{1}{\sqrt{2}}\right)^k$, which proves $R(k,k) > \frac{k \cdot 2^{k/2}}{2e}$.

We gave too much away in our inequalities— need sharper inequalities to actually prove Erdős's statement.

We have seen

$$\frac{k \cdot 2^{k/2}}{2e} < R(k,k) < \frac{c}{\sqrt{k}} 2^{2k}.$$

It follows that

$$\sqrt{2} \le \liminf_{k\to\infty} R(k,k)^{1/k} \le \limsup_{k\to\infty} R(k,k)^{1/k} \le 4.$$

<u>Conj</u> (Erdős, 1947, \$100.00)

$\lim_{k\to\infty} R(k,k)^{1/k}$ exists.

<u>Problem</u> (Erdős, 1947, \$250.00)

Determine $\lim_{k\to\infty} R(k,k)^{1/k}$ if it exists.

Fan Chung, Ron Graham will pay Erdős's rewards. See "Erdős on Graphs."

R(3, k): There exists a constant c such that

$$\frac{ck^2}{\log k} < R(3,k) < (1+o(1))\frac{k^2}{\log k}.$$

[$f = o(g)$ means $\lim f/g = 0$.]

The first inequality above is due to J.H. Kim, 1994; the second is due to Shearer 1983, improving Ajtai, Komlos, Szemerédi.

R(3,3,…,3)

Let $f(k) = R(\underbrace{3,3,\ldots,3}_{k\text{ times}})$.

EX
(i) $f(k)$ is supermultiplicative: $f(x+y) \geq f(x)f(y)$.

(ii) It follows that $\lim_{k\to\infty} f(k)^{1/k}$ exists (possibly infinite).

Problem (Erdős, \$100.00)
Determine whether this limit is finite or infinite.

Problem (Erdős, \$250.00)
Determine the limit (if it is finite).

Van der Waerden's Theorem (1927)

$\forall r,k\ \exists N$ such that $\forall f:[N]\to[r]$ [an r-coloring] there exists a k-term monochromatic arithmetic progression.

Problem (Erdős, \$5000.00)
Show that any sequence of positive integers $a_1 < a_2 < \cdots$ such that $\sum_{i=1}^{\infty}\frac{1}{a_i}$ diverges contains a k-term arithmetic progression for all k.

Note: This would imply that the primes contain arbitrarily long arithmetic progressions. But this fact about the primes is now known (due to Green and Tao).

Question Does a monochromatic arithmetic progression appear (as in Van der Waerden's theorem) in the most frequent color?

Defn $A \subseteq \mathbb{N}$ has positive upper density if

$$\limsup_{N\to\infty}\frac{|A\cap[N]|}{N} > 0.$$

Szemerédi's Theorem If A has positive upper density, then A contains a k-term arithmetic progression for all k.

Note: A having positive upper density does not imply that A contains an infinite arithmetic progression.

History

- Conjecture by Erdős and Turán (1936)
- Proved by Roth for $k=3$ (1952)
- Proved for $k=4$ (1969) } Szemerédi
- ... for all k (1974) } Szemerédi
- Alternate proof using ergodic theory, Furstenberg (1977).

Mon 31 Aug 2009

Hales – Jewett

$[t]^n = \{(a_1, \dots, a_n) : a_i \in \{1, \dots, t\}\}$

Defn A line in $[t]^n$ is a set of elements $x_1, \dots, x_t$ where $x_i = (x_{i,1}, x_{i,2}, \dots, x_{i,n})$ such that for each coordinate j, $1 \le j \le n$, we have $x_{1,j} = x_{2,j} = \dots = x_{t,j}$ or $x_{ij} = i$ for $i = 1, \dots, t$.

Hales–Jewett theorem (1963)
$\forall\, r, t\ \exists\, n$ such that $\forall\, f : [t]^n \to [r]$ there is a monochromatic line.

Note: Hales–Jewett $\Rightarrow$ van der Waerden.
Consider the map

$$[t]^n \longrightarrow \{0, \dots, t^n - 1\}$$
$$(x_1, x_2, \dots, x_n) \longmapsto \sum_{i=1}^{n} (x_i - 1)\, t^{i-1}$$

$\boxed{\text{Ex}}$ Lines in $[t]^n$ are mapped to arithmetic progressions in $\{0, \dots, t^n - 1\}$.

Proof (Shelah, 1987)

Defn $(x_1, \dots, x_n) \in [t]^n$ is a Shelah point if there exist $0 \le i < j \le n$ such that

$$x_k = \begin{cases} t-1, & \text{if } k \le i; \\ s, & \text{if } i < k \le j, \quad \text{for some } s; \\ t, & \text{if } k > j. \end{cases}$$

$$y_\ell = (y_{\ell,1}, y_{\ell,2}, \dots, y_{\ell,n})$$

Defn $y_1, \dots, y_t \in [t]^n$ form a Shelah line if there exist $0 \le i < j \le n$ such that

$$y_{\ell,k} = \begin{cases} t-1, & \text{if } k \le i; \\ \ell, & \text{if } i < k \le j; \\ t, & \text{if } k > j. \end{cases}$$

Defn Suppose $n = n_1 + n_2 + \dots + n_s$ and L_j is a Shelah line in $[t]^{n_j}$ for $j = 1, \dots, s$. Then $\underbrace{L_1 \times L_2 \times \cdots \times L_s}_{\text{this means concatenation}}$ is a Shelah s-space.

$$\underbrace{t-1, t-1, \square, \square, \square, t, t}_{n_1} \mid \underbrace{t-1, t-1, t-1, \triangle, \triangle, t, t}_{n_2} \mid \underbrace{t-1, \lozenge, \lozenge, t, t, t}_{n_3} \xrightarrow{\text{canonical map}} (\square, \triangle, \lozenge)$$

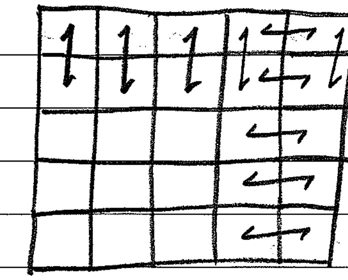
Note:

1. Number of points in a Shelah s-space $= t^s$.

in $[t]^n$:

2. Number of Shelah lines $= \binom{n+1}{2}$
3. Number of Shelah points $\le \binom{n+1}{2} t$

Defn A coloring $f: [t]^n \longrightarrow [r]$ is fliptop if whenever $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ satisfy $x_j = y_j$ for $j \ne i$ and $\{x_i, y_i\} = \{t-1, t\}$ (for some i) then $f(x) = f(y)$.



Fliptop coloring: Squares connected with $\longleftrightarrow$ are colored the same.

Defn Let $L_1 \times L_2 \times \cdots L_s$ be a Shelah s-space with canonical map

$$\varphi : L_1 \times L_2 \times \cdots \times L_s \longrightarrow [t]^s.$$

A coloring f of $L_1 \times L_2 \times \cdots \times L_s$ is fliptop if the coloring g of $[t]^s$ given by

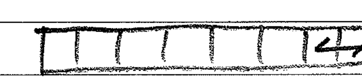
$$g(x) = f(\varphi^{-1}(x))$$

is fliptop.

Defn If $f: [t]^n \longrightarrow [c]$ is a coloring and $S \subseteq [t]^n$ is a Shelah s-space then S is fliptop with respect to f if the induced coloring is fliptop.

Lemma If $n \ge c$ and $f: [t]^n \longrightarrow [c]$ then there exists a Shelah line that is fliptop with respect to f.

$$t-1,\ t-1,\ \nabla,\ \nabla,\ \nabla,\ \dots,\ \nabla,\ t,\ t,\ t$$



Pf For $0 \le i \le n$ define

$$y_i = (x_{i,1}, x_{i,2}, \dots, x_{i,n})$$

by $$x_{i,j} = \begin{cases} t-1, & \text{if } j \le i; \\ t, & \text{if } j > i. \end{cases}$$

By pigeonhole, $\exists$ $a < b$ such that

$$f(y_a) = f(y_b).$$

y_a and y_b are the last points in some Shelah line. ∎

Theorem Let r, s, t be fixed positive integers. Define $n_1, n_2, \dots, n_s$ by

$$n_1 = r^{t^{s-1}}$$

$$A_i = \left[\prod_{j \le i} \binom{n_j+1}{2}\right] t^{s-1} \quad \text{for } i = 1, \dots, s-1$$

$$n_{i+1} = r^{A_i} \quad \text{for } i = 1, \dots, s-1.$$

If $n = n_1 + n_2 + \cdots + n_s$ and

$$f : [t]^n \longrightarrow [r]$$

then there is a Shelah s-space (respecting this choice of parameters) that is fliptop with respect to f.

Proof View $[t]^n$ as $[t]^{n_1} \times [t]^{n_2} \times \cdots \times [t]^{n_s}$ and write $y \in [t]^n$ as $(y_1, \dots, y_s)$ where $y_i \in [t]^{n_i}$.

Note: $\left| \{ (y_1, \dots, y_i) : y_j \text{ is a Shelah point in } [t]^n \text{ for } j = 1, \dots, i \} \right|$

$$\le \left[\prod_{j=1}^{i} \binom{n_j+1}{2}\right] t^i.$$

We define an equivalence relation on $[t]^{n_s}$ by

$$y_s \sim x_s \iff f((y_1, \dots, y_{s-1}, y_s)) = f((y_1, \dots, y_{s-1}, x_s))$$

for all Shelah points $y_1, \dots, y_{s-1}$ (where $y_i \in [t]^{n_i}$).

Number of equivalence classes $\le r^{[\prod_{j=1}^{s-1} \binom{n_j+1}{2}] t^{s-1}}$

$$= r^{A_{s-1}} = n_s.$$

By the lemma, there exists a Shelah line $L_s \subseteq [t]^{n_s}$ that is fliptop with respect to this coloring.

Wed
2 Sept
2009

Shelah point: $(t-1, t-1, \ldots, t-1, s, s, \ldots, s, t, t, \ldots t)$

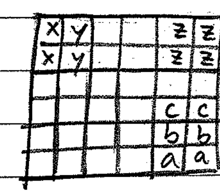
Shelah line: $(t-1, t-1, \ldots, t-1, \square, \square, \ldots, \square, t, t, \ldots, t)$

Shelah s-space: $L_1 \times L_2 \times \cdots \times L_s$
where L_i is a Shelah line for each i.

Canonical map $\varphi: L_1 \times L_2 \times \cdots \times L_s \longrightarrow [t]^s$

Hales–Jewett: $\forall r, t\ \exists n$ s.t.
for all $f: [t]^n \to [r]$ $\exists$ a monochromatic line.

fliptop $f: [t]^n \longrightarrow [r]$



$L_1 \times L_2 \times \cdots \times L_s$ is fliptop w.r.t. f

or

f is fliptop w.r.t. $L_1 \times L_2 \times \cdots \times L_s$

Lemma If $n \geq c$ and $f: [t]^n \longrightarrow [c]$ then there exists a Shelah line that is fliptop w.r.t. f.

Theorem Let r, s, t be fixed positive integers. Define $n_1, \ldots, n_s$ by

$$n_1 = r^{t^{s-1}},$$

$$A_i = \left[\prod_{j \leq i} \binom{n_j + 1}{2} \right] t^{s-1},$$

$$n_{i+1} = r^{A_i}.$$

If $n = n_1 + \cdots + n_s$ and $f: [t]^n \longrightarrow [r]$ then there is a Shelah s-space that is fliptop with respect to f.

Note: The dimensions of the lines that compose this Shelah s-space are $n_1, n_2, \ldots, n_s$.

Proof (continued)
Suppose $L_{i+1}, \ldots, L_s$ have been determined. We define an equivalence relation on $[t]^{n_i}$ by setting $y_i \sim x_i$ iff

$$f(a_1, a_2, \ldots, a_{i-1}, y_i, z_{i+1}, \ldots, z_s)$$

$$= f(a_1, a_2, \ldots, a_{i-1}, x_i, z_{i+1}, \ldots, z_s)$$

for all Shelah points $a_j \in [t]^{n_j}$ for $j = 1, \ldots, i-1$ and all $z_j \in L_j$ for $j = i+1, \ldots, n$.

Number of choices for a_j

$= $ number of Shelah points in $[t]^n$

$\leq \binom{n_j+1}{2} t.$

Number of choices for z_j

$=$ number of points in L_j

$= t.$

Number of equivalence classes

$$\leq r^{[\prod_{j=1}^{i-1}\binom{n_j+1}{2}]t^{s-1}} = r^{A_{i-1}} = n_i.$$

By the lemma, there is a Shelah line L_i that is fliptop w.r.t. this "coloring."

Consider $L_1 \times L_2 \times \cdots \times L_s$.
We claim that this is fliptop w.r.t. f.

Consider $(x_1, x_2, \ldots, x_s)$, $(y_1, y_2, \ldots, y_s) \in L_1 \times L_2 \times \cdots \times L_s$ such that $x_j = y_j$ for all $i \neq j$, the "middle" positions of x_i are $t-1$ and the "middle" positions of y_i are t.

Note that
(i) for $j<i$, $x_j = y_j$ are Shelah points;
(ii) for $j>i$, $x_j = y_j$ is in L_j.

Since L_i is fliptop w.r.t. the coloring at the ith step we have
$f((x_1, \ldots, x_s)) = f((y_1, \ldots, y_s))$. $\square$

<u>Proof of Hales–Jewett</u>

Let $HJ(r,t)$ be the minimum n such that any $f : [t]^n \to [r]$ has a monochromatic line.

To show: These numbers exist.

We go by induction on t.

$HJ(r,1) = 1$.

Suppose $HJ(r, t-1)$ exists.
Set $s = HJ(r, t-1)$ and let n be given by the previous theorem.

Consider $f : [t]^n \to [r]$. By the theorem, there is a Shelah s-space $L_1 \times L_2 \times \cdots \times L_s$ which is fliptop w.r.t. f.

Consider the coloring $g : [t-1]^s \to [r]$ defined by $g(x) = f(\varphi^{-1}(x))$, where $\varphi : L_1 \times \cdots \times L_s \to [t]^s$ is the canonical map.

By the inductive assumption, there is a line $w_1, \ldots, w_{t-1} \in [t-1]^s$ that is monochromatic under g. So,

$$\varphi^{-1}(w_1), \varphi^{-1}(w_2), \ldots, \varphi^{-1}(w_{t-1})$$

is monochromatic under f.

Since f is fliptop w.r.t. $L_1 \times L_2 \times \cdots \times L_s$, the extension of this set to a line in $[t]^n$ is monochromatic. $\square$

Wed
9 Sept
2009

A quick review of enumeration

Functions

Let N, R be sets such that $|N| = n$ and $|R| = r$, $N = \{x_1, \ldots, x_n\}$.

$$R^N := \{\text{functions } f: N \to R\}$$
$$\cong \{\text{vectors indexed by } N \text{ with entries in } R\}$$

$$f: N \to R \longleftrightarrow (f(x_1), f(x_2), \ldots, f(x_n))$$

$$|R^N| = r^n.$$

Number of injections $f: N \to R$:
$$r(r-1)(r-2)\cdots(r-n+1) =: (r)_n$$

Number of bijections $= n!$ (when $n = r$)

Number of surjections $= ?$

Stirling numbers of the second kind

Let $S(n,r)$ be the number of <u>unordered</u> partitions of N into r nonempty parts.

<u>e.g.</u> $S(5,2) = \binom{5}{1} + \binom{5}{2} = 15.$
(1 and 4) (2 and 3)

Number of surjections $f: N \to R$ is $S(n,r) \cdot r!$.

(To get a surjection:
1. Partition N into r nonempty parts.
2. Form a bijection between the parts and R.)

R is a set and $|R| = r$.

$$2^R := \{A : A \subseteq R\} \cong \{0,1\}^R$$

$A \subseteq R \longleftrightarrow 1_A$ (indicator function of A)

$$1_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

$|2^R| = 2^r$.

$\binom{R}{k} = \{k\text{-element subsets of } R\}$

$\left|\binom{R}{k}\right| =: \binom{r}{k}$

Let $T = \{(x_1, \dots, x_k) : x_i \in R \text{ for } i = 1, \dots, k \text{ and } x_1, \dots, x_k \text{ distinct}\}$

$$\cong \{\text{injections } f : \underbrace{[k]}_{\{1,\dots,k\}} \to R\}$$

so $|T| = (r)_k$.

On the other hand, $|T| = k!\binom{r}{k}$:
We get an ordered k-tuple by
1. choosing a k-set,
2. ordering it.

So, $\binom{r}{k} = \frac{|T|}{k!} = \frac{(r)_k}{k!} = \frac{r!}{k!(r-k)!}$.

<u>Note:</u>

(i) We get $\binom{r}{k} = \binom{r}{r-k}$ via the <u>bijection</u> $A \longleftrightarrow \overline{A}$.

(ii) We have shown $\sum_{k=0}^{r} \binom{r}{k} = 2^r$.

(iii) If $k < r/2$ then

$$k!(r-k)! \geq (k+1)!(r-k-1)!$$

$$\Rightarrow \binom{r}{k} \leq \binom{r}{k+1}.$$

Thus,

$$\binom{r}{0} \leq \binom{r}{1} \leq \binom{r}{2} \leq \cdots \leq \binom{r}{\lfloor r/2 \rfloor} = \binom{r}{\lceil r/2 \rceil} \geq \cdots \geq \binom{r}{r}.$$

<u>Defn</u> A sequence $\{a_k\}_{k=0}^{r}$ is <u>unimodal</u> if $\exists \ell$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_\ell \geq a_{\ell+1} \geq \cdots \geq a_r.$$

A sequence $\{a_k\}_{k=0}^{r}$ is <u>log-concave</u> if $a_i > 0 \ \forall i$ and $a_k^2 \geq a_{k-1} a_{k+1}$.

$\boxed{\text{EX}}$

1. The sequence of binomial coefficients is log-concave.
2. log-concave $\Rightarrow$ unimodal.

Binomial Theorem For $n \in \mathbb{N}$,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

(over any field).

Pf Expand

$$\underbrace{(x+y)(x+y)(x+y)\cdots(x+y)}_{n \text{ terms}}. \quad \square$$

Note: This also holds for $n \in \mathbb{C}$ if

1. replace $\sum_{i=0}^{n}$ with $\sum_{i=0}^{\infty}$
2. $\binom{n}{i} = \frac{n(n-1)(n-2)\cdots(n-i+1)}{i!}$
3. $|x/y| < 1$.

e.g. 1. $\sum_{k=0}^{r} \binom{r}{k} = \sum_{k=0}^{r} 1^k 1^{r-k} \binom{r}{k}$

$= (1+1)^r = 2^r$.

2. $\sum_{k=0}^{r} (-1)^k \binom{r}{k} = (-1+1)^r = 0$.

Alternate proof of (2): Work in $\mathbb{F}_2^n$.
Let $v \in \mathbb{F}_2^n$ be a vector with an odd number of 1's.

$$\underbrace{\sum_{k=0}^{r} (-1)^k \binom{r}{k}}_{\sum_{A \in \binom{R}{k}} (-1)^k} = \sum_{A \subseteq R} (-1)^{|A|}$$

$$= \sum_{u \in \mathbb{F}_2^r} (-1)^{(\#\ 1\text{'s in } u)}$$

$$= \sum_{u \in \mathbb{F}_2^r} (-1)^{(\#1\text{'s in } u+v)}$$

The map $g: \mathbb{F}_2^r \to \mathbb{F}_2^r$, $u \mapsto u+v$ is a bijection.

$$= \sum_{u \in \mathbb{F}_2^r} (-1)(-1)^{(\#1\text{'s in } u)}$$

$$= (-1) \sum_{u \in \mathbb{F}_2^r} (-1)^{(\#1\text{'s in } u)}. \quad \square$$

A couple of binomial identities

1. $\binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k}$
2. Vandermonde convolution

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}$$

Pf 1. Partition $\binom{X \cup Y}{k}$, $|X| = x$, $|Y| = y$, where X, Y disjoint.

Pf 2.

$$\sum_{n=0}^{x+y} \binom{x+y}{n} t^n = (1+t)^{x+y}$$

$$= (1+t)^x (1+t)^y = \left(\sum_{k=0}^{x} \binom{x}{k} t^k\right)\left(\sum_{\ell=0}^{y} \binom{y}{\ell} t^\ell\right)$$

So, $\sum_{n=0}^{x+y} \binom{x+y}{n} t^n = \sum_{n=0}^{x+y} \left[\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}\right] t^n$

but the coefficients should be equal term by term.

Generating Functions

Convention: $a_n \in \mathbb{C}$, $n \in \mathbb{N} = \{\text{nonnegative integers}\}$.

Defn
1. The ordinary generating function for $\{a_i\}_{i=0}^{\infty}$ is
$$\sum_{n\geq 0} a_n x^n.$$
2. The exponential generating function for $\{a_i\}_{i=0}^{\infty}$ is
$$\sum_{n\geq 0} \frac{a_n x^n}{n!}.$$

We can consider these as either
1. functions of x, or
2. formal objects.

For (2) we consider, for a field $\mathbb{F}$,
$$[\![\mathbb{F}]\!] = \left\{ \sum_{n=0}^{\infty} f_n x^n : f_i \in \mathbb{F} \right\}$$
as a ring.
$$\sum_{n=0}^{\infty} f_n x^n + \sum_{n=0}^{\infty} g_n x^n = \sum_{n=0}^{\infty} (f_n + g_n) x^n$$
$$\left(\sum_{n=0}^{\infty} f_n x^n\right)\left(\sum_{n=0}^{\infty} g_n x^n\right) = \sum_{k=0}^{\infty} \left[\sum_{\ell=0}^{k} f_\ell g_{k-\ell}\right] x^k$$

Note: Two elements of $[\![\mathbb{F}]\!]$ are equal iff all coefficients are equal.

By the way, $[\mathbb{F}]$ is the ring of polynomials.

Examples

1. $a_n = 1$ for all n.
$$\sum_{n\geq 0} x^n = \frac{1}{1-x}$$
Note:
(a) This is true as a function of x for $x \in \mathbb{C}$, $|x| < 1$.

(b) It is also true in $[\![\mathbb{C}]\!]$ as
$$(1-x)(1+x+x^2+\cdots) = 1.$$
So, $1-x$ is the multiplicative inverse of $\sum_{n=0}^{\infty} x^n$.

2. Defn A weak r-composition of an integer n is an ordered $\mathbb{N}$-sequence $(a_1, a_2, \ldots, a_r)$ such that $\sum_{i=1}^{r} a_i = n$.

