Continuity of utility-maximization with respect to preferences

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Abstract

This paper provides an easy verifiable regularity condition under which the investor's utility maximizer depends continuously on the description of her preferences in a general incomplete financial setting. Specifically, we extend the setting of Jouini and Napp (2004) to 1) noise generated by a general continuous semi-martingale and 2) the market price of risk is allowed to be a general adapted process satisfying a mild integrability condition. This extension allows us to obtain positive results for both the mean-reversion model of Kim & Omberg (1996) and the stochastic volatility model of Heston (1993). Finally, we provide an example set forth in Samuelsen's complete financial model illustrating that without imposing additional regularity the continuity property of the investor's optimizer can fail.

Key Words: Continuous semi-martingales, market price of risk process, expected utility theory, stability of optimizers.

Mathematics Subject Classification (2000): 91B16, 91B28

1 Introduction and summary

We assume that the investor's preferences can be described via a von Neumann-Morgenstern utility function U and that the investor seeks to maximize expected utility of terminal wealth W where W is replicable by trading in the financial market at costs x. In other words, the investor seeks a random variable $\widehat{W}$ that satisfies

$$u(x) \triangleq \sup_W \mathbb{E}[U(W)] = \mathbb{E}[U(\widehat{W})],$$

where the supremum is taken over all terminal wealths available to the investor through trading with initial costs at most x. An important piece of input data is the description of the investor's risk preferences given by U. The specific U to be used in any given implementation is ultimately an empirical task and hence is subject to measurement errors. We are therefore naturally led to the following question:

> *"How are the investor's optimizer $\widehat{W}$ and the corresponding value function u affected by a small misspecification of the investor's utility function U?"*

In order to quantify this question, we need to describe the domain in which U varies and the domains in which $\widehat{W}$ and u vary. Following [JN04], we place a growth condition on the possible choices for U, $U(\cdot) \leq \overline{U}(\cdot)$ for some universal upper utility bound $\overline{U}$. Since both U and u are real valued functions it is natural to consider pointwise convergence, however, for the optimal terminal wealth $\widehat{W}$ several possible domains are available. [JN04] establish almost surely convergence as well as $\mathbb{L}^1$-convergence of $\widehat{W}$. It is a main contribution of this paper to show that by relaxing the notion of convergence to $\mathbb{L}^0$ equipped with the topology induced by convergence in probability, we can obtain continuity properties of both u and $\widehat{W}$ in a much more general setting than [JN04]:

1. We replace [JN04]'s uniform boundedness condition placed on the market price of risk process with an easy verifiable integrability condition. This relaxation allows us to examine e.g., the complete mean-reversion model presented in [W02].

2. Perhaps more importantly, we relax the complete Brownian financial setting of [JN04] to a general continuous incomplete semi-martingale framework. This allows us to examine two of the most prominent and most widely applied models in finance: Heston's stochastic volatility model [Hes93] and Kim & Omberg's mean-reversion model [KO96]. In the first model volatility is a non-traded asset whereas in the latter model the drift component is a non-traded asset and hence both models are incomplete but embedded in the framework of the present paper.

In the complete Brownian financial setting, [KLS87] and [CH89] developed what has become known in the literature as the martingale method which allows an explicit characterization of the value function u and the optimal terminal wealth $\widehat{W}$ in terms of the pricing kernel present in the economy. We apply the martingale method to exemplify that without further regularity imposed on the set of possible utility functions U, the quantities u and $\widehat{W}$ can fail to depend continuously on the description of the investor's preferences modeled by U. This illustrates the need for a regularity condition like $U(\cdot) \leq \overline{U}(\cdot)$ suggested by [JN04] and adopted in this paper. Based on the general existence result of [KS99], we are able to prove continuity of both the investor's value function u as well as the investor's optimal terminal wealth $\widehat{W}$ within this class of utility functions and provided that the market price of risk process satisfies a simple and easy verifiable integrability condition.

In a discrete setting, the case where the investor's utility function is defined over $\mathbb{R}$ and with prices processes being bounded, [CR05] study a similar continuity problem as we do. Their setting is not embedded in our analysis as our approach fundamentally hinges on the assumption of noise generated by a continuous semi-martingale and on the assumption that the investor's utility function is defined over the positive real numbers.

