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21-257, Methods and Models for Optimization, Spring 1999

Review for Test 2
Solutions

1. Consider the following LP.

$$\begin{array}{rrcrcr} \text{Maximize:} & 40x_1 & + & 30x_2 & & \\ \text{Subject to:} & x_1 & + & x_2 & \leq & 55 \\ & 2x_1 & + & 3x_2 & \leq & 130 \\ & 60x_1 & + & 10x_2 & \leq & 1100 \\ & 10x_1 & + & 20x_2 & \leq & 900 \\ & \multicolumn{5}{l}{x_1 \geq 0, x_2 \geq 0} \end{array}$$

Insert slack variables and set up a simplex tableau (but do not solve the LP).
Solution.

$$\begin{array}{rrcrcccccccccr} \text{Maximize:} & 40x_1 & + & 30x_2 & & & & & & & & & & \\ \text{Subject to:} & x_1 & + & x_2 & + & s_1 & & & & & & & = & 55 \\ & 2x_1 & + & 3x_2 & & & + & s_2 & & & & & = & 130 \\ & 60x_1 & + & 10x_2 & & & & & + & s_3 & & & = & 1100 \\ & 10x_1 & + & 20x_2 & & & & & & & + & s_4 & = & 900 \\ & \multicolumn{13}{l}{x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0, s_4 \geq 0} \end{array}$$

and the simplex tableau is

x_1	x_2	s_1	s_2	s_3	s_4	
1	1	1	0	0	0	55
2	3	0	1	0	0	130
60	10	0	0	1	0	1100
10	20	0	0	0	1	900
-40	-30	0	0	0	0	0

2. Consider the following tableau.

x_1	x_2	x_3	s_1	s_2	s_3	
1	2	0	0	$-\frac{1}{2}$	0	120
0	-3	1	1	0	0	55
0	5	2	0	5	1	104
0	2	-3	0	-1	2	1700

Apply one step of the simplex algorithm to this tableau.
Solution. The basic variables corresponding to rows 1, 2 and 3 in this tableau are x_1, s_1, and s_3 respectively. The most negative element in the objective row is (-3), corresponding to x_3. Therefore, x_3 is the entering variable. Replacement quantities can be computed for rows 2 and 3, and these are $55/1 = 55$ and $104/2 = 52$. The smaller one is 52, corresponding to row 3 (basic

variable s_3). Thus, the departing variable is s_3 (row 3). The pivoting element is then in row 3, in the column corresponding to x_3. We then do the pivoting: First, we do $R_3/2 \to R_3$:

x_1	x_2	x_3	s_1	s_2	s_3	
1	2	0	0	$-\frac{1}{2}$	0	120
0	-3	1	1	0	0	55
0	$\frac{5}{2}$	1	0	$\frac{5}{2}$	$\frac{1}{2}$	52
0	2	-3	0	-1	2	1700

Then, we procced further to turn all the entries below and above the pivot element into zeros. We do $R2 - R3 \to R2, R4 + 3R3 \to R4$:

x_1	x_2	x_3	s_1	s_2	s_3	
1	2	0	0	$-\frac{1}{2}$	0	120
0	$-\frac{11}{2}$	0	1	$-\frac{5}{2}$	$-\frac{1}{2}$	3
0	$\frac{5}{2}$	1	0	$\frac{5}{2}$	$\frac{1}{2}$	52
0	$\frac{19}{2}$	0	0	$\frac{13}{2}$	$\frac{7}{2}$	1856

Incidentally, since all of the elements in the objective row are non-negative, this is the optimal tableau.

3. In the process of solving some LPs, the following tableaux were obtained.

(a)

x_1	x_2	x_3	s_1	s_2	
1	3	0	1	2	16
0	-2	1	1	1	7
0	-3	0	-2	1	26

(b)

x_1	x_2	s_1	s_2	s_3	
1	0	$-\frac{1}{2}$	0	1	10
0	1	-1	0	$\frac{3}{2}$	15
0	0	-5	1	0	25
0	0	-1	0	2	1020

(c)

x_1	x_2	x_3	s_1	s_2	s_3	
1	0	0	1	1	-1	11
0	0	1	$-\frac{1}{2}$	1	0	12
0	1	0	0	3	2	36
0	0	0	1	3	5	7090

For each of these, determine the situation that it describes, namely, one of the following:

- the optimal tableau has been reached;
- the problem has no upper bound;
- at least one more step of the simplex algorithm can be made.