Prop The number of weak r-compositions of n is $\binom{n+r-1}{r-1}$.

Pf 1. We form a bijection.
$$\{ v \in \{0,1\}^{n+r-1} : (\#1\text{'s in } v) = r-1 \}$$
$$\longleftrightarrow \{ \text{weak } r\text{-compositions of } n \}.$$

$$\underbrace{0}_{a_1}\,1\,\underbrace{0\,0}_{a_2}\,1\,\underset{a_3=0}{\uparrow}\,1\,\underbrace{0\,0\,0}_{a_4}\,1\,\underbrace{0\,0\,0\,0\,0}_{a_5}$$

$\square$

Pf 2. Let r be fixed. Let b_n be the number of weak r-compositions of n.

$$\sum_{n=0}^{\infty} b_n x^n = (1+x+x^2+\cdots)^r$$

$$\left[\begin{array}{l} \text{e.g., } r=3: \\ (1+x+x^2+\cdots)^3 = 1\cdot1\cdot1 + 1\cdot1\cdot x + 1\cdot x\cdot 1 + x\cdot1\cdot1 \\ \qquad + 1\cdot1\cdot x^2 + 1\cdot x\cdot x^2 + \cdots \end{array}\right]$$

$$= \left(\frac{1}{1-x}\right)^r$$

Now
$$\frac{1}{(1-x)^r} = \frac{1}{(r-1)!}\frac{d^{r-1}}{dx^{r-1}}\left(\frac{1}{1-x}\right)$$
$$= \frac{1}{(r-1)!}\frac{d^{r-1}}{dx^{r-1}}\sum_{m=0}^{\infty} x^m$$
$$= \frac{1}{(r-1)!}\sum_{m=r-1}^{\infty} (m)_{r-1}\, x^{m-(r-1)}$$
$$= \sum_{n=0}^{\infty} \frac{(n+r-1)_{r-1}}{(r-1)!}\, x^n \qquad [n=m-r+1]$$
$$= \sum_{n=0}^{\infty} \binom{n-r+1}{r-1} x^n. \qquad \square$$

Mon 14 Sept 2009

Example 3. How many ways are there to walk up n stairs going 1 or 2 steps at a time?
i.e., the number of ways to write
$n = e_1 + e_2 + \cdots + e_k$, k varies, $e_i \in \{1,2\}$.
A recurrence: $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$,
$a_0 = 1$, $a_1 = 1$.
So these are the Fibonacci numbers.

Consider the generating function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + x + \sum_{n=2}^{\infty} a_n x^n$$
$$= 1 + x + x\sum_{n=2}^{\infty} a_n x^{n-1} + x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$
$$= 1 + x + x(f(x) - 1) + x^2 f(x).$$

So, $f(x)[x^2 + x - 1] = -1$,
$$f(x) = \frac{-1}{x^2+x-1} = \frac{1}{1-x-x^2}$$

Use partial fractions.

$$1 - x - x^2 = (1-\alpha x)(1-\beta x)$$
where $\alpha = \frac{1+\sqrt5}{2}$, $\beta = \frac{1-\sqrt5}{2}$

So, $f(x) = \frac{1}{1-x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$
$$= A\sum_{n=0}^{\infty} \alpha^n x^n + B \sum_{n=0}^{\infty} \beta^n x^n.$$

So, $a_n = A\alpha^n + B\beta^n \to 0$.

Notation If $f(x) = \sum_{n\geq 0} a_n x^n$ we write $[x^n]f := a_n$.

Example 4. $x_1, x_2, \ldots, x_n$ are variables with a nonassociative product, e.g., $(x_1x_2)x_3 \neq x_1(x_2x_3)$.

a_n = # of possible values for $x_1 x_2 \ldots x_n$
= # of ways to "bracket" the product.

e.g. $a_1 = a_2 = 1$, $a_3 = 2$, $a_4 = 5$: $\begin{cases} (x_1x_2x_3)x_4 \to 2 \\ (x_1x_2)(x_3x_4) \to 1 \\ x_1(x_2x_3x_4) \to 2 \end{cases}$

A recurrence:

$$a_n = \sum_{k=1}^{n-1} a_k a_{n-k} \quad \text{for } n \geq 2.$$

Consider the generating function $g(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$[x^n]\, g^2 = \sum_{k=0}^{n} a_k a_{n-k}$$

So, if we set $a_0 = 0$ we have

$$[x^n]\, g^2 = a_n \quad \text{for } n \geq 2.$$

So,

$$g = g^2 \underbrace{+\,x}_{\substack{\text{because recurrence works for } n\geq 2; \\ g \text{ has linear term, but } g^2 \text{ does not.}}}$$

In other words

$$g^2 - g + x = 0.$$

$$g(x) = \frac{1 \pm \sqrt{1-4x}}{2}$$

[Note: A parenthesized expression is a string of "(", ")", "x_i", so $a_n \leq \binom{3n}{n,n,n}$, which is exponentially bounded, so $g(x)$ has a nonzero radius of convergence.]

Since $g(0) = a_0$ we have

$$g(x) = \frac{1 - \sqrt{1-4x}}{2} = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} (-4)^n \binom{1/2}{n} x^n$$

so

$$a_n = -\frac{1}{2}\binom{1/2}{n}(-4)^n \quad \text{for } n \geq 1$$

$$\boxed{EX} \; = \frac{1}{n}\binom{2n-2}{n-1} \qquad \text{"Catalan numbers."}$$

Note: Multiplying exponential generating functions.

$$\left(\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n\right)\left(\sum_{n=0}^{\infty} \frac{b_n}{n!} x^n\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{a_k}{k!} \cdot \frac{b_{n-k}}{(n-k)!}\right) x^n$$
$$= \sum_{n=0}^{\infty} \underbrace{\left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right)}_{c_n} \frac{x^n}{n!}$$

e.g. A, B disjoint alphabets,
$\mathcal{A} \subseteq A^* = \{\text{finite strings in } A\}$,
$\mathcal{B} \subseteq B^*$
languages.

(continued →)

$a_n = \#$ words in $\mathcal{A}$ of length n
$b_n = \#$ words in $\mathcal{B}$ of length n

f_a, f_b are the corresponding exponential generating functions

$$f_c = f_a f_b$$

Then
$$n!\,[x^n]\, f_c = \#\text{ of "shuffles" of } \mathcal{A} \text{ and } \mathcal{B},$$
i.e., $\omega \in (A \cup B)^*$ such that $\omega|_A \in \mathcal{A}$ and $\omega|_B \in \mathcal{B}$.

__Formal power series__

$\mathbb{C}[[x]]$ — product, sum

__Inverses:__ $g(x) = f(x)^{-1}$ means $g(x)f(x) = 1$.

__e.g.__ $(1-x)^{-1} = 1 + x + x^2 + \cdots$

__Prop__ $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has an inverse if and only if $a_0 \neq 0$. Also, inverses are unique.

__Pf__ We want $g(x) = \sum_{n=0}^{\infty} b_n x^n$ s.t. $gf = 1$.

$$f(x)\,g(x) = \sum_{n=0}^{\infty} \underbrace{\left[\sum_{k=0}^{n} a_k b_{n-k}\right]}_{= \begin{cases} 1, & \text{if } n=0; \\ 0, & \text{if } n>0. \end{cases}} x^n$$

So $b_0 = a_0^{-1}$ and

$$\sum_{k=0}^{n} a_k b_{n-k} = 0$$
$$\Rightarrow b_n = -a_0^{-1} \sum_{k=1}^{n} a_k b_{n-k}. \quad \square$$

__Compositions of functions__

$$f(g(x)) = ?$$

__E.g.__ 1. $f(x) = \frac{1}{1-x}$, $g(x) = 1+x$.

$$f(g(x)) \stackrel{?}{=} \sum_{n=0}^{\infty} (1+x)^n$$
$$[x^m](f \circ g) = ?$$

This is nonsense. All coefficients should be determined by finite computations.

2. $f(x) = e^x$, $g(x) = 1+x$

$$f(g(x)) \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{(1+x)^n}{n!}$$

Also no good formally (even though we could make sense of the coefficients as infinite sums).

So always take $g(0)=0$; then

$$f(g(x)) = \sum_{n\geq 0} a_n g(x)^n$$

and

$$[x^m](f\circ g) = [x^m]\sum_{n=0}^{m} a_n g(x)^n.$$

Powers

Let $f \in \mathbb{C}[[x]]$.

f^k for $k\in\mathbb{N}$: ok (repeated multiplication, no problems)

$f^{1/k}$ for $k\in\mathbb{N}$?

Potential problems:
(i) might not exist
(ii) might not be unique.

e.g. $f^3 = x$? No cube root of x in $\mathbb{C}[[x]]$.

Prop Let $f = \sum_{n\geq m} a_n x^n$, $a_m \neq 0$. ← "leading" coefficient, i.e., smallest m for which this is true.

$\exists$ kth root of f $\iff$ $k \mid m$.

If so, there are exactly k kth roots.

Pf $\boxed{EX}$. (Similar to proof of inverses.)

Wed 16 Sept 2009

$f=\sum a_n x^n \in \mathbb{C}[[x]]$

- f has an inverse $\iff a_0\neq 0$
- $f(g(x))$ makes sense if $\underbrace{g(0)=0}_{\text{means } g(x)=\sum_{n=1}^{\infty} b_n x^n}$
- $f^{1/k}$ exists $\iff$ a_m is the first nonzero coefficient and $k\mid m$.

When $f^{1/k}$ exists, there are exactly k such formal power series.

Recall: $e^{2\pi i j/k}$ for $j=0,1,\ldots,k-1$ are the kth roots of unity.

$$g^k = f \implies \left[e^{2\pi i j/k} g\right]^k = f.$$

We usually assume a_m (the first nonzero coefficient of f) is in $\mathbb{R}^+$. Then we define $f^{1/k}$ to be the kth root of f whose first nonzero coefficient is in $\mathbb{R}^+$.

Proposition

① $(f^j)^{1/k} = (f^{1/k})^j =: f^{j/k}$

$\boxed{EX}$ ② $(f^{-1})^{j/k} = (f^{j/k})^{-1} =: f^{-j/k}$

for $j,k\in\mathbb{P}$ (positive integers) assuming these exist.

Proofs

Proposition ① $(f^p)^q = f^{pq}$
② $f^p f^q = f^{p+q}$
③ $(fg)^p = f^p g^p$

for $p,q\in\mathbb{Q}$

Convention: $f^0 = 1$

Formal derivative

Operator $D: \mathbb{C}[\![x]\!] \to \mathbb{C}[\![x]\!]$

$$\sum_{n=0}^{\infty} a_n x^n \mapsto \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Proposition

(a) $D(f+g) = Df + Dg$

(b) $D(fg) = fDg + gDf$

(c) $D(f^{\alpha}) = \alpha f^{\alpha-1} D(f)$ for $\alpha \in \mathbb{Q}$, assuming f^{α} exists.

(d) $D(f(g)) = Df(g) \cdot Dg$

Proof $\boxed{\text{EX}}$

Back to counting

Example. $a_n = \#$ partitions of $[n]$ such that each part has size 1 or 2

$= |\{\sigma \in S_n : \sigma^2 = 1\}|$ (involutions).

A recurrence: $a_n = a_{n-1} + (n-1) a_{n-2}$ for $n \geq 2$; setting $a_0 = 1$, $a_1 = 1$

We consider the exponential generating function

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

$$= 1 + x + \sum_{n=2}^{\infty} \frac{a_{n-1} + (n-1)a_{n-2}}{n!} x^n$$

So $Df = 1 + \sum_{n=2}^{\infty} \frac{a_{n-1}}{(n-1)!} x^{n-1} + x \sum_{n=2}^{\infty} \frac{a_{n-2}}{(n-2)!} x^{n-2}$

$$Df = 1 + (f - 1) + xf$$

$$Df = f \cdot (1 + x)$$

$$\frac{Df}{f} = 1 + x$$

So, $f = e^{x + x^2/2}$.

That works!

Suppose $g \in \mathbb{C}[\![x]\!]$ with $g(0) = 0$.

Then $\frac{Df}{f} = Dg$, $f(0) = 1$ has the unique solution

$$f = \exp(g).$$

Pf $\exp(g)$ is a solution of the equation $Df = f Dg$ by the chain rule.
And the solution of this equation is unique:

$$g = \sum_{n=1}^{\infty} b_n x^n$$

$$f = \sum_{n=0}^{\infty} a_n x^n$$

$$(n+1)\, a_{n+1} = [x^n] Df = [x^n] f Dg$$

$$= \sum_{k=0}^{n} a_{n-k} (k+1) b_{k+1}. \qquad \square$$

Recall:

$$\sum \frac{c_n}{n!} x^n = \left(\sum \frac{a_k}{k!} x^k\right)\left(\sum \frac{b_\ell}{\ell!} x^\ell\right)$$

$$\Longrightarrow \quad c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i} \qquad (\star)$$

Example $d_n = \#$ of derangements of $[n]$
$= |\{ \sigma \in S_n : \sigma(i) \neq i \ \forall i \}|$

Set $d_0 = 1$
We have $d_1 = 0$
$d_2 = 1$
$d_3 = 2$

Take $D(x) = \sum_{n=0}^{\infty} \frac{d_n}{n!} x^n.$

Let $P(x) =$ exponential generating function for the number of all permutations of $[n]$

$$= \sum_{n=0}^{\infty} \frac{\#\text{ permutations of } [n]}{n!} x^n = \frac{1}{1-x}.$$

From the fact about multiplying e.g.f.'s we have

$$P(x) = D(x)\, \underbrace{e^x}_{\text{e.g.f. for the sequence of all 1's}}$$

[Note: We are constructing a permutation by partitioning $[n]$ into two sets and applying a derangement to one set and the identity map to the other:
in $(\star)$, $c_n = \#$ permutations of $[n]$
$a_i = \#$ derangements of $[i]$
$b_{n-i} = \#$ identity maps on $[n-i]$.]

So, $D(x) = e^{-x} \cdot \frac{1}{1-x}$

$$\frac{d_n}{n!} = [x^n]\left(e^{-x} \cdot \frac{1}{1-x}\right) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Thus, $d_n = n! \underbrace{\sum_{k=0}^{n} \frac{(-1)^k}{k!}}_{\to \frac{1}{e}.}$

The exponential formula

If $g(x) = \sum_{n=1}^{\infty} \frac{b_n}{n!} x^n$ and $f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$

and $f = \exp(g)$,

then, for $n \geq 1$,

$$a_n = \underbrace{\sum_{\pi \vdash [n]}}_{\pi \text{ is a partition of } [n]} b_1^{c_1(\pi)} b_2^{c_2(\pi)} \cdots b_n^{c_n(\pi)}$$

where $c_i(\pi) = \#$ of parts of size i.

In other words,

$$a_n = \sum_{\pi \vdash [n]} \prod_{\substack{\text{parts} \\ B \text{ of } \pi}} b_{|B|}.$$

e.g. $g(x) = x + x^2/2$ $\left(\text{i.e., } b_i = \begin{cases} 1, & \text{if } i \in \{1, 2\}; \\ 0, & \text{otherwise.} \end{cases}\right)$

If $\sum \frac{a_n}{n!} x^n = f(x) = \exp(x + x^2/2)$

then $a_n = \#$ of partitions of n in which each part has size 1 or 2.

Mon
21 Sept
2009

- Homework 2 due Friday
- No class on October 5, 7

The exponential formula

If $g(x) = \sum_{n=1}^{\infty} b_n x^n/n!$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n/n!$ and $f(x) = e^{g(x)}$, then for $n \geq 1$

$$a_n = \sum_{\pi \vdash [n]} b_1^{c_1(\pi)} b_2^{c_2(\pi)} \cdots b_n^{c_n(\pi)},$$

where $c_i(\pi)$ is the number of parts of π with i elements.

In other words, $a_n = \sum_{\pi \vdash [n]} \prod_{\substack{\text{blocks } B \\ \text{of } \pi}} b_{|B|}$.

Proof

$$\left(\sum_{i=1}^{\infty} b_i \frac{x^i}{i!}\right)^k = \sum_{n=k}^{\infty} \Bigg(\sum_{\substack{\ell_1, \ell_2, \ldots, \ell_k \in \mathbb{P} \\ \ell_1 + \ell_2 + \cdots + \ell_k = n}} \prod_{i=1}^{k} \frac{b_{\ell_i}}{\ell_i!} \Bigg) x^n$$

$$= \sum_{n=k}^{\infty} \Bigg(\sum_{\substack{\ell_1, \ell_2, \ldots, \ell_k \in \mathbb{P} \\ \ell_1 + \ell_2 + \cdots + \ell_k = n}} \binom{n}{\ell_1, \ell_2, \ldots, \ell_k} \prod_{i=1}^{k} b_{\ell_i} \Bigg) \frac{x^n}{n!}$$

$$\frac{a_n}{n!} = [x^n] \left(\sum_{k=0}^{\infty} \frac{\left(\sum_{i=1}^{\infty} b_i x^i/i!\right)^k}{k!} \right)$$

$$\text{so } a_n = \sum_{k=1}^{n} \Bigg(\underbrace{\sum_{\substack{\ell_1, \ell_2, \ldots, \ell_k \in \mathbb{P} \\ \ell_1 + \ell_2 + \cdots + \ell_k = n}}}_{\substack{\text{Note this sums over} \\ \text{all ordered } k\text{-partitions} \\ \text{of } [n] \text{, which necessitates} \\ \text{the } 1/k!.}} \binom{n}{\ell_1, \ell_2, \ldots, \ell_k} \prod_{i=1}^{k} b_{\ell_i} \Bigg) \frac{1}{k!}$$

$\square$

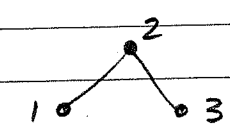
Definitions

graph, cycle, connected, tree, forest, subgraph, spanning tree, component

Recall: If $G=(V,E)$ is a graph, every spanning tree on G has $|V|-1$ edges.

Cayley's formula: The number of spanning trees on $\{1,2,\dots,n\}$ is n^{n-2}.

e.g. $n=3$:



Proof Let $t(n) = \#$ of trees on $[n]$
$f(n) = \#$ of forests on $[n]$
$t_r(n) = \#$ of rooted trees on $[n]$
$f_r(n) = \#$ of rooted forests on $[n]$

Note: Everything is labelled.

Def: (i) rooted tree: some vertex is declared the root.
(ii) rooted forest: a root is specified in each component.

Note: $t_r(n) = n\,t(n)$

Set $t(0) = t_r(0) = 0$
$f(0) = f_r(0) = 1$

Let $T(x) = \sum_{n=0}^{\infty} t(n) \frac{x^n}{n!}$, $F(x) = \sum_{n=0}^{\infty} f(n) \frac{x^n}{n!}$.

Note: By the exponential formula, $F(x) = e^{T(x)}$.

The rooted versions contain some more information. Let

$$T_r(x) = \sum_{n=0}^{\infty} t_r(n) \frac{x^n}{n!}, \quad F_r(n) = \sum_{n=0}^{\infty} f_r(n) \frac{x^n}{n!}.$$

We have $F_r(x) = e^{T_r(x)}$, as before.

Claim: $t_r(n+1) = (n+1) f_r(n)$.

Pf: 1. Choose root vertex r among $[n+1]$.
2. Choose a rooted forest on $[n+1] \setminus \{r\}$.
3. Join the roots of all components to r.
(This is a bijection.) $\square$

So,
$$\begin{aligned} T_r(x) &= \sum_{n=1}^{\infty} \frac{t_r(n)}{n!} x^n \\ &= \sum_{s=0}^{\infty} \frac{(s+1) f_r(s)}{(s+1)!} x^{s+1} \qquad [n=s+1] \\ &= x \sum_{s=0}^{\infty} \frac{f_r(s)}{s!} x^s \\ &= x F_r(x) = x e^{T_r(x)}. \end{aligned}$$

So $T_r(x) = x e^{T_r(x)}$.

Lagrange Inversion Formula

Let $H(x) = \sum_{n=0}^{\infty} b_n x^n$. The equation

$$Y(x) = x\, H(Y(x))$$

has a unique solution $Y(x) \in \mathbb{C}[\![x]\!]$ where

$$[x^n]\, Y = \frac{1}{n} [x^{n-1}]\, H^n(x).$$

e.g. If $H(x) = \exp(x)$ then

$$[x^n]\, Y = \frac{1}{n} [x^{n-1}]\, e^{nx} \qquad [\text{note } (e^x)^n = e^{nx} \text{ in } \mathbb{C}[\![x]\!].]$$

$$\overset{(*)}{=} \frac{1}{n}\left(\frac{n^{n-1}}{(n-1)!}\right) = \frac{n^{n-1}}{n!}.$$

$\boxed{EX}$ Make sure $(*)$ computes in $\mathbb{C}[\![x]\!]$.

So, $t_r(n) = n^{n-1}$ and $t(n) = n^{n-2}$.

Note: The uniqueness in the Lagrange Inversion Formula follows as usual:

Set $Y(x) = \sum_{n=1}^{\infty} a_n x^n$.

$$a_n = [x^{n-1}]\, H(Y(x))$$

$$= \underbrace{[x^{n-1}] \sum_{k=1}^{n-1} b_k \left(\sum_{\ell=1}^{\infty} a_\ell x^\ell\right)^k}_{\text{does not involve } a_\ell \text{ for } \ell \geq n.}$$

So a_n is uniquely determined by $a_1, \ldots, a_{n-1}$.

The Prüfer correspondence

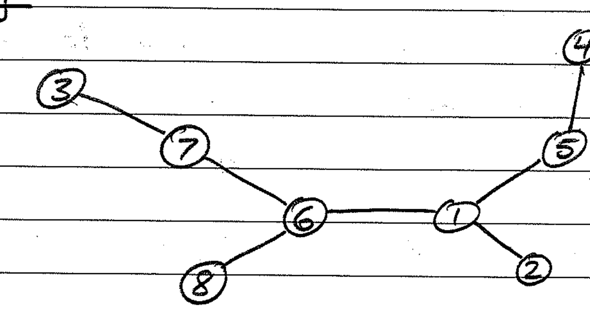
(a combinatorial proof of Cayley's theorem)

A bijection $\varphi : \{\text{labelled trees on } [n]\} \longrightarrow [n]^{n-2}$.

Let T be a tree on $[n]$. We construct $\varphi(T)$ by iterating the following rule:

1. Delete the leaf with the smallest label.
2. Record the label of its neighbor in the sequence.

e.g.



$\mapsto (1,7,5,1,6,6)$

Wed
23 Sept
2009

No class Oct 5,7.