A related stability question is that of robust utility maximization: here the idea is for a fixed utility function to create decision rules that takes into account several possible specifications of the financial market. The seminal work [GS89] provides a set of axioms which the investor's preference order has to satisfy in order to be numerical represented as a worst-case scenario. Instead of providing a complete overview of this literature we refer to [Mae04] and [Con06] as well as the textbook [FS04] for more discussions on robust utility maximization.

Another part of the literature considers the entire description of the financial market as input data to the utility maximization problem and provides conditions under which the investor's optimizer depends continuously on this description. One application of this analysis is the passage from discrete time models to continuous time models which is particular relevant for numerical implementations, see e.g., [He91] and [DP92] as well as the extensive monograph [Pri03]. Our analysis builds on the results of [LŽ06] where a condition is derived under which both u and $\widehat{W}$ depend continuously on the specification of the market price of risk process for a given and fixed utility function.

The paper is organized as follows: the next section describes the financial market, the main theorem is stated and several examples are provided to illustrate both the scope and the limitations of this result whereas the last section contains all the proofs.

2 Formulation and results

2.1 The financial market

Our analysis is based on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ for a filtration $\mathbb{F} \triangleq (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the "usual conditions" of right-continuity and completeness where $T > 0$ denotes the investor's time horizon. To keep the notation as simple as possible, we only consider a market consisting of a numéraire security carrying a zero interest rate as well as one risky security with dynamics

$$dS_t \triangleq \lambda_t d\langle M\rangle_t + dM_t, \quad t \in (0,T], \quad S_0 \triangleq 1. \tag{2.1}$$

Here, M denotes a continuous local martingale with quadratic variation process $\langle M\rangle$. We refer to λ as the market price of risk process and we will assume throughout the paper that λ satisfies the following regularity condition:

Assumption 2.1. The market price of risk process λ of (2.1) is adapted to the filtration $\mathbb{F}$ and satisfies:

$$\int_0^T \lambda_u^2 d\langle M\rangle_u < \infty, \ \mathbb{P}\text{-almost surely.}$$

Given this assumption, we can follow [Sch95] and define the minimal martingale density Z by the Doléans-Dade exponential $\mathcal{E}(\cdot)$

$$Z_t \triangleq \mathcal{E}(-\lambda \cdot M)_t \triangleq \exp\left(-\int_0^t \lambda_u dM_u - \frac{1}{2}\int_0^t \lambda_u^2 d\langle M\rangle_u\right), \ t \in [0,T]. \tag{2.2}$$

The connection between the concept of no arbitrage - or the concept of no free lunch with vanishing risk - and the assumption that the dynamics of the financial securities are given by (2.1) has a long history within the field of mathematical finance and we only report that Assumption 2.1 is slightly stronger than the assumption of no arbitrage, see the discussions in [DS95a].

Example 2.2. A popular model is to let $\mathbb{F}$ be the standard filtration generated by two Brownian motions (B, W) on the interval $[0, T]$. The dynamics of the risky security is then taken to be

$$dS_t \triangleq S_t(\mu_t dt + \sigma_t dB_t), \quad t \in (0, T], \quad S_0 \triangleq 1, \tag{2.3}$$

where the adapted processes μ (drift) and $\sigma > 0$ (volatility) are allowed to depend on both B and W. In this case, we see that $M_t = (S\sigma \cdot B)_t$ and the market price of risk process is given by $\lambda = \mu/S\sigma^2$. However, given the Brownian setting (2.3), the market price of risk is often refereed to as the process $\tilde{\lambda} \triangleq \mu/\sigma$, see e.g., the textbook [Duf01]. We will refer to λ as the market price of risk in the general setting (2.1) and we refer to $\tilde{\lambda} \triangleq \mu/\sigma$ as the market price of risk in the Brownian setting (2.3). Given the dynamics (2.3), we see that Assumption 2.1 is satisfied if $\int_0^T \tilde{\lambda}_u^2 du < \infty$, $\mathbb{P}$-a.s., in which case $Z_t = \mathcal{E}(-\lambda \cdot M)_t = \mathcal{E}(-\tilde{\lambda} \cdot B)_t$ is a well-defined strictly positive super martingale for $t \in [0, T]$.