Give reasons to support your answer.
Solution.

(a) Since there are negative elements in the objective row, the tableau cannot be optimal. The column with the most negative number (-3) in the objective row is the column corresponding to x_2. Since it has non-negative elements, one more step of the simplex algorithm can be done.

(b) Since there is a negative element in the objective row, the tableau cannot be optimal. The column with the negative number (-1) in the objective row is the column corresponding to s_1. Since it has no non-negative elements, the problem has no upper bound.

(c) Since all of the elements in the objective row are non-negative, this is the optimal tableau. The optimal solution can be obtained at this point by setting the basic variables x_1, x_3, and x_2 equal to the corresponding right-hand sides.

4. Consider the following LP. Insert slack, surplus, and artificial variables and determine the artificial objective function for the two-phase method. Set up the corresponding tableau for the two-phase method.

$$\begin{array}{rrcrcrcl} \text{Maximize:} & 10x_1 & + & 40x_2 & + & 20x_3 & & \\ \text{Subject to:} & 60x_1 & + & 30x_2 & + & 10x_3 & \leq & 1200 \\ & x_1 & + & 2x_2 & & & \geq & 1 \\ & x_1 & + & x_2 & + & 3x_3 & = & 8 \\ & \multicolumn{7}{l}{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0} \end{array}$$

Solution. We insert a slack variable s_1 into the first constraint, a surplus varible s_2 into the second constraint, and artifitial variables a_2 and a_3 into the second and third constraints respectively. The set of constraints then becomes:

$$\begin{array}{rcrcrcrcrcl} 60x_1 & + & 30x_2 & + & 10x_3 & + & s_1 & & & & = 1200 \\ x_1 & + & 2x_2 & & & & & -s_2 & +a_2 & & = 1 \\ x_1 & + & x_2 & + & 3x_3 & & & & & + \; a_3 & = 8 \\ \multicolumn{11}{l}{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0, s_2 \geq 0, a_2 \geq 0, a_3 \geq 0} \end{array}$$

From the second and third equations we can express

$$a_2 = 1 - x_1 - 2x_2 + s_2$$

and

$$a_3 = 8 - x_1 - x_2 - 3x_3.$$

The artificial objective function is equal to the negative sum of the artfitial variables, expressed in terms of the other variables:

$$\begin{aligned} -a_2 - a_3 &= -(1 - x_1 - 2x_2 + s_2) - (8 - x_1 - x_2 - 3x_3) \\ &= 2x_1 + 3x_2 + 3x_3 - s_2 - 9 \end{aligned}$$

The tableau for the two-phase method then is:

x_1	x_2	x_3	s_1	s_2	a_2	a_3	
60	30	10	1	0	0	0	1200
1	2	0	0	-1	1	0	1
1	1	3	0	0	0	1	8
-10	-40	-20	0	0	0	0	0
-2	-3	-3	0	1	0	0	-9

5. Consider the following minimizing problem.

$$\begin{array}{rrcrcrcr} \text{Minimize:} & 2y_1 & + & 5y_2 & + & 4y_3 & & \\ \text{Subject to:} & 3y_1 & + & 2y_2 & + & 5y_3 & \leq & 30 \\ & 2y_1 & + & 4y_2 & - & 3y_3 & \geq & 40 \\ & 2y_1 & + & 6y_2 & + & 4y_3 & = & 80 \\ & \multicolumn{7}{l}{y_1 \geq 0, y2 \text{ unrestricted}, y_3 \geq 0} \end{array}$$

Formulate the dual maximizing problem.
Solution: According to the dual formation rules, the dual problem is

$$\begin{array}{rrcrcrcr} \text{Maximize:} & 30x_1 & + & 40x_2 & + & 80x_3 & & \\ \text{Subject to:} & 3x_1 & + & 2x_2 & + & 2x_3 & \leq & 2 \\ & 2x_1 & + & 4x_2 & + & 6x_3 & = & 5 \\ & 5x_1 & - & 3x_2 & + & 4x_3 & \leq & 4 \\ & \multicolumn{7}{l}{x_1 \leq 0, x_2 \geq 0, x_3 \text{ unrestricted}} \end{array}$$