The Prüfer correspondence

$$\varphi: \{\text{labelled trees on } [n]\} \longrightarrow [n]^{n-2}$$

Let T be a tree. We generate $\varphi(T)$ by iterating the following ($n-2$ times):

1. Remove the leaf in T with the smallest label.
2. Record the neighbor of this leaf in $\varphi(T)$.

Observations:

(i) An edge is left over at the end.

(ii) $d(x) = 1 + (\#\text{ times } x \text{ appears in } \varphi(T))$

(iii) If x is not a leaf, $d(x)$ does not decrease until x appears in $\varphi(T)$.

The inverse

Let $(a_1, \dots, a_{n-2}) \in [n]^{n-2}$.
Let $X = [n]$ (the active vertices),
$T =$ empty graph.
For $i = 1, \dots, n-2$:
Let $x = \min\{y \in X : \nexists\, j \geq i \text{ s.t. } a_j = y\}$.
Add $\{x, a_i\}$ to T.
Remove x from X.
Add the "edge" X to T (X will be a set of two vertices).

Lagrange Inversion Formula

Let $H(x) = \sum_{n=0}^{\infty} b_n x^n$. The unique solution $Y(x) \in \mathbb{C}[[x]]$ of the equation

$$Y(x) = x H(Y(x))$$

satisfies

$$[x^n]\, Y(x) = \frac{1}{n} [x^{n-1}]\, H^n(x).$$

An aside: More generally, if $f(x) \in \mathbb{C}[[x]]$ then

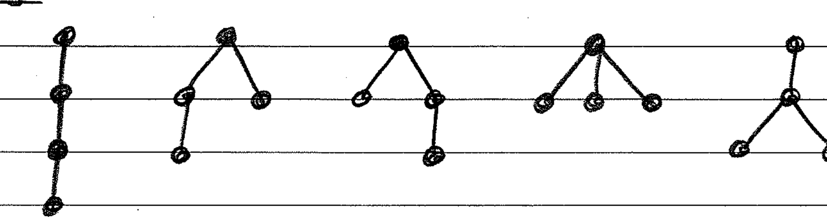
$$[x^n]\, f(Y(x)) = \frac{1}{n} [x^{n-1}]\, Df \cdot H^n(x).$$

e.g. For the Lagrange inversion formula as stated, take $f(x) = x$.

Proof.

Defn A plane tree is a rooted tree with a (left-to-right) ordering on the children of every vertex.

e.g. Plane trees with 4 vertices.



[see next page] → $u_0 u_1^3$ $\quad u_0^2 u_1 u_2$ $\quad u_0^2 u_1 u_2$ $\quad u_0^3 u_3$ $\quad u_0^2 u_1 u_2$

If τ is a plane tree,

$$s_i(\tau) = \#\text{ vertices in } \tau \text{ with } i \text{ children}$$

$$n(\tau) = \#\text{ vertices in } \tau$$

<u>Note:</u> $\sum_{i\geq 0} s_i(\tau) = n(\tau)$

$$\sum_{i\geq 0} i s_i(\tau) = n(\tau) - 1$$

We set

$P(x)$ = formal power series for # of plane trees, keeping track of degrees

$$= \sum_{\text{plane trees } \tau} \left[\prod_{i\geq 0} u_i^{s_i(\tau)}\right] x^{n(\tau)}$$

We view this as an element of

$$\left(\mathbb{C}[u_0, u_1, u_2, \ldots]\right)[\![x]\!].$$

<u>e.g.</u> $[x^4]\,P(x) = u_0 u_1^3 + 3u_0^2 u_1 u_2 + u_0^3 u_3.$

Let $U(x) = \sum_{i=0}^{\infty} u_i x^i$

(view this as an element of $\mathbb{C}[u_0, u_1, u_2, \ldots][\![x]\!]$).

<u>Claim 1:</u> $P(x) = x\,U(P(x)).$

<u>Claim 2:</u> $[x^n]\,P(x) = \frac{1}{n}[x^{n-1}]\,U^n(x).$

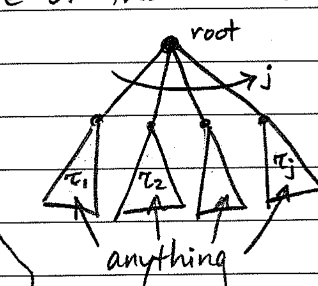
By evaluating (i.e., setting $u_i = b_i$ for $i = 0, 1, 2, \ldots$), we recover the Lagrange inversion formula from these two claims.

<u>Proof of Claim 1</u>

$$P(x) = \sum_{\text{plane trees } \tau} \left[\prod_{i\geq 0} u_i^{s_i(\tau)}\right] x^{n(\tau)}$$

$$= \sum_{j=0}^{\infty} \underbrace{\sum_{\substack{\text{plane trees } \tau \\ \deg(\text{root}(\tau)) = j}} \left[\prod_{i\geq 0} u_i^{s_i(\tau)}\right] x^{n(\tau)}}$$

One of these looks like:



so

$$\prod_{i\geq 0} u_i^{s_i(\tau)} x^{n(\tau)} = x u_j \cdot \prod_{k=1}^{j} \underbrace{\Big[\prod_{x\in\tau_k} u_{\overset{\text{number of children of } x}{d_\tau(x)-1}}\Big] x^{n(\tau_k)}}_{\text{the term in } P(x) \text{ that comes from } \tau_k}$$

$$= \sum_{j=0}^{\infty} x u_j P^j(x)$$

$$= x \sum_{j=0}^{\infty} u_j P^j(x)$$

$$= x\,U(P(x)).$$

Proof of Claim 2

Note that Claim 2 is equivalent to

$$\sum_{\substack{\text{plane trees } \tau \\ n(\tau)=n}} \prod_{i\geq 0} u_i^{s_i(\tau)} = \frac{1}{n} \sum_{\substack{k_1,k_2,\ldots,k_n\in\mathbb{N} \\ k_1+k_2+\cdots+k_n=n-1}} \prod_{j=1}^{n} u_{k_j}$$

(includes 0)

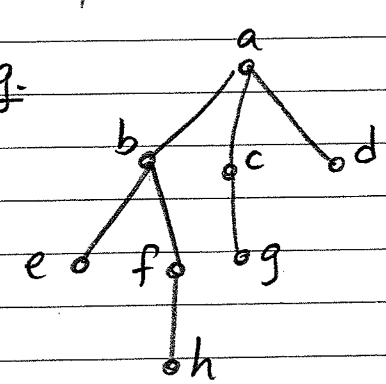
e.g. $n=4$.

$$LHS = u_0 u_1^3 + 3u_0^2 u_1 u_2 + u_0^3 u_3$$

$$RHS = \frac{1}{4}\left[4u_0u_1^3 + 12u_0^2u_1u_2 + u_0^3u_3\right]$$

Defn The postorder of vertices in plane tree τ.
To generate this order, recursively do the following: Suppose the subtrees of the root of τ are $\tau_1, \ldots, \tau_j$ (in order).
List (in postorder) the vertices of τ_1, then τ_2, then τ_3, ..., then τ_j, and finally list the root.

e.g.



Postorder: e, h, f, b, g, c, d, a.

PDS: $(0,0,1,2,0,1,0,3)$.

The postorder degree sequence (PDS) of a plane tree τ is

$$PDS(\tau) = (d_1(\tau), d_2(\tau), \ldots, d_{n(\tau)}(\tau))$$

where $d_i(\tau)$ = # of children of the ith vertex in the postorder.

Idea: Cyclically shift the PDS.

$$(0,0,1,2,0,1,0,3) \longrightarrow (3,0,0,1,2,0,1,0)$$

Mon 28 Sept 2009

Plane tree: a rooted tree with an ordering on the children of every vertex.

$s_i(\tau)$ = # of vertices with i children
$n(\tau)$ = # of vertices in τ.

In $\mathbb{C}[u_0, u_1, u_2, \ldots][[x]]$,

$$P(x) = \sum_{\tau} \Big(\prod_i u_i^{s_i(\tau)} \Big) x^{n(\tau)}$$

$$U(x) = \sum_{n \geq 0} u_n x^n$$

Claim 1: $P(x) = x\, U(P(x))$. ✓

Claim 2: $[x^n] P(x) = \frac{1}{n} [x^{n-1}] U^n(x)$.

It remains to prove Claim 2.

Note: Claim 2 is equivalent to

$$\sum_{\tau: n(\tau)=n} \prod_i u_i^{s_i(\tau)} = \frac{1}{n} \Big[\sum_{\substack{k_1, \ldots, k_n \in \mathbb{N} \\ k_1 + \cdots + k_n = n-1}} \prod_{i=1}^n u_{k_i} \Big],$$

which is equivalent to

$$\sum_{\tau: n(\tau)=n} u_{d_1(\tau)} u_{d_2(\tau)} \cdots u_{d_n(\tau)} = \frac{1}{n} \sum_{\substack{k_1 + \cdots + k_n \in \mathbb{N} \\ k_1 + \cdots + k_n = n-1}} \prod_{i=1}^n u_{k_i}$$

where $(d_1(\tau), \ldots, d_n(\tau))$ is the postorder degree sequence of τ.

Note: $PDS(\tau) = PDS(\tau') \iff \tau = \tau'$

Definition Given $\vec{k} = (k_1, \ldots, k_n)$, a cyclic shift of $\vec{k}$ is a sequence of the form $(k_i, k_{i+1}, \ldots, k_n, k_1, k_2, \ldots, k_{i-1})$.

Proposition If $\vec{k} \in \mathbb{N}^n$ and $\sum_{i=1}^n k_i = n-1$ then exactly one cyclic shift of $\vec{k}$ is a postorder degree sequence of a plane tree.

The note and proposition imply the Lagrange inversion formula.

Proof Let $\mathcal{A} = \{ \vec{y} \in \mathbb{N}^n : \sum_{i=1}^n y_i = n-1 \}$.

For $x, y \in \mathcal{A}$ we write $x \sim y$ if x is a cyclic shift of y. This defines a partition of $\mathcal{A}$.

Let $\mathcal{A}' \subseteq \mathcal{A}$ contain a unique representative of each part in this partition (i.e., each equivalence class). So,

$$\frac{1}{n} \sum_{y \in \mathcal{A}} \prod_{i=1}^n u_{y_i} = \sum_{y \in \mathcal{A}'} \prod_{i=1}^n u_{y_i}. \quad (*)$$

For each $y \in \mathcal{A}'$ let τ_y be the plane tree that has $PDS(\tau_y) = y$ (or some cyclic shift of y).

$$(*) = \sum_{\tau: n(\tau)=n} \prod_{i=1}^n u_{d(i)}. \qquad \square$$

Inclusion/Exclusion

$A_1, \ldots, A_m \subseteq \Omega$. For $I \subseteq [m]$ set

$$A_I = \begin{cases} \bigcap_{i \in I} A_i, & \text{if } I \neq \emptyset; \\ \Omega, & \text{if } I = \emptyset. \end{cases}$$

Then $\left| \bigcap_{i=1}^m \overline{A_i} \right| = \left| \overline{\bigcup_{i=1}^m A_i} \right| = \sum_{I \subseteq [m]} (-1)^{|I|} |A_I|.$

Proof For each $x \in \Omega$, set $N_x = \{ i : x \in A_i \}$.

$$\sum_{I \subseteq [m]} (-1)^{|I|} |A_I| = \sum_{I \subseteq [m]} \sum_{x \in A_I} (-1)^{|I|}$$
$$= \sum_{x \in \Omega} \sum_{I : x \in A_I} (-1)^{|I|}$$
$$= \sum_{x \in \Omega} \sum_{I \subseteq N_x} (-1)^{|I|}$$
$$= \sum_{x \in \Omega} \sum_{\ell=0}^{|N_x|} (-1)^{\ell} \binom{|N_x|}{\ell} \qquad [\ell = |I|]$$
$$= \left| \{ x \in \Omega : N_x = \emptyset \} \right|$$
$$= \left| \overline{\bigcup_{i=1}^m A_i} \right|. \qquad \square$$

Because $\sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} = \begin{cases} 1, & \text{if } k=0; \\ 0, & \text{if } k>0. \end{cases}$

$\| \quad (1+(-1))^k$

Example. Derangements. $D_n = \left| \{ \sigma \in S_n : \sigma(i) \neq i \ \forall i \} \right|$.

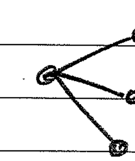
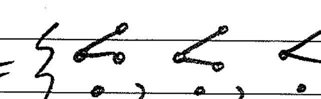
Let $\Omega = S_n = \{ \text{permutations of } [n] \}$.
For $i = 1, \ldots, n$ let $A_i = \{ \sigma \in S_n : \sigma(i) = i \}$.

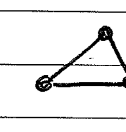
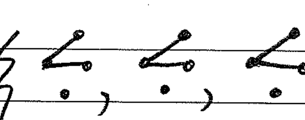
$$D_n = \left| \overline{\bigcup_{i=1}^n A_i} \right| = \sum_{I \subseteq [n]} (-1)^{|I|} |\underbrace{A_I}_{\{\sigma \in S_n : \sigma(i) = i \ \forall i \in I\}}|.$$
$$= \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)!$$
$$= n! \left(\sum_{j=0}^{n} \frac{(-1)^j}{j!} \right).$$

The edge reconstruction hypothesis

Defn For graphs $G = (V,E)$ and $H = (W,F)$ an isomorphism is a bijection $\varphi : V \to W$ such that
$\{x,y\} \in E \iff \{ \varphi(x), \varphi(y) \} \in F$.

Consider
$\mathcal{L}(G)$ = multiset of isomorphism types of $\underbrace{G-e}_{\text{means same vertex set, remove } e \text{ from edge set}}$ for $e \in E$.

e.g. $G_1 =$  $\mathcal{L}(G_1) = \{$  $\}$

$G_2 =$  $\mathcal{L}(G_2) = \{$  $\}$

Edge Reconstruction Problem

Does $\mathcal{L}(G)$ determine G?
We say G is edge-reconstructible if the answer is yes.

Conjecture If $G \notin \{G_1, G_2,$  $, \overline{K_n}\}$, up to the addition of isolated vertices, then G is edge-reconstructible.

Theorem (Lovász 1972)
If $G=(V,E)$ and $|E| > \frac{1}{2}\binom{|V|}{2}$, then G is edge-reconstructible.

Theorem (Müller 1977)
If
$$2^{|E|-1} > |V|!$$
then $G=(V,E)$ is edge-reconstructible.

Wed
30 Sept
2009

Theorem (Lovász 1972) Let $G=(V,E)$ be a graph. Then $|E| > \frac{1}{2}\binom{|V|}{2} \Rightarrow G$ is edge-reconstructible.

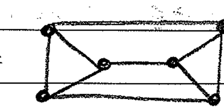
Proof Suppose $G=(V,E)$ and $G'=(V,E')$ [so G and G' have the same vertex set] and $\mathcal{L}(G) = \mathcal{L}(G')$.

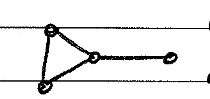
Set $E = \{e_1, \dots, e_m\}$, $E' = \{f_1, \dots, f_m\}$,

$$G_i = G - e_i, \qquad G_i' = G' - f_i.$$

Defn For $G=(V,E)$ and $H=(W,F)$ with $|V|=|W|$ let

$$N(H,G) = \#\text{ labelled copies of } H \text{ in } G$$
$$= \left|\left\{\sigma : W \to V \;\middle|\; \begin{array}{l}\sigma \text{ is a bijection,} \\ \{x,y\} \in F \Rightarrow \{\sigma(x),\sigma(y)\} \in E\end{array}\right\}\right|$$

e.g. $G =$ 

$H =$ 

$$N(H,G) = 2 \cdot 3 \cdot 2 \cdot 2 = 24$$

Claim If $H=(W,F)$ and $|F| < |E|$ then $N(H,G) = N(H,G')$.

Pf
$$\sum_{i=1}^{m} N(H,G_i) = N(H,G)\big[|E| - |F|\big]$$
$$\Vert$$
$$\sum_{i=1}^{m} N(H,G_i') = N(H,G')\big[|E'| - |F|\big] \quad \blacksquare$$

Note
$N(G,\overline{G}) = 0$ [since $|E| > \frac{1}{2}\binom{|V|}{2}$]

$N(G',\overline{G}) = 0$

Now, for $I \subseteq [m]$ let

$$H_I = (V, \{e_i : i \in I\}).$$

e.g., $H_{[m]} = G$.

Define
$$\Omega = \{\sigma : V \to V \text{ bijections}\}.$$

For $i \in [m]$, $A_i = \{\sigma \in \Omega : \underbrace{e_i \in \sigma(E)}_{\substack{e_i \text{ is the image} \\ \text{of some edge}}}\}$

$$A_I = \{\sigma \in \Omega : e_i \in \sigma(E)\ \forall i \in I\}$$
$$= \bigcap_{i \in I} A_i$$

Note: $|A_I| = N(H_I, G)$.

Pf: Consider the inverse of a map $\sigma \in A_I$. ∎

We have

$$0 = N(G,\overline{G}) = \left|\overline{\bigcup_{i=1}^{m} A_i}\right| = \sum_{I \subseteq [m]} (-1)^{|I|} |A_I|$$
$$= \sum_{I \subseteq [m]} (-1)^{|I|} N(H_I, G).$$

So, $0 = \sum_{I \subseteq [m]} (-1)^{|I|} N(H_I, G)$.

We can do the same, replacing G with G'.
Set $A_i' = \{\sigma \in \Omega : e_i \in \sigma(E')\}$. Now,

$$A_I' = N(H_I, G') \quad \text{[again considering the inverse].}$$

$$0 = N(G',\overline{G}) = \left|\overline{\bigcup_{i=1}^{m} A_i'}\right| = \sum_{I \subseteq [m]} (-1)^{|I|} |A_I'|$$
$$= \sum_{I \subseteq [m]} (-1)^{|I|} N(H_I, G').$$

Furthermore, if $I \neq [m]$ then $N(H_I, G') = N(H_I, G)$.
Since the sums are equal,

$$N(G,G) = N(H_{[m]}, G) = N(H_{[m]}, G') = N(G,G').$$

And $N(G,G) \geq 1$. ∎

Defn The permanent of a matrix $M = (m_{ij})_{ij=1}^{n}$ is

$$\operatorname{per}(M) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} m_{i,\sigma(i)}.$$

e.g. $\operatorname{per}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad + bc.$

Defn A matching in a graph $G = (V,E)$ is a set of edges $X \subseteq E$ such that
$$e, f \in X \implies e \cap f = \emptyset.$$

X is a perfect matching if $|X| = \frac{|V|}{2}$.

Defn Suppose $G=(V,E)$ is a bipartite graph with bipartition $\{u_1,\dots,u_k\}$, $\{v_1,\dots,v_\ell\}$. The bipartite adjacency matrix is the matrix

$$A=(a_{ij})_{\substack{i=1,\dots,k\\ j=1,\dots,\ell}}$$

where

$$a_{ij}=\begin{cases}1, & \text{if } \{u_i,v_j\}\in E;\\ 0, & \text{if } \{u_i,v_j\}\notin E.\end{cases}$$

Note: If G is a bipartite graph with bipartite adjacency matrix $A=(a_{ij})$ and bipartition V_1, V_2 with $|V_1|=|V_2|$, then

$$G \text{ has a perfect matching} \iff \begin{array}{c}\exists\, \sigma\in S_n \text{ s.t.}\\ a_{i,\sigma(i)}=1\\ \text{for } i=1,\dots,n\end{array} \iff \operatorname{per}(A)>0.$$

How do we compute per(M)?

The formula in the definition has
$n!-1$ additions, $n!(n-1)$ multiplications.

A form of inclusion–exclusion:
Let Ω be a set, $A_1,\dots,A_n\subseteq I$. Define A_I for $I\subseteq[n]$ as usual.

If $f:2^\Omega\to\mathbb{R}$ satisfies $f(X)=\sum_{x\in X}f(x)$ and $f(\emptyset)=0$,

then $f\left(\Omega\setminus\left(\bigcup_{i=1}^n A_i\right)\right)=\sum_{I\subseteq[n]}(-1)^{|I|}f(A_I)$.

Due to Ryser:

Let $\Omega=[n]^{[n]}=\{\sigma:[n]\to[n]\}$.
For $i=1,\dots,n$ let

$$A_i=\{\sigma\in\Omega \;:\; i\notin \operatorname{Im}(\sigma)\}.$$

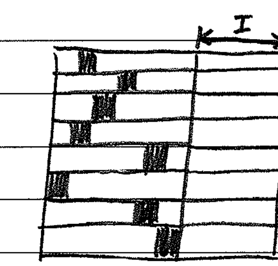
For $\sigma\in\Omega$, set $f(\sigma)=\prod_{i=1}^n m_{i,\sigma(i)}$.

Note that
$$f\left(\Omega\setminus\left(\bigcup_{i=1}^n A_i\right)\right)=f(S_n)=\operatorname{per}(M).$$

By inclusion–exclusion,

$$\operatorname{per}(M)=\sum_{I\subseteq[n]}(-1)^{|I|}f(A_I).$$

$\{\sigma\in\Omega : \sigma(i)\notin I\ \forall i\in[n]\}$



We claim that

$$f(A_I)=\prod_{i=1}^n\left(\sum_{j\notin I}m_{ij}\right).$$

So
$$\operatorname{per}(M)=\sum_{I\subseteq[n]}(-1)^{|I|}\prod_{i=1}^n\left(\sum_{j\notin I}m_{ij}\right).$$

We have
$$2^n\cdot(n-1)+2^n\geq \#\text{ of additions},$$
$$2^n\cdot n\geq \#\text{ of multiplications}.$$

Mon
12 Oct
2009

Recall: A matching in a graph $G=(V,E)$ is a collection of pairwise disjoint edges.

Example. Let Y be a set, $\mathcal{A}=(A_i \subseteq Y \mid i \in J)$. A system of distinct representatives (or a traversal) is a collection $(a_i \mid i \in J)$ of distinct elements of Y such that $a_i \in A_i$ for all $i \in J$.

Consider the bipartite graph with bipartition J, Y with an edge $\{i,x\}$ if $x \in A_i$. There exists a traversal iff there is a matching that "covers" every $i \in J$.

Definition Such a matching is J-perfect.

Theorem (Hall's theorem)
Let $G=(V,E)$ be a bipartite graph with bipartition X,Y. Then there exists an X-perfect matching if and only if

$$\underbrace{|N(A)|}_{\text{the neighborhood of } A} \geq |A| \qquad \forall A \subseteq X.$$

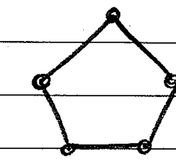
$N(A) = \{y \in Y : \exists x \in A \text{ s.t. } \{x,y\} \in E\}$

Definition Let $G=(V,E)$ be a graph.