2.2 The investor's problem

The investor's risk preferences are modeled by a utility function U defined on the positive semi-axis. We will call a continuous differentiable strictly increasing, concave real-valued function U that satisfies the Inada conditions

$$\lim_{z \downarrow 0} U'(z) = +\infty, \quad \lim_{z \uparrow +\infty} U'(z) = 0,$$

as well as the reasonable asymptotic elasticity condition $AE[U] < 1$, where

$$AE[U] \triangleq \limsup_{z \uparrow +\infty} \frac{zU'(z)}{U(z)},$$

for a utility function and we write $U \in \mathcal{U}$. The investor's initial wealth is denoted by $x > 0$ and we will call a predictable process H a strategy if the stochastic integral $(H \cdot S)_t$ is well-defined for $t \in [0, T]$ in which case we write

$H \in L(S)$. Furthermore, if $x + (H \cdot S)$ is a non-negative process we call H x-admissible and we write $\mathcal{X}(x)$ for the set of replicable terminal wealths $W = x + (H \cdot S)_T$ for H x-admissible. The investor is assumed to maximize expected utility of terminal wealth over x-admissible strategies:

$$u(x;U) \triangleq \sup_{W \in \mathcal{X}(x)} \mathbb{E}[U(W)]. \tag{2.4}$$

Assumption 2.1 and the main result in [KS99] grant the following:

Theorem 2.3 (Kramkov-Schachermayer)**.** *For any $U \in \mathcal{U}$ with $u(x,U) < \infty$ there exists a unique element $\widehat{W} = \widehat{W}(x,U) \in \mathcal{X}(x)$ that optimizes* (2.4)*,*

$$u(x;U) = \mathbb{E}[U(\widehat{W})].$$

The main result in [KS99] is stated under the assumption that there exists an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $(H \cdot S)$ is a local $\mathbb{Q}$-martingale for any 1-admissible strategy H, however, as we prove[1] their result remains valid under our weaker Assumption 2.1. Proposition 1 of [LW00] provides an analogue result for the complete Brownian model (2.3). Alternatively, we can make the strictly stronger assumption that the minimal martingale measure exists, i.e., $\mathbb{E}[Z_T] = 1$ where Z is defined by (2.2). However, this assumption would exclude a model like the following.

Example 2.4. Let $\mathbb{F}$ be the standard filtration generated by a Brownian motion B on the interval $[0,T]$. We define the risky security by

$$dS_t \triangleq S_t(\tilde{\lambda}_t dt + dB_t), \quad t \in (0,T], \quad S_0 \triangleq 1.$$

The market price of risk process $\tilde{\lambda}$ is modeled by the 3-dimensional Bessel process, $\tilde{\lambda}_t \triangleq \frac{1}{R_t}$ where R solves the following stochastic differential equation

$$dR_t \triangleq dB_t + \frac{1}{R_t} dt, \quad t \in (0,T], \quad R_0 \triangleq 1,$$

see e.g., [KS88] p.158. This specification constitutes an example of a complete market with the property that the density $Z_t \triangleq \mathcal{E}(-\tilde{\lambda} \cdot B)_t$ is a strict local martingale, $\mathbb{E}[Z_T] < 1$. To see this, we apply Itô's Lemma to get that

[1] Thanks to Dmitry Kramkov for pointing out this relaxation.

$d\left(\frac{\tilde{\lambda}_t}{Z_t}\right) = 0$ for all $t \in [0,T]$, hence $Z = \tilde{\lambda}$. From e.g., [KS88], Exercise 3.36.i on p.168 we know that $\tilde{\lambda}$ is a strict local martingale and the claim follows.