$\nu(G)$ = matching number of G
= cardinality of a maximum matching in G

$\tau(G)$ = vertex cover number of G
$= \min\{|W| : W \subseteq V \text{ and } e \cap W \neq \emptyset \ \forall e \in E\}$
[such a W is a vertex cover]

e.g.



C_5 $\qquad \nu(C_5) = 2$
$\qquad \tau(C_5) = 3$

Note: $\nu(G) \leq \tau(G) \ \forall G$ (this is easy)

Theorem (König — Egerváry theorem)
$\nu(G) = \tau(G)$ for G bipartite.

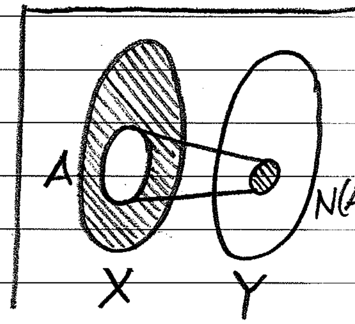
$\boxed{\text{EX}}$ König — Egerváry $\Longrightarrow$ Hall.

Pf. Let $G=(V,E)$ be a bipartite graph with bipartition X, Y.

For $A \subseteq X$ the deficiency of A is

$$\delta(A) = |A| - |N(A)|.$$

Let $D(G) = \max\limits_{A \subseteq X} \delta(A)$.



Claim 1: $\nu(G) \geq |X| - D(X)$.

Claim 2: $\tau(G) \leq |X| - D(X)$.

Pf Choose $A \subseteq X$ such that $D(X) = \delta(A)$.
Note that $N(A) \cup (X \setminus A)$ is a vertex cover.

Therefore,

$$\begin{aligned}\tau(G) &\le |N(A)| + \big[|X| - |A|\big] \\ &= |X| - \big[|A| - |N(A)|\big] \\ &= |X| - \delta(A) \\ &= |X| - D(X). \qquad \blacksquare\end{aligned}$$

<u>Pf of Claim 1</u> (using Hall's theorem)
We define a bipartite graph G' that contains G by introducing a new set Y' of $D(X)$ vertices and all edges of the form $\{x, y'\}$ where $x \in X$ and $y' \in Y'$.

Let $A \subseteq X$.

$$\begin{aligned}\underbrace{|N_{G'}(A)|}_{\text{neighborhood of } A \text{ in } G'} &= |N_G(A)| + |Y'| \\ &= |A| - \delta(A) + |Y'| \\ &= |A| - \delta(A) + D(X) \\ &\ge |A|.\end{aligned}$$

So, G' has an X-perfect matching. So,

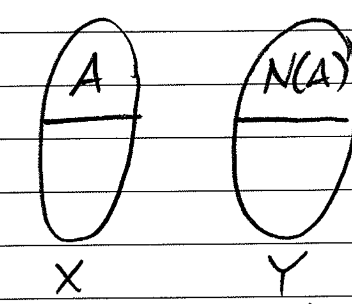
$$\nu(G) \ge |X| - |Y'| = |X| - D(X). \qquad \square$$

<u>Proof of Hall's theorem</u>
$G = (V, E)$ is bipartite with bipartition X, Y.
Suppose $|N(A)| \ge |A| \ \forall A \subseteq X$.

Go by induction on $|E|$.

<u>Case 1</u>: $|N(A)| > |A|$ for all $A \subsetneq X$, $A \neq \emptyset$.
Remove an edge and apply the inductive assumption. (Be careful with $A = X$.)

<u>Case 2</u>: There exists $A \subseteq X$ such that $A \neq X$, $A \neq \emptyset$, and $|N(A)| = |A|$.



[For $G = (V, E)$ a graph and $X \subseteq V$, the induced subgraph $G[X] = (X, E \cap \binom{X}{2})$.]

Consider the graph $H_1 = G[A \cup N(A)]$.
Note that this is a bipartite graph that satisfies Hall's condition. So, by induction, there is an A-perfect matching M_1 in H_1.

Now define $H_2 = G[(X \setminus A) \cup (Y \setminus N(A))]$.
This is bipartite. Consider $B \subseteq X \setminus A$.

$$N_{H_2}(B) = N_G(A \cup B) \setminus N_G(A)$$

so
$$\begin{aligned}|N_{H_2}(B)| &= |N_G(A \cup B)| - |N_G(A)| \\ &\ge |A| + |B| - |N_G(A)| = |B|.\end{aligned}$$

By induction, H_2 has an $(X\setminus A)$-perfect matching M_2.

Then $M_1 \cup M_2$ is an X-perfect matching in G. $\square$

Wed 14 Oct 2009

This class now lasts an extra 15 minutes, until 5:00, to make up for the two lost days last week.

<u>Theorem</u> If G is a bipartite graph then

$$\underset{\substack{\uparrow \\ \text{matching number}}}{\nu(G)} = \underset{\substack{\uparrow \\ \text{vertex cover number}}}{\tau(G)}.$$

<u>Theorem</u> (Tutte 1947)
Let $G=(V,E)$ be a graph. For $S \subseteq V$ let

$$q(S) = \text{number of odd components of } G[V\setminus S].$$

Then G has a perfect matching if and only if $q(S) \le |S|$ for all $S \subseteq V$.

Matchings in hypergraphs

$\mathcal{H}$ Let $\mathcal{H}$ be a k-uniform, k-partite hypergraph.

"hypergraph" = collection of sets

"k-uniform" = every set has size k

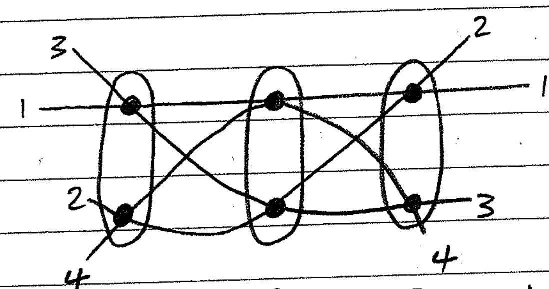
"k-partite" = $V = A_1 \dot\cup A_2 \dot\cup \cdots \dot\cup A_k$ and each hyperedge has one vertex in each part

Note: For hypergraphs we write $\mathcal{H}$ for the collection of hyperedges. So, $\mathcal{H} \subseteq \binom{V}{k}$ if $\mathcal{H}$ is k-uniform.

Ryser's conjecture: If $\mathcal{H}$ is a k-uniform, k-partite hypergraph then

$$(k-1)\underbrace{\nu(\mathcal{H})}_{\text{matching number}} \geq \underbrace{\tau(\mathcal{H})}_{\text{vertex cover number}}$$

e.g.



Lines: hyperedges
Ovals: partite sets

This is 3-uniform, 3-partite.
$\nu(\mathcal{H}) = 1, \quad \tau(\mathcal{H}) = 2.$

k=2: Ryser's conjecture = König–Egerváry theorem.

k=3: Proved 1999 by Aharoni, using methods introduced by Aharoni, Haxell.

k≥4: Open.

Extremal Combinatorics

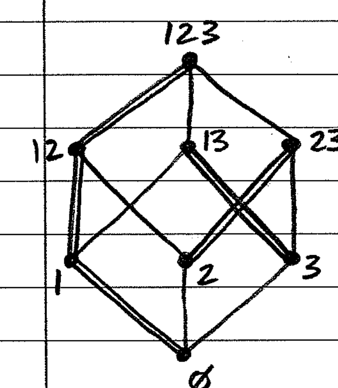
Defn $\mathcal{A} \subseteq 2^{[n]}$ is an antichain if $A, B \in \mathcal{A}$, $A \neq B \implies A \not\subseteq B$.

$\mathcal{C} = \{C_1, C_2, \dots, C_k\} \subseteq 2^{[n]}$ is a chain if $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_k$.

Sperner's theorem (1928)
If $\mathcal{A} \subseteq 2^{[n]}$ is an antichain then

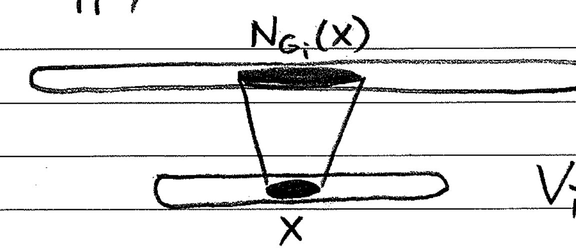
$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Pf 1. We partition $2^{[n]}$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains.



Let G_i be the bipartite graph with bipartition V_i, V_{i+1} where $V_i = \binom{[n]}{i}$, and in which $\{A, B\}$ is an edge for $A \in V_i$, $B \in V_{i+1}$ iff $A \subseteq B$.

ETS: If $i < n/2$ then G_i has a V_i–perfect matching.
We apply Hall's theorem. Let $X \subseteq V_i$.



V_{i+1} ← vertices here have degree $i+1$

V_i ← vertices here have degree $n-i$

So
$$|X|\,(n+i) = \text{number of edges in } G_i[X \cup N_{G_i}(X)]$$
$$\leq |N_{G_i}(X)|\,(i+1).$$

Since $n-i \geq i+1$ we have $|X| \leq |N_{G_i}(X)|$.
So by Hall's theorem, G_i has a perfect matching. $\square$

Pf 2 For each $\sigma \in S_n$ let

$$\mathcal{A}_\sigma = \{A \in \mathcal{A} : A = \{\sigma(1), \sigma(2), \dots, \sigma(\ell)\} \text{ for some } \ell\}.$$

Note that since the sets that are initial with respect to σ, that is, the sets in $\mathcal{A}_\sigma$, form a chain, we have $|\mathcal{A}_\sigma| \leq 1$.

$$n! \geq \sum_{\sigma \in S_n} |\mathcal{A}_\sigma| = \sum_{k=0}^{n} \left|\mathcal{A} \cap \binom{[n]}{k}\right| (n-k)!\, k!$$
$$= \sum_{A \in \mathcal{A}} (n-|A|)!\, |A|!$$

Thus,
$$1 \geq \sum_{A \in \mathcal{A}} \frac{(n-|A|)!\,|A|!}{n!} = \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}}. \qquad \text{[LYM inequality]}$$

Since $\binom{n}{|A|} \leq \binom{n}{\lfloor n/2 \rfloor}$ we have

$$1 \geq \frac{|\mathcal{A}|}{\binom{n}{\lfloor n/2 \rfloor}}. \qquad \square$$

Defn $\mathcal{F} \subseteq 2^{[n]}$ is intersecting if
$A, B \in \mathcal{F} \Rightarrow A \cap B \neq \emptyset$.

Extremal questions on intersecting collections

0. What is $\max\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]} \text{ intersecting}\}$?

Note:
(i) $|\mathcal{F}| \leq 2^{n-1}$ because
$|\mathcal{F} \cap \{A, \bar{A}\}| \leq 1 \;\; \forall A \subseteq [n]$.

(ii) Suppose $\mathcal{F}$ is maximal intersecting.
Then $|\mathcal{F}| = 2^{n-1}$.

Pf Suppose $A \notin \mathcal{F}$. So there is $B \in \mathcal{F}$ s.t. $B \cap A = \emptyset$, i.e., $B \subseteq \bar{A}$.
Since $\mathcal{F}$ is maximal, $\bar{A} \in \mathcal{F}$. $\square$

1. What is $\max\{|\mathcal{F}| : \mathcal{F} \subseteq \binom{[n]}{k}, \mathcal{F} \text{ intersecting}\}$, given k?

(a) If $2k > n$ then $\max = \binom{n}{k}$.

(b) If $2k = n$ then $\max = \frac{1}{2}\binom{n}{k} = \binom{n-1}{k-1}$
[from each complement pair, choose one].

(c) If $2k < n$ then $\max = \binom{n-1}{k-1}$.
[Erdős–Ko–Rado theorem]

Note For case (c), $\max \geq \binom{n-1}{k-1}$: Consider

$$\mathcal{F}_x = \{ A \in \binom{[n]}{k} : x \in A \}.$$

Thm (Erdős–Ko–Rado) Let $2k < n$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Furthermore, if $|\mathcal{F}| = \binom{n-1}{k-1}$ then $\mathcal{F} = \mathcal{F}_x$ for some $x \in [n]$.

Pf 1.

Defn: shifting. For $A \in \binom{[n]}{k}$ and $i<j$ set

$$S_{ij}(A) = \begin{cases} A, & \text{if } A \cap \{i,j\} \neq \{j\}; \\ (A \setminus \{j\}) \cup \{i\}, & \text{if } A \cap \{i,j\} = \{j\}. \end{cases}$$

Furthermore, set

$$S_{ij}(\mathcal{F}) = \{ S_{ij}(A) : A \in \mathcal{F} \} \cup \{ A : S_{ij}(A) \neq A \text{ and } S_{ij}(A) \in \mathcal{F} \}.$$

Note If $A \in \mathcal{F}$ and $A \notin S_{ij}(\mathcal{F})$ then $A \cap \{i,j\} = \{j\}$ and $(A \setminus \{j\}) \cup \{i\} \notin \mathcal{F}$.

We say that $\mathcal{F}$ is shifted if

$$S_{ij}(\mathcal{F}) = \mathcal{F} \quad \forall\, i<j.$$

Observations

1. This process (repeatedly shifting) terminates since $\sum_{A \in \mathcal{F}} \sum_{\ell \in A} \ell$ decreases.

2. The shifted set is not necessarily unique.

3. $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$.

Claim 1: $\mathcal{F}$ intersecting $\Longrightarrow S_{ij}(\mathcal{F})$ intersecting.

Pf: Assume for the sake of contradiction that $A, B \in S_{ij}(\mathcal{F})$ and $A \cap B = \emptyset$.
We may assume $A \cap \{i,j\} = \{i\}$ and $B \cap \{i,j\} = \{j\}$.
Note that $S_{ij}(B) \in \mathcal{F}$ and $B \in \mathcal{F}$. Further, $A \in \mathcal{F}$ or $(A \setminus \{i\}) \cup \{j\} \in \mathcal{F}$.
But $A \cap B = \emptyset$ and $S_{ij}(B) \cap S_{ji}(A) = \emptyset$.
This is a contradiction. $\square$

Claim 2: If $\mathcal{F}$ is shifted and $A, B \in \mathcal{F}$, then $A \cap B \neq \{n\}$.

Then go by induction on n.

Mon 19 Oct 2009

Continuing proof of Erdős–Ko–Rado theorem from last time.

We may assume $S_{in}(\mathcal{F}) = \mathcal{F} \quad \forall\, i<n$.
To show: $|\mathcal{F}| \le \binom{n-1}{k-1}$
We go by induction on n.

Claim 2: If $A,B \in \mathcal{F}$ then $A \cap B \ne \{n\}$.

Pf Assume for the sake of contradiction that $A,B \in \mathcal{F}$ and $A \cap B = \{n\}$. Since $2k<n$, $\exists\, j \in [n]$ s.t. $j \notin A \cup B$. Since $S_{jn}(\mathcal{F}) = \mathcal{F}$, we have $S_{jn}(A) \in \mathcal{F}$. But $S_{jn}(A) \cap B = \emptyset$. $\lightning$ $\square$

Define
$$\mathcal{F}_0 = \{A \in \mathcal{F} : n \notin A\},$$
$$\mathcal{F}_1 = \{A \setminus \{n\} : A \in \mathcal{F},\ n \in A\}.$$

$\mathcal{F}_0$ is an intersecting collection in $\binom{[n-1]}{k}$.
$\mathcal{F}_1$ is an intersecting family in $\binom{[n-1]}{k}$, by Claim 2.
So, by induction,
$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| \overset{(\star)}{\le} \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}.$$

Note:
(i) We can take $n=3$ as a base case.

(ii) If $2k=n-1$ then we note that $|\mathcal{F}_0| \le \binom{n-2}{k-1}$ directly, since at most one of A and $[n-1]\setminus A$ is in $\mathcal{F}_0$.

In the case of equality we use the following:

Claim 3: If $S_{ij}(\mathcal{F}) = \mathcal{F}_x$ for some $x \in [n]$, then $\mathcal{F} = \mathcal{F}_x$.

Pf $\boxed{EX}$

So, to achieve equality in $(\star)$ we need

Case 1: $2k < n-1$.

$$\mathcal{F}_0 = \mathcal{F}_x, \qquad \mathcal{F}_1 = \mathcal{F}_y,$$

and if $x \ne y$ then the families contain nonintersecting sets.

Case 2: $2k = n-1$.

$$\mathcal{F}_1 = \mathcal{F}_y \text{ for some } y \in [n-1].$$

So, $\{A \in \binom{[n]}{k} : n, y \in A\} \subseteq \mathcal{F}$.

Furthermore, $\forall\, B \in \binom{[n-1]}{k} = \binom{[2k]}{k}$, either $B \in \mathcal{F}$ or $[2k] \setminus B \in \mathcal{F}$. The one that contains y must be in $\mathcal{F}$, to maintain the intersecting property with $\mathcal{F}_1$.

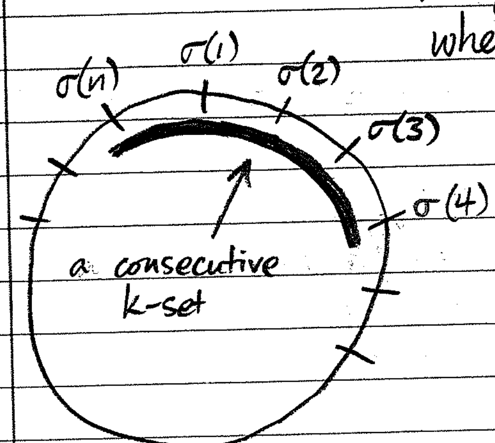
$\square$

Proof 2 (Katona "circle method")

For $\sigma \in S_n$ let

$$\mathcal{F}_\sigma = \{ A \in \mathcal{F} : A \text{ is } \underbrace{\text{consecutive}}_{\substack{\exists\, i \text{ s.t.} \\ A = \{\sigma(i), \sigma(i+1), \dots, \sigma(i+k-1)\},}} \text{ w.r.t. } \sigma \}$$

where addition is taken modulo n.



Note:

(i) There are n k-sets that are consecutive with respect to σ.

(ii) $|\mathcal{F}_\sigma| \le k$.

For a fixed $A \in \mathcal{F}_\sigma$ there are $2(k-1)$ sets in $\binom{[n]}{k}$ that are consecutive with respect to σ and intersect A. This collection of sets can be partitioned into $k-1$ pairs such that the sets in each pair do not intersect.

Each $A \in \mathcal{F}$ is included in $\mathcal{F}_\sigma$ for $n \cdot k!(n-k)!$ different $\sigma \in S_n$. So

$$|\mathcal{F}| \cdot n \cdot k!(n-k)! = \sum_{\sigma \in S_n} |\mathcal{F}_\sigma| \le k \cdot n!.$$

So, $$|\mathcal{F}| \le \frac{(n-1)!}{(k-1)!(n-k!)} = \binom{n-1}{k-1}.$$

In the case of equality we have

$$|\mathcal{F}_\sigma| = k \quad \forall\, \sigma \in S_n.$$

$$\left[\begin{array}{cccccccc} \sigma(n-2) & \sigma(n-1) & \sigma(n) & \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \sigma(5) \leftarrow A \\ & \boxed{R} & \boxed{R} & \boxed{R} & \boxed{R} & \boxed{L} & \boxed{L} & \boxed{L} \end{array} \right]$$

It follows that $\mathcal{F}_\sigma = \underbrace{\binom{[n]}{k}_\sigma}_{\substack{\text{the } k\text{-sets} \\ \text{consecutive} \\ \text{w.r.t. } \sigma}} {}_x$

but the x's may vary with σ.

<u>Claim:</u> If $\sigma = \rho \underbrace{(x,j)}_{\text{a transposition}}$ and $\mathcal{F}_\sigma = \binom{[n]}{k}_{\sigma x}$ then $\mathcal{F}_\rho = \binom{[n]}{k}_{\rho x}$.

<u>Pf</u> $\boxed{\text{EX}}$

Since transpositions of the form (x,j) generate S_n, we have $\mathcal{F} = \mathcal{F}_x$. $\square$

Corollary (EKR)

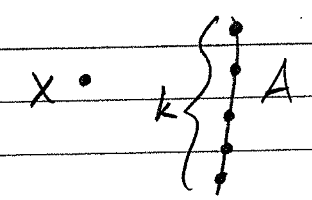
If $\mathcal{F} \subseteq \binom{[n]}{\leq k} = \bigcup_{i \leq k} \binom{[n]}{i}$ is intersecting,

then $|\mathcal{F}| \leq \sum_{i=1}^{k} \binom{n-1}{i-1}$. [No restriction on k]

Pf $\boxed{\text{EX}}$

More extremal questions on intersecting collections

2. An alternate intersecting family:



All k-sets that contain x and touch A, and A.

Number of such sets: $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$

For k fixed, $\binom{n-1}{k-1} = \frac{n^{k-1}}{(k-1)!} + O(n^{k-2})$

$$\binom{n-1}{k-1} - \binom{n-k-1}{k-1} = O(n^{k-2})$$

Theorem (Hilton — Milner 1967)

If $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting and $\bigcap_{A \in \mathcal{F}} A = \emptyset$,
then
$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Comment: This is the first instance of combinatorial stability.

3. Theorem (Katona)

Let $\mathcal{F} \subseteq 2^{[n]}$. We say $\mathcal{F}$ is t-intersecting if $A, B \in \mathcal{F} \Rightarrow |A \cap B| \geq t$.
This implies

$$|\mathcal{F}| \leq \begin{cases} \sum_{i \geq (n+t)/2} \binom{n}{i}, & \text{if } n \equiv t \pmod 2; \\ \sum_{i \geq (n+t)/2} \binom{n}{i} + \binom{n-1}{(n+t-1)/2}, & \text{if } n \not\equiv t \pmod 2. \end{cases}$$

Question: How large can a t-intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ be?

Note: We can always get $|\mathcal{F}| = \binom{n-t}{k-t}$ for fixed k, t.

Let
$$\mathcal{F}_i = \{ A \in \binom{[n]}{k} : A \cap [t+2i] \geq t+i \}.$$

Note
(i) $\mathcal{F}_i$ is t-intersecting.
(ii) $|\mathcal{F}_0| = \binom{n-t}{k-t}$.

Theorem (Ahlswede, Khachatrian 1999, conjectured by Frankl)
$\forall\, n, k, t\ \exists\, i$ such that

$$\mathcal{F} \subseteq \binom{[n]}{k} \text{ } t\text{-intersecting} \Rightarrow |\mathcal{F}| \leq |\mathcal{F}_i|$$

with equality iff $\mathcal{F}$ is isomorphic to $\mathcal{F}_i$.

Wed 21 Oct 2009

4. Chvátal's conjecture (1972)
Let $I \subseteq 2^{[n]}$ be an ideal (i.e., if $A \in I$ and $B \subseteq A$ then $B \in I$). If $\mathcal{F} \subseteq I$ is intersecting, then there exists $x \in [n]$ such that
$$|\mathcal{F}| \le |\{A \in I : x \in A\}|.$$

Note: EKR would be a special case.

Kruskal — Katona

Defn The (lower) shadow of $\mathcal{F} \subseteq 2^{[n]}$ is

$$\partial\mathcal{F} = \{A \subseteq [n] : \exists\, B \in \mathcal{F} \text{ s.t. } \underbrace{A \lessdot B}_{\substack{\text{"A covers B":} \\ A \subseteq B \text{ and } |B \setminus A| = 1.}}\}$$

Question: Given $\mathcal{F} \subseteq \binom{[n]}{k}$ with $|\mathcal{F}| = m$, what is the minimum possible value of $|\partial\mathcal{F}|$?

Defn: Reverse lexicographic (colex) order on k-sets:

$$A <_{RL} B \iff \max(\underbrace{A \triangle B}_{\substack{\text{symmetric} \\ \text{difference}}}) \in B$$

$$\iff \sum_{i \in A} 2^i < \sum_{i \in B} 2^i$$

e.g. $k=3$:
123, 124, 134, 234, 125, ...

Note This does not depend on the size of the ground set.

Defn $\mathcal{F} \subseteq \binom{[n]}{k}$ is initial in $<_{RL}$ if $A <_{RL} B$ and $B \in \mathcal{F}$ implies $A \in \mathcal{F}$.

$\boxed{\text{EX}}$

1. initial in $<_{RL}$ $\Rightarrow$ shifted $\not\Rightarrow$ initial in $<_{RL}$.
$[S_{ij}(\mathcal{F}) = \mathcal{F}\ \forall i<j]$

2. $\mathcal{F}$ initial $\Longrightarrow$ $\partial\mathcal{F}$ initial.

Theorem (Kruskal 1963, Katona 1968)
Among $\mathcal{F} \subseteq \binom{[n]}{k}$ with $|\mathcal{F}| = m$, $|\partial\mathcal{F}|$ is minimized when $\mathcal{F}$ is initial in $<_{RL}$.

Proof Induction on n.

Let $\mathcal{F}_i = \{A \in \mathcal{F} : i \in A\}$,
$\mathcal{F}_{\bar{i}} = \{A \in \mathcal{F} : i \notin A\}$,
$\mathcal{F}_{i'} = \{A \setminus \{i\} : A \in \mathcal{F}_i\}$.

Let $\mathcal{G}_i = \{$ first $|\mathcal{F}_i|$ sets in $<_{RL}$ that contain $i\}$

$\mathcal{G}_{\bar{i}} = \{$ first $|\mathcal{F}_{\bar{i}}|$ sets in $<_{RL}$ that do not contain $i\}$

$\mathcal{G}_{i'} = \{A \setminus \{i\} : A \in \mathcal{G}_i\}$.

We define the i-compression of $\mathcal{F}$ to be

$$C_i = C_i(\mathcal{F}) = \mathcal{G}_i \cup \mathcal{G}_{\bar{i}} = \mathcal{G}.$$

Clearly, $|\mathcal{G}| = |\mathcal{F}|$.

<u>Claim</u> $|\partial\mathcal{G}| \le |\partial\mathcal{F}|$.

Pf Observations:

(1) $|\mathcal{F}_{i'}| = |\mathcal{G}_{i'}|$

(2) $|\partial\mathcal{G}_{i'}| \le |\partial\mathcal{F}_{i'}|$

Note $\mathcal{G}_{i'}$ is initial in $\binom{[n]\setminus\{i\}}{k-1}$.
So this observation follows by induction.

(3) $|\partial\mathcal{G}_{\top}| \le |\partial\mathcal{F}_{\top}|$

since $\mathcal{G}_{\top}$ is initial in $\binom{[n]\setminus\{i\}}{k}$.

(4) $|(\partial\mathcal{G}_{\top}) \cup \mathcal{G}_{i'}| = \max\{|\partial\mathcal{G}_{\top}|, |\mathcal{G}_{i'}|\}$

since both $\partial\mathcal{G}_{\top}$ and $\mathcal{G}_{i'}$ are initial in $\binom{[n]\setminus\{i\}}{k-1}$.

With these observations in hand, we have

$$|\partial\mathcal{G}| = \underbrace{|\partial\mathcal{G}_{i'}|}_{|\{A\in\partial\mathcal{G} : i\in A\}|} + |\underbrace{(\partial\mathcal{G}_{\top}) \cup \mathcal{G}_{i'}}_{\{A\in\partial\mathcal{G} : i\notin A\}}|$$

$$= |\partial\mathcal{G}_{i'}| + \max\{|\partial\mathcal{G}_{\top}|, |\mathcal{G}_{i'}|\}$$

$$\le |\partial\mathcal{F}_{i'}| + \max\{|\partial\mathcal{F}_{\top}|, |\mathcal{F}_{i'}|\}$$

$$\le \underbrace{|\partial\mathcal{F}_{i'}|}_{|\{A\in\partial\mathcal{F} : i\in A\}|} + |\underbrace{(\partial\mathcal{F}_{\top}) \cup \mathcal{F}_{i'}}_{\{A\in\partial F : i\notin A\}}|$$

$$= |\partial\mathcal{F}|. \qquad \blacksquare$$

Now we may assume $C_i(\mathcal{F}) = \mathcal{F} \quad \forall i. \quad (\lozenge)$

<u>To show:</u> $\mathcal{F}$ is initial in $<_{RL}$.
Is this true? (Not always. But if so we're done.)

Suppose $A <_{RL} B$, $A \notin \mathcal{F}$ and $B \in \mathcal{F}$.
Note that if $\exists\, i \in A \cap B$ then we get a contradiction of $(\lozenge)$ by considering the compression C_i. Similarly we have a contradiction if $\exists\, i \notin A \cup B$. Thus, we have $A = \overline{B}$ (so $n = 2k$) and hence $A <^{\cdot}_{RL} B$ (i.e., A is the immediate predecessor of B in the linear ordering $<_{RL}$). So

$$\mathcal{F} = \{C : C <_{RL} A\} \cup B.$$

This can happen only if

$$\mathbb{1}_A = (\underbrace{0,0,0,\dots,0}_{k-1}, \underbrace{1,1,1,\dots,1}_{k}, 0),$$

$$\mathbb{1}_B = (\underbrace{1,1,1,\dots,1}_{k-1}, \underbrace{0,0,0,\dots,0}_{k}, 1).$$

Note that

$$\partial\mathcal{F} = \binom{[n-1]}{k-1} \cup \underbrace{\partial(\{B\})}_{\substack{\text{this has } k-1 \text{ sets} \\ \text{not in } \binom{[n-1]}{k-1}}}$$

and

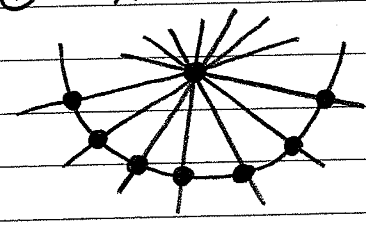
$$\partial\big((\mathcal{F}\setminus\{A\})\cup\{B\}\big) = \binom{[n-1]}{k-1}. \qquad \square$$

Design-type restrictions on intersections and linear-algebraic methods.

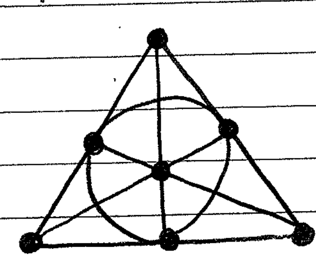
Thm (Fischer's inequality)
If $\mathcal{F} = \{A_1, A_2, \dots, A_m\} \subseteq 2^{[n]}$ satisfies $|A_i \cap A_j| = \lambda \ \ \forall i \neq j$ and $|A_i| > \lambda$ for some $\lambda \geq 0$ then $m = |\mathcal{F}| < n$.

Examples

① $\lambda = 1$



[points are elements; curves are sets]



Fano plane

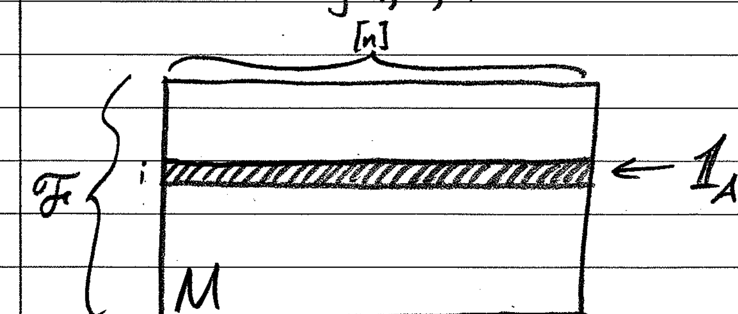
② Projective plane.

A projective plane consists of a set X of points and a set $L \subseteq 2^X$ of lines such that
(i) any two points define a line
$(\forall x \neq y \in X \ \ \exists!\, \ell \in L \ \ \{x,y\} \subseteq \ell)$;
(ii) any two lines intersect in exactly one point
$(\forall \ell \neq m \in L \ \ \exists!\, x \in X \ \ x \in \ell \cap m)$;
(iii) there exists a quadrilateral (four points, no three of which lie on a line).

$\boxed{\text{EX}}$ For any projective plane there exists a q such that $|X| = |L| = q^2 + q + 1$.

Defn Let M be the incidence matrix of $\mathcal{F} = \{A_1, \dots, A_m\}$, that is,

$$M = (m_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \quad \text{where } m_{ij} = \begin{cases} 1, & \text{if } j \in A_i; \\ 0, & \text{if } j \notin A_i. \end{cases}$$



Note: $$MM^T = \begin{bmatrix} |A_1| & \lambda & \lambda & \cdots \\ \lambda & |A_2| & \lambda & \cdots \\ \lambda & \lambda & |A_3| & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Mon
26 Oct
2009

Proving Fischer's inequality (continued)

Pf 1 We have

$$MM^T = \begin{bmatrix} |A_1| & \lambda & \lambda & \cdots \\ \lambda & |A_2| & \lambda & \cdots \\ \lambda & \lambda & |A_3| & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{m\times m}$$

Claim: $\operatorname{rank}(MM^T) = m.$

Suppose the claim holds; then
$$n \geq \operatorname{rank}(M) \geq \operatorname{rank}(MM^T) = m.$$

Pf of Claim:

$$\det\begin{bmatrix} |A_1| & \lambda & \lambda & \cdots & \lambda \\ \lambda & |A_2| & \lambda & \cdots & \lambda \\ \lambda & \lambda & |A_3| & \cdots & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \lambda & \cdots & |A_m| \end{bmatrix}$$

$$= \det\begin{bmatrix} |A_1| & \lambda & \lambda & \cdots & \lambda \\ \lambda-|A_1| & |A_2|-\lambda & 0 & \cdots & 0 \\ \lambda-|A_1| & 0 & |A_3|-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda-|A_1| & 0 & 0 & \cdots & |A_m|-\lambda \end{bmatrix}$$

$$= \prod_{i=1}^{m} \underbrace{(|A_i|-\lambda)}_{>0} \left[1 + \lambda \sum_{i=1}^{m} \underbrace{\frac{1}{|A_i|-\lambda}}_{>0}\right]$$

$\boxed{\text{EX}}$

$> 0.$ □

Pf 2 It suffices to show that the rows of M are linearly independent.

Let $v_1, \dots, v_m$ be the rows of M.

Suppose
$$\sum_{i=1}^{m} \alpha_i v_i = 0. \qquad (\alpha_i \in \mathbb{R})$$

Consider
$$\left\langle \sum_{i=1}^{m} \alpha_i v_i, \sum_{i=1}^{m} \alpha_i v_i \right\rangle = 0.$$

On the other hand,
$$\left\langle \sum_{i=1}^{m} \alpha_i v_i, \sum_{i=1}^{m} \alpha_i v_i \right\rangle = \sum_{i,j} \alpha_i \alpha_j \langle v_i, v_j \rangle$$
$$= \sum_{i=1}^{m} \alpha_i^2 |A_i| + 2\sum_{i<j} \alpha_i \alpha_j \lambda$$
$$= \sum_{i=1}^{m} \alpha_i^2 \underbrace{(|A_i|-\lambda)}_{>0} + \lambda \underbrace{\left(\sum_{i=1}^{m} \alpha_i\right)^2}_{\geq 0}$$

So $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0.$ □

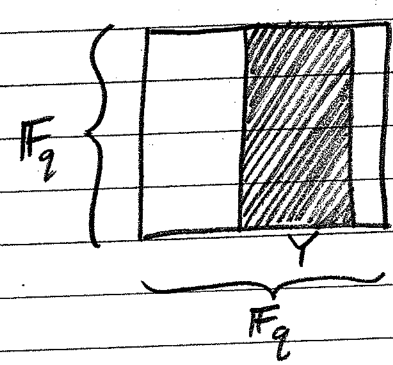
Ray-Chaudhuri, Wilson Theorem (1975)

Suppose $s \le k$, $L \subseteq \mathbb{N}$, $|L| = s$.
If $\mathcal{F} \subseteq \binom{[n]}{k}$ and
$A, B \in \mathcal{F}$, $A \ne B \Rightarrow |A \cap B| \in L$ — In this case we say that $\mathcal{F}$ is "L-intersecting."
then $|\mathcal{F}| \le \binom{n}{s}$.

Examples

① $k = s$, $L = \{0, \dots, k-1\}$, $\mathcal{F} = \binom{[n]}{k}$.

② Suppose $k \le q$ where q is a prime.
Let $Y \subseteq \mathbb{F}_q$ with $|Y| = k$.
($\mathbb{F}_q$: finite field)
ground set: $Y \times \mathbb{F}_q$ (so $n = kq$)



sets: for $f \in \mathbb{F}_q[x]$ define

$$A_f = \{(x, f(x)) : x \in Y\} \quad [\text{"the graph of } f\text{"}]$$

Note: $|A_f| = k$

Now, let $\mathcal{F} = \{A_f : \deg(f) \le s-1\}$.

$\mathcal{F}$ is $\{0, \dots, s-1\}$ - intersecting.

We have $|\mathcal{F}| = q^s = \left(\frac{n}{k}\right)^s$.

The bound from RCW is $\binom{n}{s}$ "$\approx$" $\left(\frac{ne}{s}\right)^s$.

Non-uniform modular RCW theorem (Deza, Frankl, Singhi 1983)
Let p be prime, $L \subseteq \mathbb{Z}_p$, $|L| = s$.
If $\mathcal{F} \subseteq 2^{[n]}$ such that
① $|A| \notin L \pmod p$ $\forall A \in \mathcal{F}$;
② $|A \cap B| \in L \pmod p$ $\forall A, B \in \mathcal{F}$, $A \ne B$
then

$$|\mathcal{F}| \le \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{s} =: \binom{n}{\le s}.$$

$\boxed{\text{EX}}$ What does this give in the settings of Fischer's inequality and RCW?

Proof (Alon, Babai, Suzuki 1991)
Outline:
1. Find a vector space V s.t. $\dim(V) \le \binom{n}{\le s}$.
2. Find an injection $\varphi : \mathcal{F} \to V$.
3. Show $\mathrm{Im}(\varphi)$ is linearly independent in V.

V will be a subspace of $\mathbb{F}_p[x_1, \dots, x_n]$.

Defn For $g \in \mathbb{F}_p[x_1, \dots, x_n]$, let $\hat{g}$ be the polynomial where we replace $x_i^{a_i}$ with x_i (for $a_i \ge 1$).

e.g., $g = x_1^5 x_3^3 + x_2 x_4^{12}$

$\hat{g} = x_1 x_3 + x_2 x_4$

Note: If $x \in \{0,1\}^n$ then $g(x) = \hat{g}(x)$.

Defn For $v \in \mathbb{F}_p^n$ define $p_v \in \mathbb{F}_p[x_1, \dots, x_n]$ to be
$$p_v(x) = \prod_{\ell \in L} \big(\underbrace{\langle v, x \rangle}_{\substack{\text{standard}\\\text{inner}\\\text{product}}} - \ell \big).$$

Note: We think of $x = (x_1, \dots, x_n)$.

Now, let $\mathcal{F} = \{A_1, A_2, A_3, \dots\}$ and set
$$v_i = \mathbb{1}_{A_i}, \qquad p_i = p_{v_i}.$$

Note

① $\hat{p}_i(v_j) = p_i(v_j) \begin{cases} = 0, & \text{if } i \neq j; \\ \neq 0, & \text{if } i = j. \end{cases}$

② $\deg(\hat{p}_i) \leq \deg(p_i) = s$.

We consider the map
$$\begin{aligned} \varphi : \mathcal{F} &\longrightarrow \mathbb{F}_p[x_1, \dots, x_n] \\ A_i &\longmapsto \hat{p}_i. \end{aligned}$$

Claim 1: φ is an injective map and $\mathrm{Im}(\varphi)$ is linearly independent.

Pf Suppose $\alpha_1, \alpha_2, \dots \in \mathbb{F}_p$ such that
$$\sum_i \alpha_i \hat{p}_i = 0.$$

Then we have
$$0 = \Big(\sum_i \alpha_i \hat{p}_i\Big)(v_j) = \sum_i \alpha_i \hat{p}_i(v_j) = \alpha_j \hat{p}_j(v_j).$$

Since $\hat{p}_j(v_j) \neq 0$, we have $\alpha_j = 0$. This holds for all j. ■

Claim 2: There exists a subspace W of $\mathbb{F}_p[x_1, \dots, x_n]$ such that $\mathrm{Im}(\varphi) \subseteq W$ and $\dim(W) = \binom{n}{\leq s}$.

Pf Let W be the space spanned by
$$\Big\{ \prod_{i \in A} x_i \ : \ A \in \binom{[n]}{\leq s} \Big\}.$$

Then W contains $\varphi(A_i)$ for all $A_i \in \mathcal{F}$. Furthermore, W has the desired dimension. ■

□

An application: Constructive lower bounds on Ramsey numbers.

Thm (Nagy 1972) $\quad R(t, t+1) \geq \binom{t-1}{3}$. $\sim t^3 = e^{3 \log t}$

Pf Let $V = \binom{[t-1]}{3}$. Define $f : \binom{V}{2} \to \{\text{Red}, \text{Blue}\}$ by
$$f(\{A, B\}) = \begin{cases} \text{Red}, & \text{if } |A \cap B| = 1; \\ \text{Blue}, & \text{otherwise}. \end{cases}$$

A red K_t consists of sets $A_1, A_2, \dots, A_t \in \binom{[t-1]}{3}$ such that $|A_i \cap A_j| = 1 \ \forall\, i \neq j$.

This does not exist, by Fischer's inequality.

A blue K_{t+1} consists of $B_1, \dots, B_{t+1} \in \binom{[t-1]}{3}$ such that
$$|B_i \cap B_j| \equiv 0 \pmod 2 \quad \forall\, i \neq j.$$
By modular RCW with $p = 2$, $L = \{0\}$, such a collection does not exist. $\square$

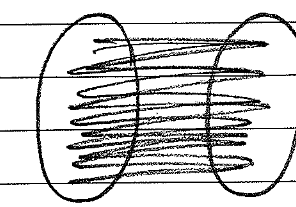
<u>Open:</u> Find an explicit construction that gives $R(t,t) > c^t$ for some $c > 1$.

Wed
28 Oct
2009

<u>Turán-type questions</u>

What is
$$\max \{ |E| : G=(V,E) \text{ a graph with } |V|=n \text{ and } G \supseteq K_{s+1} \}$$

<u>e.g.</u> $s = 2$: no triangle.



$G = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$

Is this optimal? $(\Diamond)$

<u>Notation</u> Turán function. F is a graph.

$$ex(n,F) = \max\{ |E| : G=(V,E) \text{ has } |V|=n \text{ and } G \not\supseteq F \}.$$

<u>e.g.,</u> the question $(\Diamond)$ asks, $ex(n, K_3) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$?

<u>Defn</u> A <u>complete multipartite graph</u> is a graph $G = (V,E)$ for which there is a partition $V = V_1 \,\dot\cup \cdots \dot\cup\, V_k$ such that $\{x,y\} \in E \iff \exists\, i,j$ s.t. $i \neq j$, $x \in V_i$, $y \in V_j$.

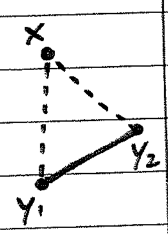
<u>Defn</u> A <u>Turán graph</u> is a complete multipartite graph where the partition is an equipartition:
$$\big||V_i| - |V_j|\big| \leq 1 \quad \forall\, i,j.$$
In other words, $|V_i| = \left\lfloor \frac{n+i-1}{k} \right\rfloor$.

Let $T_k(n)$ be the Turán graph with n vertices and k parts.

<u>Theorem</u> (Turán 1943; $r=3$ was Mantel 1907)
For any $n \geq r \geq 3$,
$$ex(n, K_{r+1}) = \#\text{ edges in } T_r(n).$$
Furthermore, $T_r(n)$ is the only graph that achieves this bound.

<u>Proof</u> (Bollobás 1960)
Let G be a maximum K_{r+1}-free graph.
It is <u>enough to show</u> that G is complete multipartite. $\hookrightarrow \boxed{EX}$
In other words, it is enough to show that non-adjacency is an equivalence relation.
Assume for the sake of contradiction that non-adjacency in G is not an equivalence relation. Then

$$\exists\ x, y_1, y_2 \text{ such that } \{x, y_1\}, \{x, y_2\} \notin E$$
$$\text{but } \{y_1, y_2\} \in E.$$



<u>Case 1:</u> $d(y_i) > d(x)$.

Consider the graph given by
(i) remove x;
(ii) insert a "duplicate" of y_i — a vertex $\hat{y}_i$ such that
$\{\hat{y}_i, z\} \in E \iff \{y_i, z\}$.
(Note that $\{y_i, \hat{y}_i\}$ is not an edge.)