Even though free lunches with vanishing risk exist, it can still be the case that $u(x, U) < \infty$ for an unbounded utility function U in this setting. One concrete exemplification is given by the *log*-investor, $U(x) \triangleq \log(x)$, leading to the Growth Optimal Portfolio (GOP). The value function (2.4) can be represented by

$$u(x, \log) = \log(x) + \frac{1}{2}\mathbb{E}\left[\int_0^T \tilde{\lambda}_u^2 du\right]. \tag{2.5}$$

This follows since a *log*-investor optimally invests a proportion of $\tilde{\lambda}$ in the risky security S. The finiteness of the integral present in (2.5) can be deduced from e.g., [KS88], Exercise 3.36.ii on p.168, hence we have $u(x, \log) < \infty$. For more information about the relevancy of the GOP in settings where the minimal density Z lacks the martingale property, we refer to the discussions in the recent textbook [PH06].

2.3 Problem formulation

Problem 2.5. Given a sequence of utility functions $\{U^n\}_{n\in\mathbb{N}_0} \subseteq \mathcal{U}$ with corresponding optimizers $\widehat{W}^n = \widehat{W}(x, U^n)$, $n = 0, 1, ...$, does pointwise convergence of $U^n \to U^0$ imply the following:

1. Convergence of the value functions, $u(x; U^n) \to u(x; U^0)$ for all x?

2. Convergence of the optimal terminal wealths, $\widehat{W}^n \to \widehat{W}^0$ in $\mathbb{L}^0_+$?

As usual $\mathbb{L}^0$ denotes the space of real-valued $\mathbb{F}$-measurable random variables equipped with the metrizable topology induced by convergence in probability and $\mathbb{L}^0_+$ denotes its positive cone.

2.4 Regularity condition

As exemplified in Section 2.7, pointwise convergence of U^n to U^0 alone is not sufficient to ensure that the corresponding optimizers converge, hence additional regularity needs to be imposed. Inspired by the condition on p.136 in [JN04], we introduce the following subset $\overline{\mathcal{U}}$ of $\mathcal{U}$:

Definition 2.6. For a fixed utility function $\overline{U} \in \mathcal{U}$ we will say that $U \in \mathcal{U}$ belongs to $\overline{\mathcal{U}}$ if $U(x) \leq \overline{U}(x)$ for all $x \geq 0$.

Example 2.7. Following [JN04], one choice of $\overline{U}$ is given by a power utility function or what is called CRRA-utility in the financial literature. Specifically, we define the function $\overline{U} = \overline{U}(\delta, k_1, k_2)$ by

$$\overline{U}(x) \triangleq k_1 + k_2 x^{\delta}, \quad x \geq 0,$$

for constants $k_1 \geq 0, k_2 > 0$ and $\delta \in (0,1)$. Often we will take $k_1 \triangleq 0$ and $k_2 \triangleq \frac{1}{\delta}$ in which case $(1-\delta)$ is refereed to as the coefficient of constant relative risk aversion (CRRA) for the investor with preferences $\overline{U}$.

2.5 Main result

In order to state our main result, we recall the definition of the Fenchel-Legendre transform - or simply the convex conjugate - of a utility function $U \in \mathcal{U}$:

$$V(y) \triangleq \sup_{z>0} U(z) - zy, \quad y > 0. \tag{2.6}$$

The following theorem constitutes our main result:

Theorem 2.8. *Let $\overline{U} \in \mathcal{U}$ be a utility function with conjugate $\overline{V}$ and let Z be given by* (2.2) *and assume that $\overline{V}(Z_T)$ is integrable, $\mathbb{E}[|\overline{V}(Z_T)|] < \infty$. Then for any $U \in \overline{\mathcal{U}}$ we have $u(x;U) < \infty$ and consequently the optimizer $\widehat{W} = \widehat{W}(x,U)$ exists. Furthermore, the following mappings*

$$(0,\infty) \times \overline{\mathcal{U}} \ni (x,U) \to u(x;U) \in \mathbb{R},$$
$$(0,\infty) \times \overline{\mathcal{U}} \ni (x,U) \to \widehat{W}(x,U) \in \mathbb{L}^0_+$$

are jointly continuous when $\overline{\mathcal{U}}$ is equipped with the topology of pointwise convergence and when $\mathbb{L}^0$ is equipped with the topology induced by convergence in probability.

Given the super martingale property of the minimal martingale density Z defined by (2.2) we have $\mathbb{E}[Z_T] \leq 1$ and since $\overline{V}$ is a convex function, we trivially see that the negative part of $\overline{V}(Z_T)$ is always integrable, hence for the conclusions of Theorem 2.8 to hold it suffices that positive is integrable, i.e., $\mathbb{E}[\overline{V}^+(Z_T)] < \infty$.

The following result is an incomplete continuous analogue of the setting in [JN04].

Corollary 2.9. *Assume that the market price of risk λ is a bounded process or in the Brownian setting* (2.3)*, assume that $\tilde{\lambda} \triangleq \mu/\sigma$ is a bounded process. Then for the specification $\overline{U}(x) \triangleq k_1 + k_2 x^\delta$ with $k_1 \geq 0, k_2 > 0$ and $\delta \in (0,1)$ the conclusions of Theorem 2.8 holds.*

This corollary illustrates the difference between Proposition 2.5 in [JN04] and this paper: by relaxing the financial setting of [JN04] to a general incomplete continuous setting we pay the cost of using the weaker notion of convergence in probability of the optimal terminal wealth $\widehat{W}$.

2.6 Applications

The section serves to illustrate how to apply our main theorem in various often applied specifications of the financial market. In all these examples, we restrict attention to the power utility case with parameter $\delta \in (0,1)$, meaning

$$\overline{U}^\delta(x) \triangleq \frac{1}{\delta} x^\delta, \quad \overline{V}^\delta(x) = \frac{1}{\delta'} x^{-\delta'}, \ \delta' \triangleq \frac{\delta}{1-\delta} \in (0,\infty). \tag{2.7}$$

Also, throughout the remaining part of this subsection, (B, W) are two possibly correlated Brownian motions generating the filtration $\mathbb{F}$.

We start by examining a class of models, each having a stochastic rate of return leading to an unbounded market price of risk process $\tilde{\lambda}$.

Example 2.10 (Kim-Omberg models)**.** The models used by [W02] and [MS04] are complete financial models with dynamics specified by

$$dS_t \triangleq X_t S_t dt + \sigma S_t dB_t, \quad t \in (0,T], \quad S_0 \triangleq 1, \tag{2.8}$$

$$dX_t \triangleq \kappa(\alpha - X_t)dt + \beta dB_t, \quad t \in (0,T], \quad \lambda_0 > 0, \tag{2.9}$$

for positive constants σ, β, κ and α.

The complete Schwartz mean-reversion model is given by

$$dS_t \triangleq \alpha\Big(\mu - \log(S_t)\Big) S_t dt + \sigma S_t dB_t, \quad t \in (0,T], \quad S_0 \triangleq 1, \tag{2.10}$$

for positive constants α, μ and σ. It follows from Itô's formula that $\log(S_t)$ is an Ornstein-Uhlenbeck process, see p.689-690 in [BK05].

The mean reversion model originally developed by [KO96] reads

$$dS_t \triangleq X_t S_t dt + \sigma S_t dB_t, \quad t \in (0,T], \quad S_0 \triangleq 1, \tag{2.11}$$

$$dX_t \triangleq \kappa(\alpha - X_t)dt + \beta dW_t, \quad t \in (0,T], \quad \lambda_0 > 0 \tag{2.12}$$

for positive constants σ, β, κ and α. Unlike the models (2.8)-(2.9) and (2.10), this specification involves the second Brownian motion W and therefore the model (2.11)-(2.12) is incomplete.

Lemma 2.11. *Let* $\overline{U}^\delta$ *be given by* (2.7) *and consider the financial market given by* (2.3) *and assume that the market price of risk process* $\tilde{\lambda}$ *is normally distributed:*

$$\tilde{\lambda}_t \sim \mathcal{N}\big(f(t), g(t)\big),\ t \in [0, T].$$

Here f *and* g *are non-negative continuous functions with* g *non-decreasing. This includes the model specifications* (2.8)-(2.9), (2.10) *and* (2.11)-(2.12). *If the following condition holds*

$$(\delta - 1)^2 > \delta(\delta + 1)2Tg(T), \tag{2.13}$$

then the conclusions of Theorem 2.8 are valid.