This graph has more edges than G and has no K_{r+1} (because not both y_i and $\hat{y}_i$ can be part of a K_{r+1}, since $\{y_i, \hat{y}_i\}$ is not an edge). ↯

<u>Case 2:</u> $d(y_1) \leq d(x)$ and $d(y_2) \leq d(x)$.

Now consider the graph where we
(i) remove y_1, y_2;
(ii) "triplicate" x.

The change in the number of edges is
$2d(x) - (d(y_1) + d(y_2) - 1) \geq 1$.
So this graph has more edges than G and no K_{r+1}. ↯ $\square$

<u>$ex(n, F)$ for general F.</u>

<u>Recall:</u> The chromatic number of a graph $F=(V,E)$ is the smallest k such that $\exists\ f: V \to [k]$ s.t. $\{x,y\} \in E \Rightarrow f(x) \neq f(y)$.
The chromatic number of F is denoted $\chi(F)$.

<u>Note:</u> If $\chi(F) = k$, then $F \not\subseteq T_{k-1}(n)$, as an embedding of F in this graph would give a proper $(k-1)$-coloring of F.
So
$$ex(n, F) \geq \#\text{ of edges in } T_{\chi(F)-1}(n)$$
$$\geq \frac{\chi(F)-2}{\chi(F)-1}\binom{n}{2} + O(n).$$

Theorem (Erdős — Stone 1946)

$$ex(n,F) = \left(\frac{\chi(F)-2}{\chi(F)-1} + o(1)\right)\binom{n}{2}.$$

Note: If $\chi(F)=2$ (i.e., F is bipartite), then Erdős — Stone just gives the bound

$$ex(n,F) = o\left(\binom{n}{2}\right).$$

The order of magnitude of $ex(n,F)$ for F bipartite is largely open.

First question: For a given bipartite graph F, find $\alpha = \alpha(F)$ such that

$$ex(n,F) = n^{\alpha + o(1)}.$$

1. Zarankiewicz problem (1951): Determine $ex(n,F)$ for $F = K_{k,\ell}$.

- $ex(n, K_{2,2}) = \left(\frac{1}{2} + o(1)\right) n^{3/2}$

  (Kövari, Sós, Turán 1954)

- $ex(n, K_{3,3}) = \left(\frac{1}{2} + o(1)\right) n^{5/3}$

  (Brown 1966, Füredi 1996)

- $ex(n, K_{2,t}) = \frac{1}{2}\sqrt{tn^3} + O(n^{4/3})$

  (KST 1954, F 1996)

Claim: If q is a prime power and $n = 2(q^2+q+1)$ then

$$ex(n, K_{2,2}) \geq (q+1)(q^2+q+1).$$

Pf (sketch)

(Constructed using the finite field $\mathbb{F}_q$)

There exists a projective plane with q^2+q+1 points and q^2+q+1 lines.

Recall: A projective plane consists of

i. a set X of points,

ii. a set $L \subseteq 2^X$ of lines

such that

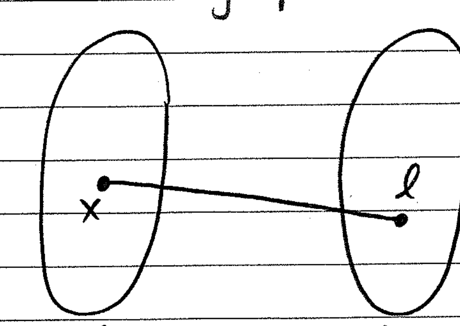
a. two lines intersect in exactly one point;

b. two points define a unique line;

c. there exists a quadrangle: four points no three of which lie on a line.

$\boxed{\text{EX}}$ Every point lies on $q+1$ lines and every line contains $q+1$ points.

Consider the graph



X (points of the proj. plane)

L (lines of the proj. plane)

$\{x,\ell\} \in E$ iff $x \in \ell$.

This contains no $K_{2,2}$, and the number of edges is $(q^2+q+1)(q+1)$. $\square$

Theorem (Füredi 1996) If q is a prime power, then
$$ex(q^2+q+1, K_{2,2}) = (q+1)(q^2+q+1).$$

Theorem (Kövari, Sós, Turán 1954)
If $s \le t$, then
$$ex(n, K_{s,t}) = O(n^{2-1/s}).$$

Conjecture If $s \le t$, then
$$ex(n, K_{s,t}) = \Theta(n^{2-1/s}).$$

Recall: Notation. Let $f,g : \mathbb{N} \to \mathbb{R}_+$.

If $f/g \to 0$, we write $f = o(g)$.
If $f < Cg$, we write $f = O(g)$.
If $f/g \to \infty$, we write $f = \omega(g)$.
If $f > Cg$, we write $f = \Omega(g)$.
If $f = O(g)$ and $f = \Omega(g)$, we write $f = \Theta(g)$.

This conjecture holds for $s=2$.
It also holds for $t > s!+1$ (Kollar, Ronyai, Szabo 1996).

Hypergraph Turán

Recall: $\mathcal{F}$ is a k-uniform hypergraph (or a k-graph) if $\mathcal{F} \subseteq \binom{V}{k}$.

For a fixed k-graph $\mathcal{F}$,
$$ex(n,\mathcal{F}) = \max\{|\mathcal{G}| : \mathcal{G} \text{ is a } k\text{-graph on } n \text{ vertices and } \mathcal{F} \not\subseteq \mathcal{G}\}.$$

Theorem For any k-graph $\mathcal{F}$,
$$\frac{ex(n,\mathcal{F})}{\binom{n}{k}} \text{ is nonincreasing in } n.$$

Pf Next time.

So,
$$\pi(\mathcal{F}) := \lim_{n\to\infty} \frac{ex(n,\mathcal{F})}{\binom{n}{k}} \text{ exists.}$$

This is the Turán density of $\mathcal{F}$.

Wed 4 Nov 2009

Hypergraph Turán problem

Let $\mathcal{F}$ be a fixed k-uniform hypergraph (a.k.a. k-graph). Let

$$ex(n,\mathcal{F}) = \max\{|\mathcal{G}| : \mathcal{G} \text{ is a } k\text{-uniform hypergraph on } n \text{ vertices such that } \mathcal{F} \not\subseteq \mathcal{G}\}.$$

Theorem For any k-graph $\mathcal{F}$, $\dfrac{ex(n,\mathcal{F})}{\binom{n}{k}}$ is nonincreasing.

Proof 1 Let $n<m$. Let $\mathcal{G}$ be a fixed k-graph on m vertices, not containing $\mathcal{F}$.

$$|\mathcal{G}|\cdot\binom{m-k}{n-k} = \#\text{ pairs } (A,X) \text{ where } X \text{ is an } n\text{-element subset of the vertex set, } A\subseteq X, \text{ and } A\in\mathcal{G}$$

$$\leq \binom{m}{n}\, ex(n,\mathcal{F}).$$

Thus $\dfrac{|\mathcal{G}|}{\binom{m}{k}} \leq \dfrac{\binom{m}{n}}{\binom{m-k}{n-k}\binom{m}{k}}\, ex(n,\mathcal{F}) = \dfrac{ex(n,\mathcal{F})}{\binom{n}{k}}$. $\square$

So, $\pi(\mathcal{F}) = \lim\limits_{n\to\infty} \dfrac{ex(n,\mathcal{F})}{\binom{n}{k}}$ exists.

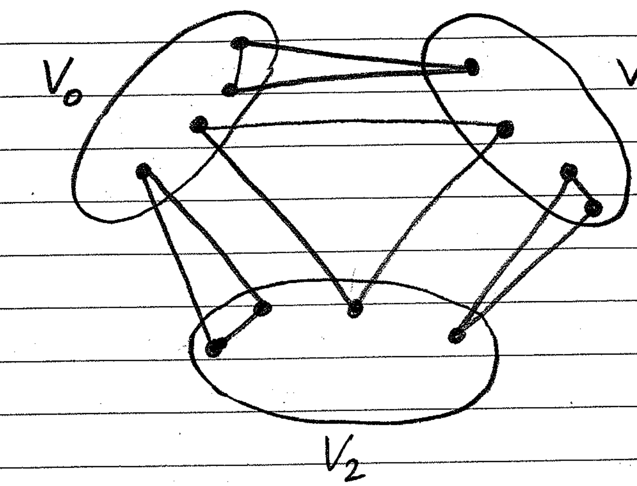
This is the Turán density of $\mathcal{F}$.

Problem (Erdős, \$500) Compute $\pi\left(\binom{[\ell]}{k}\right)$ for any $\ell > k \geq 3$.

Note: $\binom{[\ell]}{k}$ is called the complete k-graph on k vertices.

Conjecture (Turán) $\pi\left(\binom{[4]}{5}\right) = 5/9$.

Example $[n] = V_0 \,\dot\cup\, V_1 \,\dot\cup\, V_2$ an equipartition.



$$\mathcal{G} = \left\{A\in\binom{[n]}{3} : |A\cap V_i| = 1 \text{ for } i=0,1,2\right\}$$
$$\cup \left\{A\in\binom{[n]}{3} : \exists\, i \text{ s.t. } |A\cap V_i| = 2 \text{ and } |A\cap V_{i+1}| = 1\right\}$$

$\boxed{EX}$ $\dfrac{|\mathcal{G}|}{\binom{n}{3}} \longrightarrow \dfrac{5}{9}$.

Known $\dfrac{5}{9} \leq \pi\left(\binom{[4]}{3}\right) \leq \dfrac{3+\sqrt{17}}{2} \approx 0.593$

Chung, Lu 1999

Note: Kostochka, Sudakov found many examples that match the lower bound.

The Probabilistic Method

Example.

Defn A tournament is a directed graph $T=(V,A)$ (arc set: A) where $A \subseteq V \times V$ such that

① $(x,x) \notin A$

② $\forall x,y \in V,\ x \neq y,\ |A \cap \{(x,y),(y,x)\}| = 1.$

i.e., a tournament is an orientation of K_n.

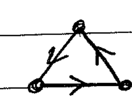
Defn For $W \subseteq V$ the subtournament induced by W is $T[W] = (W, A \cap (W \times W))$.

Defn T is transitive if $(x,y),(y,z) \in A$ implies $(x,z) \in A$.

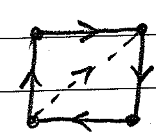
Let

$V(n) = \max\{v :$ any tournament on n vertices has a transitive subtournament on v vertices$\}$.

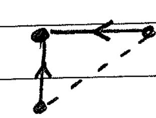
Example $V(3) = 2$



$V(4) = 3$

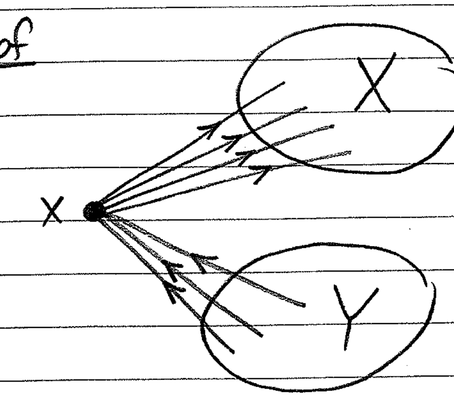


or



Proposition If $n \geq 2^k$ then $V(n) \geq k+1$.

Proof



[One of these pieces must contain at least $n/2$ vertices. Find a subtournament therein with k vertices and add x to it.]

We go by induction on k. ($k=2$ ✓)
Assume the proposition holds for $n \leq 2^{k+1}-1$.
Let T be a tournament on m vertices, with $2^{k+1} \leq m \leq 2^{k+2}-1$.
Let x be a vertex.

$X = \{y : (x,y) \in A\}, \quad Y = \{y : (y,x) \in A\}.$

By pigeonhole $|X| \geq 2^k$ or $|Y| \geq 2^k$.
By induction the induced subdigraph on this set has a transitive subtournament on k vertices. Attach x. □

Corollary $V(n) \geq \lfloor \log_2 n \rfloor + 1$.

Proposition (Erdős, Moser 1964)

$$V(n) < \lfloor 2 \log_2 n \rfloor + 1.$$

Proof Let T be a tournament on n vertices chosen uniformly at random.
Equivalently, $\Pr((x,y) \in A) = \Pr((y,x) \in A) = 1/2$ independently $\forall x,y \in V,\ x \neq y$.

$Pr(\exists$ a transitive subtournament on k vertices$)$

$$= Pr\Big(\bigcup_{A\in\binom{V}{k}} \{T[A] \text{ is transitive}\}\Big)$$
$$\le \sum_{A\in\binom{V}{k}} Pr(T[A] \text{ is transitive})$$
$$= \sum_{A\in\binom{V}{k}} \frac{k!}{2^{\binom{k}{2}}} = \binom{n}{k}\frac{k!}{2^{\binom{k}{2}}}.$$

It is enough to show

$$\binom{n}{k}\frac{k!}{2^{\binom{k}{2}}} < 1 \quad \text{for } k = \lfloor 2\log_2 n \rfloor + 1.$$

$$\binom{n}{k}\frac{k!}{2^{\binom{k}{2}}} < 1 \iff \frac{(n)_k}{2^{\binom{k}{2}}} < 1$$
$$\Leftarrow \frac{n^k}{2^{\binom{k}{2}}} < 1 \iff n < 2^{(k-1)/2}$$
$$\iff \log_2 n < \frac{k-1}{2} \iff 2\log_2 n + 1 < k. \quad \square$$

$\boxed{EX}$ We can get the statement as written (with the floor) from this.

<u>Example 2</u> The Zarankiewicz problem.

$ex(n, K_{r,r})$ — a corresponding Turán-type problem.

$Z_r(n) := \min\{N :$ G bipartite with $2n$ vertices equibipartioned and N edges $\Rightarrow K_{r,r} \subseteq G\}$.

Equivalently:

$Z_r(n) = \min\{N :$ any $n\times n$ 0–1 matrix with N ones has an $r\times r$ submatrix of all ones$\}$.

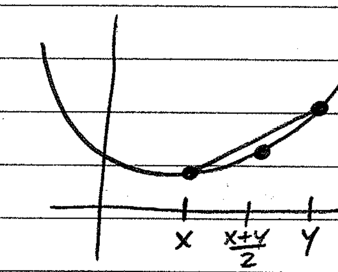
<u>$r=2$</u> upper bound.

Consider an $n\times n$ 0–1 matrix with N ones. (and no 2×2 submatrix of ones)
Let r_i = # of ones in row i.

$$\binom{n}{2} \ge \sum_{i=1}^{n}\binom{r_i}{2} \quad \leftarrow \text{[counting the number of times each pair of columns is covered by a row]}$$

$\boxed{EX}$
$$\ge n\binom{N/n}{2}$$

[Jensen's inequality]



$$f(x)+f(y) > 2f\left(\tfrac{x+y}{2}\right)$$

So
$$\frac{n(n-1)}{2} \ge n\cdot\frac{\frac{N}{n}\left(\frac{N}{n}-1\right)}{2}$$

$$n^2(n-1) \ge N(N-n) \qquad (\lozenge)$$

[Consider $N^2 - nN - n^2(n-1) = 0$ which has roots $\dfrac{n\pm\sqrt{n^2+4(n^3-n^2)}}{2}$]

$$(\lozenge) \Rightarrow N < n^{3/2}+n \Rightarrow Z_2(n) < n^{3/2}+n.$$

Mon
9 Nov
2009

Recall: $Z_2(n) = \Theta(n^{3/2})$.

Now: $Z_r(n)$ for $r \geq 3$.

<u>Upper bound</u>

Consider an $n \times n$ 0–1 matrix with no $r \times r$ submatrix of ones. Let r_i be the number of ones in row i. We have

$$\sum_{i=1}^{n} \binom{r_i}{r} \leq (r-1)\binom{n}{r}$$

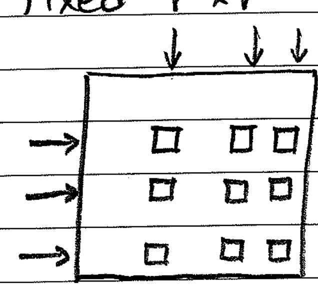
$$\underset{\boxed{EX}}{\Longrightarrow} \quad Z_r(n) \leq 2n^{2-1/r} \text{ for } n \text{ sufficiently large.}$$

<u>Lower bound</u>

We choose an $n \times n$ 0–1 matrix with N ones uniformly at random.

<u>Note</u>: This is a probability space containing $\binom{n^2}{N}$ matrices.

Consider a fixed $r \times r$ submatrix A.



$$\Pr(A \text{ is all ones}) = \frac{N}{n^2} \cdot \frac{N-1}{n^2-1} \cdot \dots \cdot \frac{N-(r^2-1)}{n^2-(r^2-1)} \leq \left(\frac{N}{n^2}\right)^{r^2}.$$

$$\Pr(\exists \text{ an } r \times r \text{ submatrix of ones})$$

$$= \Pr\Big(\bigcup_{\substack{A \\ r\times r \text{ submatrix}}} \{A \text{ is all ones}\}\Big)$$

$$\leq \sum_A \Pr(A \text{ is all ones}) \leq \binom{n}{r}^2 \left(\frac{N}{n^2}\right)^{r^2}.$$

Note that $\Pr(\exists \text{ an } r \times r \text{ submatrix of ones}) < 1$ implies $Z_r(n) > N$.

So, $\binom{n}{r}^2 \left(\frac{N}{n^2}\right)^{r^2} < 1 \Longrightarrow Z_r(n) > N$.

$\boxed{\text{Useful fact: } \binom{n}{r} \leq \left(\frac{ne}{r}\right)^r.}$

$$\binom{n}{r}^2\left(\frac{N}{n^2}\right)^{r^2} \leq \left(\frac{ne}{r}\right)^{2r}\left(\frac{N}{n^2}\right)^{r^2} = \left[\left(\frac{ne}{r}\right)^2\left(\frac{N}{n^2}\right)^r\right]^r$$

So it suffices to have

$$N < n^2\left(\frac{r}{ne}\right)^{2/r} = \left(\frac{r}{e}\right)^{2/r} n^{2-2r}.$$

<u>Thm</u> $Z_r(n) = \Omega(n^{2-2/r})$.

Plain Averaging

If X is a random variable, then $\Pr(X \geq E[X]) > 0$ and $\Pr(X \leq E[X]) > 0$.

Definition Let $G=(V,E)$ be a graph. A set $X \subseteq V$ is a *dominating set* if $X \cup N(X) = V$.

Question: How small can a dominating set be?

Proposition Let $G=(V,E)$ be a graph. Then for all $p \in (0,1)$ there exists a dominating set X such that

$$|X| \leq \sum_{v \in V} \left[p + (1-p)^{d(v)+1}\right].$$

Proof Choose a random set $U \subseteq V$ by putting each $v \in V$ into U with probability p, independently. Set

$$X = U \cup \{v \in V : v \notin U \text{ and } N(v) \cap U = \emptyset\}.$$

Note that X is (deterministically) a dominating set.

$$E[|X|] = E\Big[\sum_{v \in V} 1_{\{v \in X\}}\Big] = \sum_{v \in V} E[1_{\{v \in X\}}] \quad \text{(linearity of expectation)}$$

$$= \sum_{v \in V} \Pr(v \in X)$$

$$= \sum_{v \in V} \Big[\underbrace{p}_{\Pr(v \in U)} + \underbrace{(1-p)^{d(v)+1}}_{\Pr(v \notin U \text{ and } N(v) \cap U = \emptyset)}\Big]. \quad \square$$

Alterations

Recall: We saw that $R(k,k) \geq \dfrac{k 2^{k/2}}{2e}$.

Proof We chose a coloring $\binom{[n]}{2} = R \cup B$ by setting $\Pr(e \in R) = \Pr(e \in B) = \frac{1}{2}$ independently.

$$\Pr(\exists \text{ a monochromatic } K_k) \leq \binom{n}{k} 2 \left(\tfrac{1}{2}\right)^{\binom{k}{2}},$$

which is less than 1 for $n = \dfrac{k 2^{k/2}}{2e}$. $\square$

But wait, if we let $X = \#$ of monochromatic K_k's,

$$E[X] = \sum_{A \in \binom{[n]}{k}} \Pr\left(\tbinom{A}{2} \text{ is monochromatic}\right)$$
$$= \binom{n}{k} 2 \left(\tfrac{1}{2}\right)^{\binom{k}{2}}.$$

Now, for each coloring in the probability space, we produce a coloring with no monochromatic K_k by deleting one vertex from each monochromatic K_k. The number of vertices remaining is at least $n - X$. So

$$n - E[X] < R(k,k).$$

It remains to choose an n that maximizes $n - \binom{n}{k} 2 \left(\frac{1}{2}\right)^{\binom{k}{2}}$.

We work with the lower bound $n - \left(\frac{ne}{k}\right)^k 2 \left(\frac{1}{2}\right)^{\binom{k}{2}}$.

$$n-\left(\frac{ne}{k}\right)^k 2\left(\frac{1}{2}\right)^{\binom{k}{2}} = n-2\left[\frac{ne}{k2^{(k-1)/2}}\right]^k$$
$$= n-2\left[\frac{ne}{k2^{k/2}}\right]^k 2^{k/2}.$$

Take $n=\frac{2^{k/2}k}{e}$.

Claim: It follows that
$$R(k,k) > (1+o(1))\frac{k2^{k/2}}{e}.$$

Pf $\boxed{EX}$

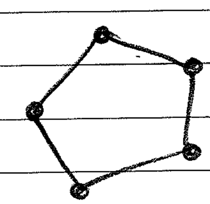
So $\sqrt{2} \le R(k,k)^{1/k} \le 4$.

Recall: Let $G=(V,E)$ be a graph.

Chromatic number:
$\chi(G) = \min\{t : \exists$ a proper coloring of G with t colors$\}$.

Clique number:
$\omega(G) = \max\{|X| : X\subseteq V$ and $G[X]$ is complete$\}$.

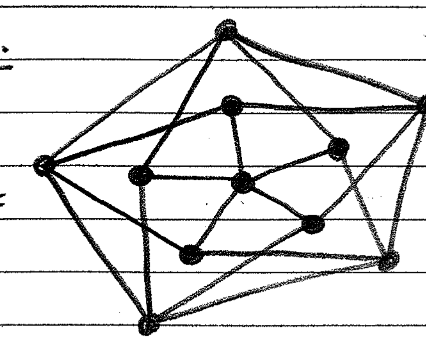
Independence number:
$\alpha(G) = \max\{|X| : X\subseteq V$ and $\binom{X}{2}\cap E=\emptyset\}$.

e.g. 

$\chi(C_5)=3$

$\omega(C_5)=2$

Q: Does a "locally sparse" graph necessarily have small chromatic number?

e.g. $H=$ 

$\chi(H)=4$ $\boxed{EX}$

$\omega(H)=2$

Defn Girth $g(G)$ = length of a smallest cycle in G.