We next turn to stochastic volatility models and the first example is taken from [FHH03] and it illustrates an application of Corollary 2.9.

Example 2.12 (Fleming & Hernández-Hernández)**.** The stock price is given by

$$dS_t \triangleq S_t(\mu dt + \sigma(\omega_t)dB_t), \quad t \in (0, T], \quad S_0 \triangleq 1,$$

for a positive and constant drift μ. The function $\sigma : \mathbb{R} \to [\sigma_{min}, \sigma_{max}]$ transforms the state-process ω into volatility and ω is typically taken to driven by also the second Brownian motion W. If we assume $0 < \sigma_{min} < \sigma_{max} < \infty$, it immediately follows that $\tilde{\lambda}_t \triangleq \frac{\mu}{\sigma(\omega_t)}$ is a bounded process, irrespectively of the state process ω. The continuity conclusion now follows from Corollary 2.9.

The last example is inspired by the celebrated stochastic volatility model due to [Hes93]. We note that in this example the market price of risk process is unbounded and that the model is phrased in terms of both Brownian motions (B, W) and is therefore also incomplete.

Example 2.13 (Heston models)**.** Assume that there is a non-traded volatility asset

$$d\omega_t \triangleq \kappa(\theta - \omega_t)dt + \beta\sqrt{\omega_t}dW_t, \quad t \in (0, T], \quad \omega_0 > 0, \tag{2.14}$$

for positive constants κ, θ and β satisfying Feller's condition $2\kappa\theta > \beta^2$. In this setting, there are several ways to model the risky security and let us mention two of them: [Liu06] and [Kra05] assume

$$dS_t \triangleq S_t(\mu'\omega_t dt + \sqrt{\omega_t} dB_t), \quad t \in (0, T], \quad S_0 \triangleq 1, \tag{2.15}$$

for a positive and constant drift μ' whereas [CV05] purpose the model

$$dS_t \triangleq S_t \left(\mu'' dt + \frac{1}{\sqrt{\omega_t}} dB_t \right), \quad t \in (0, T], \quad S_0 \triangleq 1, \tag{2.16}$$

where also μ'' is a positive constant. Therefore, given our assumption of a zero risk-free interest rate, for both specifications (2.15) and (2.16) we are led to the same form of the market price of risk process $\tilde{\lambda}$.

Lemma 2.14. *Let $\overline{U}^\delta$ be given by* (2.7) *and consider the financial market given by* (2.3) *and assume that $\tilde{\lambda}_t \triangleq \mu\sqrt{\omega_t}$ where μ is a positive constant and ω solves* (2.14) *for a set of positive constants κ, θ and β satisfying Feller's condition. This includes both model specifications* (2.15) *and* (2.16)*. If the following condition holds*

$$\mu^2 \frac{\delta(\delta + 1)}{(\delta - 1)^2} \leq \frac{\kappa^2}{2\beta^2}, \tag{2.17}$$

then the conclusions of Theorem 2.8 are valid.

2.7 The need of a regularity condition

In this section we illustrate why pointwise convergence of U^n to U^0 alone is not sufficient to reach a positive continuity result. [KK04] illustrate that for the utility function $\overline{U}^\delta$ given by (2.7) the investor's problem might not be well-posed. We would like to stress that the following example has a well-defined finite valued value function $u(x; U^k)$ with corresponding well-defined optimizers $\widehat{W}^k$, $k = 0, 1, 2....$ However, $U^n \to U^0$ pointwise but the corresponding value functions do not converge correctly, i.e., $u(\cdot; U^n) \nrightarrow u(\cdot; U^0)$.