Question: Can G simultaneously have large girth and large chromatic number?

Theorem (Erdős 1959) For all k,ℓ there exists a graph G such that $g(G)\ge \ell$ and $\chi(G)\ge k$.

(Proved constructively by Lovász in 1968.)

Wed
11 Nov
2009

Theorem (Erdős 1959) For all k, ℓ there exists a graph G such that $g(G) > \ell$ and $\chi(G) > k$.

Proof We assume G has n vertices.

Note: Let $t = n/k$. If $\alpha(G) < t$ then $\chi(G) > k$.
So it is enough to show that there exists a graph G on n vertices with
(i) $\alpha(G) < t = n/k$ and
(ii) $g(G) > \ell$.

First try.
Consider the random graph $G_{n,p}$. This has vertex set $[n]$ and $Pr(\{x,y\} \in E) = p$ independently.
(This is called the *Erdős–Rényi* random graph.)

e.g. If $H = ([n], F)$ is a graph then
$Pr(G_{n,p} = H) = p^{|F|}(1-p)^{\binom{n}{2} - |F|}$.

We consider $G_{n,p}$ and note
(i) requires p large,
(ii) requires p small.
Is there a value of p that satisfies both?
(That is, such that the probability of the associated event is greater than $\frac{1}{2}$.)

How small can p be that satisfies (i)?

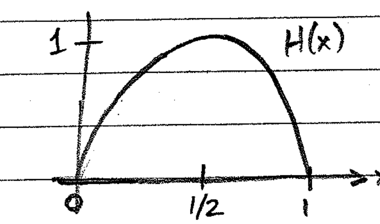
For $S \in \binom{[n]}{t}$,

$$Pr(S \text{ is an independent set in } G_{n,p}) = (1-p)^{\binom{t}{2}} \overset{(*)}{\approx} e^{-p\binom{t}{2}}.$$

Note:
- Replacing $(*)$ with $<$ is correct.
- This approximation is good for p small.

So $E[\#\text{ indep. sets of cardinality } t] \approx \binom{n}{t} e^{-p\binom{t}{2}}$. $(\Diamond)$

An approximation



The binary entropy function

$$H(x) = -x \log_2 x - (1-x)\log_2(1-x) \quad \text{for } x \in (0,1).$$

Claim: If $\alpha \in (0,1)$,

$$\binom{n}{\alpha n} = \Theta\left(\frac{1}{\sqrt{n}} 2^{H(\alpha)n}\right).$$

Pf
$$\binom{n}{\alpha n} = \frac{n!}{(\alpha n)!\,((1-\alpha)n)!}$$
[by Stirling's formula]
$$= \Theta\left(\frac{\sqrt{n}\,(n/e)^n}{\sqrt{n}\,(\alpha n/e)^{\alpha n} \cdot \sqrt{n}\,[(1-\alpha)n/e]^{(1-\alpha)n}}\right)$$
$$= \Theta\left(\frac{1}{\sqrt{n}\,(\alpha)^{\alpha n}(1-\alpha)^{(1-\alpha)n}}\right)$$
$$= \Theta\left(\frac{1}{\sqrt{n}} \cdot 2^{-(\alpha \log_2 \alpha)n} \cdot 2^{-[(1-\alpha)\log_2(1-\alpha)]n}\right). \quad \square$$

So, returning to $(\Diamond)$:

$E[\#\text{ indep. sets of cardinality } t]$

$$\approx \binom{n}{t} e^{-p\binom{t}{2}} \approx \frac{1}{\sqrt{n}} 2^{H(1/k)n} e^{-p(n/k)^2(1/2)}.$$

We want this to be small. So we want

$$(\log_2 e)\, p \left(\frac{n}{k}\right)^2 \cdot \frac{1}{2} > H\left(\frac{1}{k}\right) n.$$

So, we want $p > 2H\left(\frac{1}{k}\right) k^2 \cdot \frac{1}{n} / \log_2 e$.

(ii) Is this small enough to give

$$\Pr(g(G_{n,p}) > \ell) > \frac{1}{2} ?$$

(†) __Claim__ If $np > 2$ then

$$E[\#\text{ of } (\leq \ell)\text{-cycles in } G_{n,p}] \leq (np)^\ell.$$

__Pf__ By linearity of expectation,

$$E[\#\text{ of } (\leq \ell)\text{-cycles in } G_{n,p}]$$

$$= \sum_{j=3}^{\ell} \binom{n}{j} \frac{(j-1)!}{2} p^j \leq \sum_{j=3}^{\ell} \frac{n^j p^j}{2j}$$

$$= \sum_{j=3}^{\ell} \frac{(np)^j}{2j} \leq \ell \frac{(np)^\ell}{2\ell}. \qquad \square$$

$$\left[\text{since } \frac{np}{j+1} \cdot j > 1\right]$$

__The proof (properly)__

Take $p = \frac{\text{large constant}}{n}$ and delete one edge from each short cycle in $G_{n,p}$. Set $Y = \#$ of $(\leq \ell)$-cycles in $G_{n,p}$.
Our goal is to find a graph such that for all $S \in \binom{[n]}{t}$ we have $(\#\text{ edges in } S) > Y$.

__Markov's inequality__ If X is a random variable taking only nonnegative values, then for $\lambda > 0$

$$\Pr(X \geq \lambda) \leq \frac{E[X]}{\lambda}.$$

__Pf__ $\lambda \Pr(X \geq \lambda) \leq E[X]$. $\square$

By Markov's inequality and Claim (†),

$$\Pr(Y \geq 2(np)^\ell) \leq \frac{1}{2}.$$

So it suffices to show

$$\Pr(\underbrace{\exists\, S \in \binom{[n]}{t} \text{ s.t. } \#\text{ edges in } S \leq 2(np)^\ell}_{\text{Let this event be } \mathcal{E}.}) < \frac{1}{2}.$$

By the union bound,

$$\Pr(\mathcal{E}) \leq \underbrace{\binom{n}{t}}_{\substack{\text{specifies}\\ S}} \cdot \underbrace{\binom{\binom{t}{2}}{2(np)^\ell}}_{\substack{\text{specifies}\\ \text{edges in } S}} \cdot (1-p)^{\overbrace{\binom{t}{2} - 2(np)^\ell}^{(\Delta)}}.$$

Note We do not specify that particular edges actually appear in the events in this union.

So,

$$\Pr(\mathscr{E}) \le O\left(2^{H(1/k)n} (n/k)^{4(np)^\ell} e^{\underbrace{-p(n/k)^2(1/2)}}\right)$$

because the exponent (Δ) is $\frac{1}{2}\left(\frac{n}{k}\right)^2 - \frac{1}{2}\left(\frac{n}{k}\right) - 2(np)^\ell$

If we take $p = \frac{4k^2 H(1/k)}{n}$ then

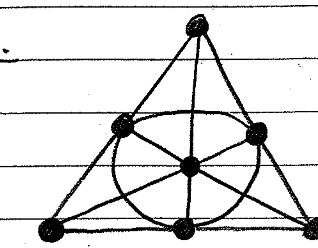
$\Pr(\mathscr{E}) \to 0$ as $n \to \infty$. $\square$

Mon 16 Nov 2009

[for Bernstein]

Definition A hypergraph $\mathcal{H}$ ($\mathcal{H} \subseteq 2^V$) has property B if it is 2-colorable; that is, if there exists a partition $V = X \cup Y$ such that $e \in \mathcal{H} \Rightarrow e \not\subseteq X$ and $e \not\subseteq Y$.

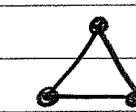
e.g.



The Fano plane is a 3-uniform (every hyperedge has cardinality 3), 3-regular (every vertex is in 3 edges) hypergraph.

$\boxed{\text{EX}}$ The Fano plane does not have property B.

e.g.



C_3 is a 2-uniform, 2-regular hypergraph that does not have property B.

Theorem If $\mathcal{H}$ is a t-uniform, t-regular hypergraph and $t \ge 10$ then $\mathcal{H}$ has property B.

Note: This actually holds for $t \ge 4$. (Thomassen 1992)

First try: Color V at random (uniformly, indep.). Let $e \in \mathcal{H}$.

$$\Pr(e \text{ is monochromatic}) = 2^{1-t}$$

big loss here

$$\Pr(\exists\, e \in \mathcal{H} : e \text{ is monochromatic}) \le |\mathcal{H}|\, 2^{1-t}$$

This is less than 1 if $|\mathcal{H}| < 2^{t-1}$.

Note: There is "a lot" of independence among these events.

Definition Let $A_1, \dots, A_m$ be events in a probability space. A dependency graph for $A_1, \dots, A_m$ is a graph with

(i) vertex set $[m]$;

(ii) A_i is mutually independent of the collection $\{A_j : j \not\sim i\} =: S$.

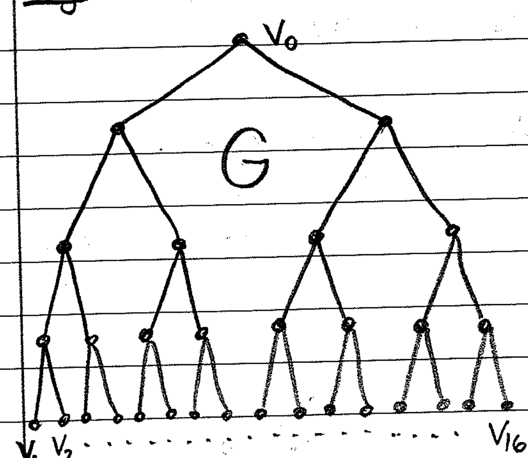
This means that for any $P, R \subseteq S$ with

$$\Pr\Big(\big(\bigwedge_{E \in P} E\big) \wedge \big(\bigwedge_{F \in R} \overline{F}\big)\Big) > 0$$

we have

$$\Pr\Big(A_i \,\Big|\, \big(\bigwedge_{E \in P} E\big) \wedge \big(\bigwedge_{F \in R} \overline{F}\big)\Big) = \Pr(A_i).$$

e.g. Percolation on a binary tree.

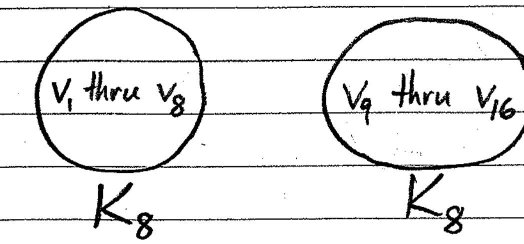


Choose a subgraph H of G by setting $\Pr(e \in H) = p$ independently for all $e \in G$.

For $i = 1, \dots, 16$, let

$A_i = \{$ the $v_0 - v_i$ path appears in $H\}$.

Then a dependency graph would be



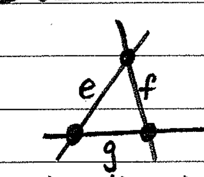
For property B in a t-uniform, t-regular hypergraph $\mathcal{H}$ let

$$A_e = \{ e \text{ is monochromatic} \}.$$

Recall: We are coloring the vertices by fair coin flips.

Note: (i) If $|e \cap f| = 1$ then A_e and A_f are independent. If $|e \cap f| \geq 2$ then A_e and A_f are dependent.

(ii) Suppose we have



Then A_e is not mutually independent of $\{A_f, A_g\}$.

We form our dependency graph on vertex set $\mathcal{H}$ by setting $e \sim f$ if $e \cap f \neq 0$.

Lovász Local Lemma

Let $A_1, \dots, A_m$ be events in a probability space such that $\Pr(A_i) < p$. If G is a dependency graph for these events and G has maximum degree d, and $4pd < 1$, then

$$\Pr\Big(\overline{\bigvee_{i=1}^{m} A_i}\Big) = \Pr\Big(\bigwedge_{i=1}^{m} \overline{A_i}\Big) > 0.$$

Proof of Theorem

$\mathcal{H}$ is a t-uniform, t-regular hypergraph. We randomly color V.

$$A_e = \{ e \text{ is monochromatic} \}$$

We apply Lovász Local Lemma with $p = 2^{1-t}$, $d = t(t-1)$. So, if $4 \cdot 2^{1-t} \cdot t(t-1) < 1$ then $\exists$ a proper 2-coloring of $\mathcal{H}$. This inequality holds for $t \geq 10$. $\square$

(Algorithmic, constructive proof by Robin Moser, ETH Zürich, 2008?.)

Brief History of Combinatorics

Erdős

↓

Turán

↓

Lovász

↓

You

(Based on Hungarian Communist propaganda?)

— Beck

Proof of Lovász Local Lemma

$$\Pr\Big(\bigwedge_{i=1}^{m} E_i\Big) = \Pr(E_1)\Pr(E_2 | E_1)\Pr(E_3 | E_1 \wedge E_2) \cdots$$

$$\Pr(A | BC) = \frac{\Pr(AB | C)}{\Pr(B | C)}$$

Claim: If $S \subseteq [m]$ and $i \notin S$ then

$$\Pr\Big[A_i \,\Big|\, \bigwedge_{j \in S} \overline{A_j}\Big] < \frac{1}{2d}.$$

Note: Claim implies Lovász Local Lemma:

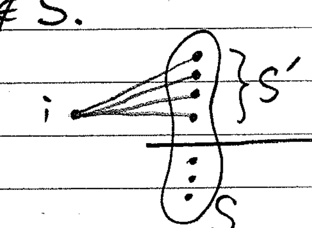
$$\Pr\Big(\bigwedge_{i=1}^{m} \overline{A_i}\Big) = \prod_{i=1}^{m} \Pr\Big(\overline{A_i} \,\Big|\, \bigwedge_{j<i} \overline{A_j}\Big) > \Big(1 - \frac{1}{2d}\Big)^m > 0.$$

Proof of Claim: We go by induction on $|S|$.

$|S| = 0$ ✓ $\quad \Big[\Pr\big(A_i \,\big|\, \bigwedge_{j \in S} \overline{A_j}\big) = \Pr(A_i) < \frac{1}{4d} \leq \frac{1}{2d}\Big]$

Now suppose S is nonempty and $i \notin S$. Define

$$S' = \{ j \in S : i \sim j \}.$$



$$\Pr\Big(A_i \,\Big|\, \bigwedge_{j \in S} \overline{A_j}\Big) = \Pr\Big(A_i \,\Big|\, \Big(\bigwedge_{j \in S'} \overline{A_j}\Big) \wedge \Big(\bigwedge_{j \in S \setminus S'} \overline{A_j}\Big)\Big)$$

$$= \frac{\Pr\Big(A_i \wedge \Big(\bigwedge_{j \in S'} \overline{A_j}\Big) \,\Big|\, \bigwedge_{j \in S \setminus S'} \overline{A_j}\Big)}{\Pr\Big(\bigwedge_{j \in S'} \overline{A_j} \,\Big|\, \bigwedge_{j \in S \setminus S'} \overline{A_j}\Big)}.$$

Note that we have

$$\Pr\Big(\bigwedge_{j\in S'} \overline{A_j} \,\Big|\, \bigwedge_{j\in S\setminus S'} \overline{A_j}\Big)$$
$$= 1 - \Pr\Big(\bigvee_{j\in S'} A_j \,\Big|\, \bigwedge_{j\in S\setminus S'} \overline{A_j}\Big)$$
$$\geq 1 - \sum_{j\in S'} \Pr\Big(A_j \,\Big|\, \bigwedge_{j\in S\setminus S'} \overline{A_j}\Big)$$
$$\underset{\text{by induction}}{\geq} 1 - \frac{|S'|}{2d} \underset{\text{since } d(i)\leq d}{\geq} \frac{1}{2}.$$

Furthermore

$$\Pr\Big(A_i \wedge \Big(\bigwedge_{j\in S'} \overline{A_j}\Big) \,\Big|\, \bigwedge_{j\in S\setminus S'} \overline{A_j}\Big)$$
$$\leq \Pr\Big(A_i \,\Big|\, \bigwedge_{j\in S\setminus S'} \overline{A_j}\Big) \underset{\text{by independence}}{=} \Pr(A_i) < p.$$

Putting it back together,

$$\Pr\Big(A_i \,\Big|\, \bigwedge_{j\in S} \overline{A_j}\Big) \leq 2p < \frac{1}{2d}. \qquad \square$$

Wed 18 Nov 2009

Lovász Local Lemma and $R(k,k)$

Previously: Lower bounds on $R(k,k)$.

Color $\binom{[n]}{2}$ at random, $\Pr(e\in R) = \Pr(e\in B) = \frac{1}{2}$ independently $\forall e \in \binom{[n]}{2}$. For $X \in \binom{[n]}{k}$ let $A_X = \{\binom{X}{2} \text{ is monochromatic}\}$.

$$\Pr\Big(\bigwedge_{X\in\binom{[n]}{k}} \overline{A_X}\Big) > 0 \implies R(k,k) > n.$$

Union bound $\longrightarrow$ $R(k,k) > \frac{1}{2e} k 2^{k/2}$

Alteration $\longrightarrow$ $R(k,k) > \frac{1}{e} k 2^{k/2}$

LLL $\rightarrow$ (i) Set $p = 2^{1-\binom{k}{2}}$.
$\Pr(A_X) = p \quad \forall X \in \binom{[n]}{k}$.

(ii) Dependency graph.
Set $A_X \sim A_Y$ if $|X\cap Y| \geq 2$.

$$d = \binom{k}{2}\binom{n}{k-2}$$

So, $4\cdot 2^{1-\binom{k}{2}}\cdot\binom{k}{2}\binom{n}{k-2} < 1$

$$\implies \Pr\Big(\bigwedge \overline{A_X}\Big) > 0 \implies R(k,k) > n.$$

It suffices to have

$$2^{2-\binom{k}{2}} k(k-1)\Big(\frac{ne}{k-2}\Big)^{k-2} < 1.$$

$$O(1)\cdot k^2\Big(\frac{k}{k-2}\Big)^{k-2}\Big(\frac{2^{-(k+1)/2}ne}{k}\Big)^{k-2} < 1$$

So $R(k,k) > (1-o(1))\frac{\sqrt{2}}{e} k 2^{k/2}$.

Second moment (method)

Notation If X is a random variable,

$$\mu_X = E[X], \qquad \sigma_X^2 = \mathrm{Var}[X] = E[(X-\mu_X)^2].$$

Chebyshev's inequality If X is a random variable and $\lambda > 0$,

$$\Pr(|X-\mu_X| \geq \lambda\sigma_X) \leq \frac{1}{\lambda^2}.$$

Proof $\Pr(|X-\mu_X| \geq \lambda\sigma_X) = \Pr\left((X-\mu_X)^2 \geq \lambda^2\sigma_X^2\right)$

$$\underset{\text{Markov's inequality}}{\leq} \frac{\sigma_X^2}{\lambda^2\sigma_X^2} = \frac{1}{\lambda^2}. \qquad \square$$

Note (second moment method)
Suppose $X_1, X_2, \ldots, X_n, \ldots$ are random variables and we would like to show $\Pr(X_n > 0) > 0$ for n sufficiently large. If we have $[\mu_{X_i} \to \infty$ and$]$ $\sigma_{X_i} = o(\mu_{X_i})$, then

$$\Pr(X_i = 0) < \Pr(|X_i - \mu_{X_i}| \geq \mu_{X_i}) \leq \left(\frac{\sigma_{X_i}}{\mu_{X_i}}\right)^2 \to 0.$$

An application

Definition A set of positive integers S has **distinct subset sums** if

$$X, Y \subseteq S \text{ and } X \neq Y \implies \sum_{x \in X} x \neq \sum_{y \in Y} y.$$

Eg. $\{1, 2, 4, 8, \ldots, 2^{n-1}\}$

$\{3, 5, 6, 7\}$

Let $f(n) = \min\{\max S : S \text{ is a set of } n \text{ positive integers with distinct subset sums}\}$.

Conj (Erdős, \$500)

$$f(n) = \Omega(2^n),$$

i.e., $\exists$ a constant c s.t. $f(n) > c2^n \ \forall n$.

Note
(i) $f(n) \leq 2^{n-1}$

(ii) $f(n) > \frac{2^n}{n}$

Pf Suppose S has distinct subset sums.

$$(\max S)n > \sum_{x \in S} n \geq 2^n - 1. \qquad \square$$

Theorem (Erdős, Moser 1956)

$f(n) = \Omega(2^n/\sqrt{n})$.

Proof Let $S = \{a_1, \ldots, a_n\}$ have distinct subset sums. Let $X_1, \ldots, X_n$ be i.i.d. random variables,

$$\Pr(X_i = 0) = \Pr(X_i = 1) = \tfrac{1}{2}.$$

Define

$$Y = \sum_{i=1}^{n} a_i X_i.$$

$$\mu_Y = \sum_{i=1}^{n} \frac{a_i}{2}$$

$$\sigma_Y^2 = \sum_{i=1}^{n} \frac{a_i^2}{4}$$

$$\boxed{\begin{aligned}\sigma_Y^2 &= E[(Y-\mu_Y)^2] \\ &= E[Y^2] - \mu_Y^2 \\ &= \sum_{i,j} a_i a_j E[X_i X_j] \\ &\quad - \sum_{i,j} a_i a_j E[X_i]E[X_j] \\ &\overset{\text{independence}}{=} \sum_i a_i^2 (E[X_i^2] - E[X_j]^2) \\ &= \sum_i \frac{a_i^2}{4}.\end{aligned}}$$

By Chebyshev,

$$\Pr(|Y-\mu_Y| \geq \lambda\sigma_Y) \leq \frac{1}{\lambda^2}$$

so

$$\Pr(|Y-\mu_Y| < \lambda\sigma_Y) \geq 1 - \frac{1}{\lambda^2}.$$

$$\frac{2\lambda\sigma_Y}{2^n} \underset{\text{pigeonhole and distinct subset sum property}}{\geq} \Pr(|Y-\mu_Y| < \lambda\sigma_Y) \underset{\text{Chebyshev}}{\geq} 1 - \frac{1}{\lambda^2}.$$

Furthermore

$$\sigma_Y^2 = \frac{1}{4}\sum_{i=1}^{n} a_i^2 < \frac{1}{4} n (\max S)^2.$$

Putting it all together,

$$\frac{2\lambda \frac{1}{2}\sqrt{n}\,\max S}{2^n} > \frac{2\lambda\sigma_Y}{2^n} \geq 1 - \frac{1}{\lambda^2}.$$

So $\max S > \frac{2^n}{\sqrt{n}}\left(\frac{1}{\lambda} - \frac{1}{\lambda^3}\right)$. □

Chernoff bound

Recall central limit theorem.

If $X_1, X_2, \ldots$ are i.i.d. random variables with mean μ and standard deviation σ, then

$$\Pr\left(\frac{X_1 + X_2 + \cdots + X_n - \mu n}{\sigma\sqrt{n}} > \lambda\right) \to \underbrace{\frac{1}{\sqrt{2\pi}}\int_\lambda^\infty e^{-t^2/2}\,dt}_{\text{normal distribution}}.$$

Chernoff bound Let $X_1, \ldots, X_n$ be i.i.d. random variables with $\Pr(X_i = 1) = \Pr(X = -1) = \frac{1}{2}$. Set $S_n = \sum_{i=1}^n X_i$. Then

$$\Pr(S_n > \lambda\sqrt{n}) < e^{-\lambda^2/2}$$

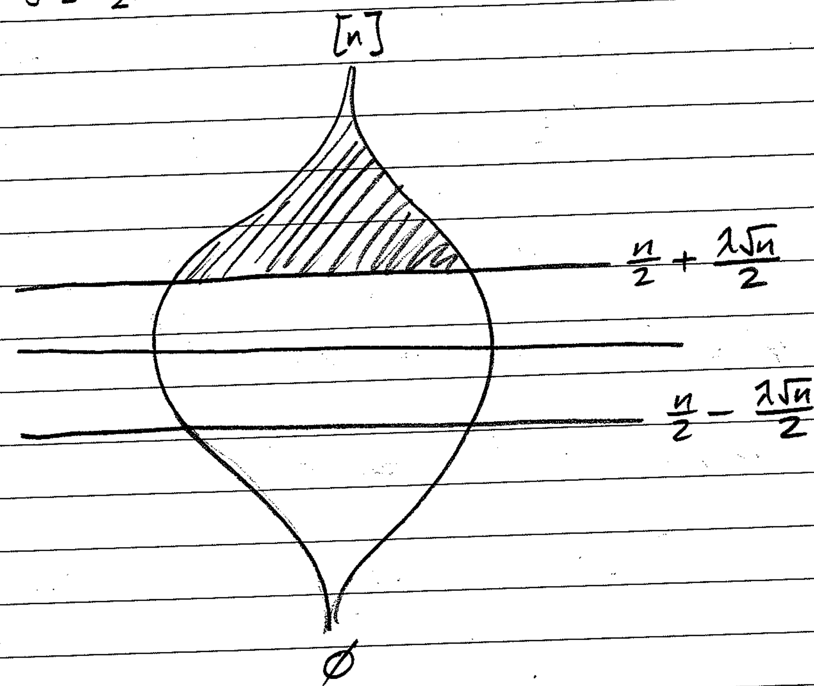
for all $\lambda > 0$.

Note: $\Pr(S > \lambda\sqrt{n}) = \frac{1}{2^n} \sum_{j=\frac{n}{2}+\frac{\lambda\sqrt{n}}{2}}^{n} \binom{n}{j}$

So, Chernoff implies

$$\sum_{j=\frac{n}{2}+\frac{\lambda\sqrt{n}}{2}}^{n} \binom{n}{j} < 2^n e^{-\lambda^2/2}.$$

poset:



Mon 23 Nov 2009

Proof of Chernoff bound Let $\alpha = \lambda\sqrt{n}$.

$$\Pr(S_n \geq \alpha) = \Pr(e^{\eta S_n} \geq e^{\eta\alpha}) \quad [\text{where } \eta > 0 \text{ is to be determined}]$$

$$\overset{\text{Markov}}{\leq} E[e^{\eta S_n}]\, e^{-\eta\alpha}$$

$$= \left(\prod_{i=1}^{n} E[e^{\eta X_i}]\right) e^{-\eta\alpha} \quad [\text{using independence}]$$

$$= \left(\frac{e^{-\eta}+e^{\eta}}{2}\right)^n e^{-\eta\alpha} \quad \left[E[e^{\eta X_i}] = \frac{e^{-\eta}+e^{\eta}}{2}\ \forall i\right]$$

$$< e^{\eta^2 n/2} e^{-\eta\alpha}. \quad \left[e^{\eta}+e^{-\eta} = 2\left(1+\frac{\eta^2}{2}+\frac{\eta^4}{4!}+\frac{\eta^6}{6!}+\cdots\right) < 2e^{\eta^2/2}\right]$$

Taking $\eta = \frac{\alpha}{n} = \frac{\lambda}{\sqrt{n}}$ we have

$$\Pr(S_n \geq \alpha) < e^{-\alpha^2/2n} = e^{-\lambda^2/2}. \quad \square$$

Defn Let $\mathcal{H}$ be a hypergraph. The discrepancy of $\mathcal{H}$ is

$$\mathrm{disc}(\mathcal{H}) = \min_{V = A \dot\cup B} \max_{e \in \mathcal{H}} \big|\, |e \cap A| - |e \cap B| \,\big|$$

$$= \min_{f: V \to \{-1,1\}} \max_{e \in \mathcal{H}} \Big| \sum_{x \in e} f(x) \Big|.$$

e.g. If $\mathcal{H}$ is t-uniform,

$$\mathcal{H} \text{ has property B} \iff \mathrm{disc}(\mathcal{H}) \leq t-2.$$

[Note $\mathrm{disc}(\mathcal{H}) = t-1$ is impossible.]

Prop (Beck) If $\mathcal{H}$ is a t-uniform, t-regular hypergraph then

$$\operatorname{disc}(\mathcal{H}) < 2\sqrt{2}\sqrt{t\log t}.$$

Conj (Komlos?) If $\mathcal{H}$ is a t-uniform, t-regular hypergraph then

$$\operatorname{disc}(\mathcal{H}) = O(\sqrt{t}).$$

Proof of Prop Choose a random $f: V \to \{-1,1\}$ by setting

$$\Pr(f(x) = -1) = \Pr(f(x) = 1) = \frac{1}{2}$$

independently $\forall x \in V$.

For each $e \in \mathcal{H}$ define A_e to be the event

$$\left|\sum_{x\in e} f(x)\right| \geq 2\sqrt{2}\sqrt{t\log t}.$$

$$p = \Pr(A_e) < 2e^{-(2\sqrt{2}\sqrt{\log t})^2/2} \quad \text{[by Chernoff]}$$

$$= \frac{2}{t^4}.$$

We apply the Lovász local lemma with a dependency graph given by

$$A_e \sim A_f \iff e \cap f \neq \emptyset.$$

This graph has degrees bounded by $d = t(t-1)$.

Since $4pd < 8\frac{t-1}{t^3} \leq 1$ we have

$$\Pr\left(\bigwedge_{e\in\mathcal{H}} \overline{A_e}\right) > 0.$$

Thus there exists $f: V \to \{-1, 1\}$ that gives $\operatorname{disc}(\mathcal{H}) < 2\sqrt{2}\sqrt{t\log t}$. $\square$

Probabilistic Combinatorics:

- Probabilistic method (proving existence)
- Random structures
- Randomized algorithms

Correlation Inequalities

A probability space is given. Events A and B are positively correlated if

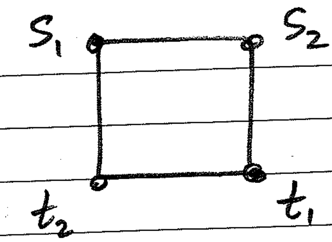
$$\Pr(A \wedge B) \geq \Pr(A)\Pr(B).$$

Equivalently, $\Pr(A|B) \geq \Pr(A)$.

e.g. G is a graph; s_1, s_2, t_1, t_2 are vertices of G. Consider the probability space on subgraphs of G generated by $\Pr(e \text{ is "open"}) = p$ for all edges e, independently. i.e., e exists in the subgraph

Let $A_i = \{\exists \text{ an "open" } s_i - t_i \text{ path}\}$ for $i = 1, 2$.

__sub-e.g.__



$$\Pr(A_1) = \Pr(A_2) = 2p^2 - p^4$$

$$\Pr(A_1 \wedge P_2) = 4p^3(1-p) + p^4.$$

__Kleitman's lemma__ (1966)
Suppose $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ are filters (i.e., closed w.r.t. supersets: $A \in \mathcal{F}$, $B \supseteq A \Rightarrow B \in \mathcal{F}$).
Then
$$2^n |\mathcal{F} \cap \mathcal{G}| \geq |\mathcal{F}| \cdot |\mathcal{G}|.$$

__Note:__ If we choose $A \subseteq [n]$ uniformly at random, then

$$\Pr(\mathcal{F}) := \Pr(A \in \mathcal{F}) = \frac{|\mathcal{F}|}{2^n}, \qquad \text{etc.}$$

and Kleitman's lemma says

$$\Pr(\mathcal{F} \wedge \mathcal{G}) \geq \Pr(\mathcal{F}) \Pr(\mathcal{G}).$$

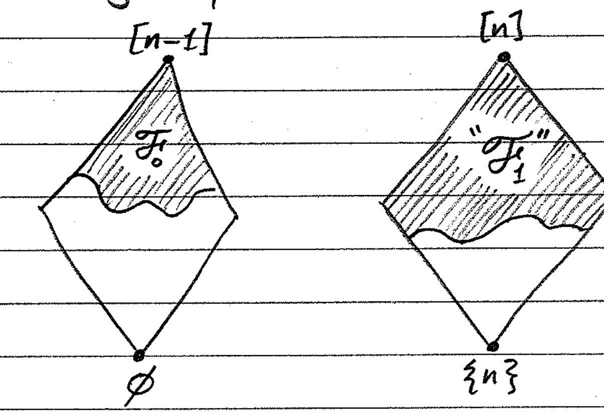
__Harris inequality__ (1960)
Suppose $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ are filters. Let $A \subseteq [n]$ be chosen at random by setting $\Pr(i \in A) = p_i$ independently for all $i \in [n]$.
Then
$$\Pr(A \in \mathcal{F} \cap \mathcal{G}) \geq \Pr(A \in \mathcal{F}) \Pr(A \in \mathcal{G}).$$

__Note:__ Setting $p_i = \frac{1}{2}$ gives Kleitman's lemma.

__Proof__ We go by induction on n. $n=1$ ✓

__$n>1$.__

poset:



Define $\mathcal{F}_0 = \{ A \in \mathcal{F} : n \notin A \}$

$\mathcal{F}_1 = \{ A \setminus \{n\} : A \in \mathcal{F} \text{ and } n \in A \}.$

Note that $\mathcal{F}_0, \mathcal{F}_1 \subseteq 2^{[n-1]}$ are filters.
Define $\mathcal{G}_0, \mathcal{G}_1$ analogously. Set $p = p_n$ and $q = 1 - p = 1 - p_n$.

__Key fact:__
$a_1 \geq a_0$ and $b_1 \geq b_0$
$\Longrightarrow (a_1 - a_0)(b_1 - b_0) \geq 0$
$\Longrightarrow a_1 b_1 + a_0 b_0 \geq a_1 b_0 + a_0 b_1$

$$\Pr(\mathcal{F}\cap\mathcal{G}) = (1-p)\Pr(\mathcal{F}_0\cap\mathcal{G}_0) + p\Pr(\mathcal{F}_1\cap\mathcal{G}_1),$$

where we view $\mathcal{F}_0, \mathcal{F}_1, \mathcal{G}_0, \mathcal{G}_1$ as events in $2^{[n-1]}$ with probabilities given by $p_1, \dots, p_{n-1}$.

By induction,

$$\Pr(\mathcal{F}\cap\mathcal{G}) \geq \underbrace{(1-p)}_{q}\Pr(\mathcal{F}_0)\Pr(\mathcal{G}_0) + p\Pr(\mathcal{F}_1)\Pr(\mathcal{G}_1).$$

To show:

$$q\Pr(\mathcal{F}_0)\Pr(\mathcal{G}_0) + p\Pr(\mathcal{F}_1)\Pr(\mathcal{G}_1)$$
$$\geq [q\Pr(\mathcal{F}_0) + p\Pr(\mathcal{F}_1)][q\Pr(\mathcal{G}_0) + p\Pr(\mathcal{G}_1)].$$

So ETS:

$$pq\Pr(\mathcal{F}_0)\Pr(\mathcal{G}_0) + pq\Pr(\mathcal{F}_1)\Pr(\mathcal{G}_1)$$
$$\geq p\Pr(\mathcal{F}_1)\cdot q\Pr(\mathcal{G}_0) + q\Pr(\mathcal{F}_0)\cdot p\Pr(\mathcal{G}_1).$$

To show this, we apply the key with
$a_0 = \Pr(\mathcal{F}_0)$, $a_1 = \Pr(\mathcal{F}_1)$,
$b_0 = \Pr(\mathcal{G}_0)$, $b_1 = \Pr(\mathcal{G}_1)$,
noting that $\Pr(\mathcal{F}_1) \geq \Pr(\mathcal{F}_0)$,
$\Pr(\mathcal{G}_1) \geq \Pr(\mathcal{G}_0)$,
as $\mathcal{F}, \mathcal{G}$ are filters. $\square$

Mon
30 Nov
2009

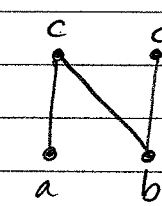
$$\left|\bigcup_{i=1}^{m} \mathcal{F}_i\right| \leq 2^n - 2^{n-m}, \quad \mathcal{F}_i \subseteq 2^{[n]} \text{ intersecting}$$

(Use correlation inequalities for this.)

An example

Suppose P is a (finite) poset. A linear extension of P is a total ordering that agrees with the poset relation.

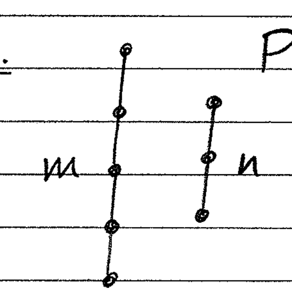
e.g.



5 linear extensions of this poset.

Notation:
$E(P) = \{\text{linear extensions of } P\}$
$e(P) = |E(P)|$

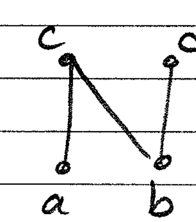
e.g.



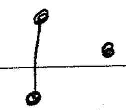
$$e(P') = \binom{m+n}{n}$$

We consider the probability space on $E(P)$ with uniform distribution.

e.g.



$$\Pr(a<b) = \frac{2}{5}$$

e.g. P'' 

$$\Pr(x<y) \in \{0, \tfrac{1}{3}, \tfrac{2}{3}, 1\}.$$

Conj ($\frac{1}{3}-\frac{2}{3}$ conjecture)
If P is a poset and P is not a chain, then there exist x, y such that

$$\Pr(x,y) \in [\tfrac{1}{3}, \tfrac{2}{3}].$$

Correlation Questions.

Let P be a poset, and let $x, y, z, w \in P$ ~~(no two of these comparable)~~.

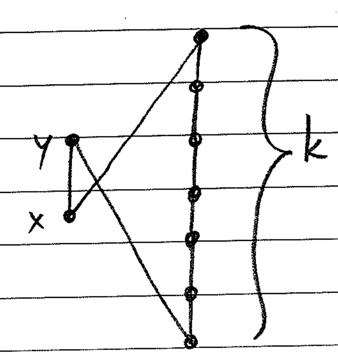
① Are $\{x<y\}$ and $\{x<z\}$ positively (i.e., nonnegatively) correlated?

② Are $\{w<x<z\}$ and $\{w<y<z\}$ positively correlated?

Answers

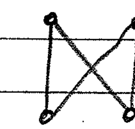
① Yes. (Known as the XYZ theorem, due to Shepp.)

② No.



$\boxed{\text{EX}}$ Find a good position for z, w.

A lattice is a poset with a
- meet: greatest common lower bound $(x \wedge y)$
- join: least common upper bound $(x \vee y)$

e.g.  not a lattice.

A lattice is distributive if meet and join distribute, i.e.,

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z), \text{ etc.}$$

FKG Inequality (1971, Fortuibi, Kastelyn, Ginibre)
Let L be a distributive lattice. Let μ be a probability measure on L such that

$$(*) \quad \mu(A)\mu(B) \le \mu(A \wedge B)\mu(A \vee B) \quad \forall A, B \in L.$$

Then if $f, g : L \to \mathbb{R}^+$ are nondecreasing [i.e., $x<y \Rightarrow f(x) \le f(y)$] then

$$E_\mu[fg] \ge E_\mu[f]\, E_\mu[g].$$

Note: For L a finite distributive lattice, L is isomorphic to some subset of $2^{[n]}$ closed with respect to intersections and unions.

For today we work with subsets of $2^{[n]}$.

Remarks

(1) Condition ⊛ is called the log-supermodular condition.

$h : L \to \mathbb{R}^+$ is supermodular if

$$h(A) + h(B) \le h(A \wedge B) + h(A \vee B).$$

(2) Verifying FKG for L a chain is left as an $\boxed{EX}$.

(3) FKG ⇒ Harris.

$$L = 2^{[n]}$$

$$\mu(A) = p^{|A|}(1-p)^{n-|A|}$$

⊛ is easy to check.

Take $f = 1_{\mathcal{F}}$, $g = 1_{\mathcal{G}}$. Then

$$E_\mu[1_{\mathcal{F}}] = Pr(\mathcal{F}), \text{ etc.}$$

(4) Every log-supermodular measure on $2^{[n]}$ is of the following form:

Take $[n] = I_1 \dot\cup I_2 \dot\cup \cdots \dot\cup I_k$ a partition, $p_i \in [0,1]$ for $i = 1, \dots, k$, and then set

$$\mu(A) = \begin{cases} \prod_{j \in J} p_j, & \text{if } A = \bigcup_{j \in J} I_j; \\ 0, & \text{otherwise.} \end{cases}$$

Pf of (4) $\boxed{EX}$

(5) If f is nondecreasing and g nonincreasing, then

$$E_\mu[fg] \le E_\mu[f]\, E_\mu[g].$$

Pf $\boxed{EX}$

Theorem (Holley 1974)
Let L be a distributive lattice, and let μ_1, μ_2 be probability measures on L.
If

$$\mu_1(A)\mu_2(B) \le \mu_1(A \vee B)\mu_2(A \wedge B) \quad \forall A, B \in L$$

and $f : L \to \mathbb{R}_+$ is nondecreasing then

$$E_{\mu_1}(f) \ge E_{\mu_2}(f).$$

Theorem (Four-functions theorem)
(Ahlswede, Daykin 1978)
Let L be a distributive lattice, and let $\alpha, \beta, \gamma, \delta : L \to \mathbb{R}_+$ be such that

$$(\star\star) \qquad \alpha(A)\beta(B) \le \gamma(A \vee B)\,\delta(A \wedge B) \quad \forall A, B \in L.$$

Then $\alpha(L)\beta(L) \le \gamma(L)\,\delta(L)$

where $f(L) := \sum_{x \in L} f(x)$.

Proof We may assume $L = 2^{[n]}$.
We go by induction on n.

Base case: $n=1$. Set $f_0 = f(\emptyset)$, $f_1 = f(\{1\})$. We have

$$\begin{aligned} \alpha_0\beta_0 &\le \gamma_0\delta_0 \\ \alpha_0\beta_1 &\le \gamma_0\delta_1 \\ \alpha_1\beta_0 &\le \gamma_0\delta_1 \\ \alpha_1\beta_1 &\le \gamma_1\delta_1 \end{aligned} \quad \Bigg\}\ \text{This is not quite what we want.}$$

and we want to show

$$(\alpha_0+\alpha_1)(\beta_0+\beta_1) \le (\gamma_0+\gamma_1)(\delta_0+\delta_1).$$

We have

$$(\gamma_0\delta_1 - \alpha_0\beta_1)(\gamma_0\delta_1 - \alpha_1\beta_0) \ge 0$$

$$\Rightarrow \gamma_0^2\delta_1^2 + \alpha_0\alpha_1\beta_0\beta_1 \ge \alpha_0\beta_1\gamma_0\delta_1 + \alpha_1\beta_0\gamma_0\delta_1.$$

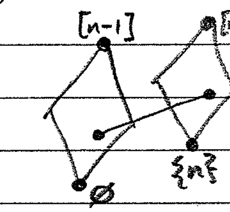
$$\Rightarrow \gamma_0^2\delta_1^2 + \alpha_0\alpha_1\beta_0\beta_1 + \alpha_0\beta_0\gamma_0\delta_1 + \alpha_1\beta_1\gamma_0\delta_1 \ge (\alpha_0+\alpha_1)(\beta_0+\beta_1)\gamma_0\delta_1$$

$$\Rightarrow \left(\delta_1 + \frac{\alpha_0\beta_0}{\gamma_0}\right)\left(\gamma_0 + \frac{\alpha_1\beta_1}{\delta_1}\right)\gamma_0\delta_1 \ge (\alpha_0+\alpha_1)(\beta_0+\beta_1)\gamma_0\delta_1$$

[Note the conclusion follows easily if $\gamma_0 = 0$ or $\delta_1 = 0$.]

$$\Rightarrow (\delta_1+\delta_0)(\gamma_0+\gamma_1) \ge (\alpha_0+\alpha_1)(\beta_0+\beta_1).$$

Inductive step For $f \in \{\alpha, \beta, \gamma, \delta\}$, $f: 2^{[n]} \to \mathbb{R}_+$, define



$$f' : 2^{[n-1]} \to \mathbb{R}_+$$

by

$$f'(A) = f(A) + f(A \cup \{n\}).$$

Note that $f'(2^{[n-1]}) = f(2^{[n]})$.
So, it remains to show that $\alpha', \beta', \gamma', \delta'$ satisfy $\star\star$. For $A, B \subseteq [n-1]$, set

$$\begin{aligned} \alpha_0 &= \alpha(A) & \alpha_1 &= \alpha(A \cup \{n\}) \\ \beta_0 &= \beta(B) & \beta_1 &= \beta(B \cup \{n\}) \\ \gamma_0 &= \gamma(A \cap B) & \gamma_1 &= \gamma((A\cap B) \cup \{n\}) \\ \delta_0 &= \delta(A \cup B) & \delta_1 &= \delta((A \cup B) \cup \{n\}). \end{aligned}$$

We have

$$\begin{aligned} \alpha_0\beta_0 &= \alpha(A)\beta(B) \\ &\le \gamma(A\cap B)\,\delta(A \cup B) = \gamma_0\delta_0. \end{aligned}$$

Similarly

$$\begin{aligned} \alpha_0\beta_1 &\le \gamma_0\delta_1, \\ \alpha_1\beta_0 &\le \gamma_0\delta_1, \\ \alpha_1\beta_1 &\le \gamma_1\delta_1. \end{aligned}$$

$\square$