To keep the setting as familiar as possible, we consider Samuelsen's model (the Black-Scholes-Merton model) where the risky security is modeled by a geometric Brownian motion (to keep it simple, we take both the drift and volatility to be one):

$$dS_t \triangleq S_t(dt + dB_t), \quad t \in (0, T], \quad S_0 \triangleq 1,$$

where B is a Brownian motion generating the filtration $\mathbb{F}$. In this complete financial setting the market price of risk process is one, $\tilde{\lambda}_t \triangleq 1$ for $t \in [0, T]$, and therefore the state price density is given by

$$Z_t \triangleq \exp\big(- B_t - 0.5t\big), \quad t \in [0, T].$$

In particular, the terminal value Z_T is *log*-normally distributed, $\mathbb{P}(Z_T \in dz) = f(z)dz$ where f denotes the *log*-normal density function. To construct the sequence of utility functions $U^k(\cdot), k = 0, 1, 2...$ we define for $n \in \mathbb{N}$, we define the mapping $g_n : (0, \frac{1}{n}) \to (0, \infty)$ by

$$g_n(a) \triangleq \int_a^{\frac{1}{n}} \left(\frac{\sqrt{n} - \frac{1}{\sqrt{a}}}{\frac{1}{n} - a} \big(z - \frac{1}{n}\big) + \sqrt{n} - \frac{1}{\sqrt{x}} \right) f(z)dz. \tag{2.18}$$

We note that g_n is via the monotone convergency theorem continuous on $(0, \frac{1}{n})$, g_n is strictly decreasing and g_n satisfies $\lim_{a \to 0} g_n(a) = \infty$. Therefore, we can find a unique point $a_n \in (0, \frac{1}{n})$ such that $g_n(a_n) = 1$ which allows us to define the sequence of conjugate functions $\{V^n\}_{n \in \mathbb{N}}$ by

$$V^n(z) \triangleq \begin{cases} \frac{1}{\sqrt{z}} & \text{for } z \geq \frac{1}{n} \\ \frac{\sqrt{n} - \frac{1}{\sqrt{a_n}}}{\frac{1}{n} - a_n} \big(z - \frac{1}{n}\big) + \sqrt{n} & \text{for } a_n < z < \frac{1}{n} \\ \frac{1}{\sqrt{z}} & \text{for } z \leq a_n. \end{cases}$$

It immediately follows that $V^n(z) \to \frac{1}{\sqrt{z}} \triangleq V^0(z)$ for all $z > 0$. The sequence of utility functions $\{U^k\}_{k \in \mathbb{N}_0}$ is obtained by conjugating V^k, $k = 0, 1, 2...$ and it is straightforward to see how to make smoothness adjustments to the above constructed sequence $\{V^n\}_{n \in \mathbb{N}}$ in order for V^n to be continuous differentiable.

We denote by $v_k(y)$ the dual value function corresponding to the utility function U^k, i.e., for $k = 0, 1, ...$ we have $v_k(y) \triangleq \mathbb{E}[V^k(yZ_T)]$. with derivatives given by

$$v_k'(y) = -\frac{1}{y^2} \int_0^\infty V^k(w) f\left(\frac{w}{y}\right) dw - \frac{1}{y^3} \int_0^\infty V^k(w) f'\left(\frac{w}{y}\right) w dw, \quad y > 0.$$

We will focus on $y \triangleq 1$ and we consider n so large that $(0, \frac{1}{n}) \subseteq \{z > 0 : f'(z) \geq 0\}$. By the construction of a_n we get the following relation for n large enough

$$\begin{aligned}-v_n'(1) &= \int_0^\infty V^n(w) f(w) dw + \int_0^\infty V^n(w) f'(w) w dw \\ &\geq 1 + \int_0^\infty V^0(w) f(w) dw + \int_0^\infty V^0(w) f'(w) w dw = 1 - v_0'(1).\end{aligned}$$

To see that the primal functions do not converge correctly we ague by contradiction. So suppose that $u(x; U^n) \to u(x; U^0)$ for $x > 0$. The convexity property implies that the derivatives converge too, $u'(x; U^n) \triangleq \frac{\partial}{\partial x} u(x; U^n) \to u'(x; U^0) \triangleq \frac{\partial}{\partial x} u(x; U^0)$ for $x > 0$. The primal value function and the dual value function are mutual conjugates, hence the inverse of $u'(\cdot, U^k)$ is given by $-v_k'(\cdot)$ for $k \in \mathbb{N}_0$. Since $u'(\cdot; U^n)$ is decreasing, we have

$$1 = u'(-v_n'(1); U^n) \leq u'(1 - v_0'(1); U^n) \to u'(1 - v_0'(1); U^0).$$

Applying the decreasing function $-v_0'(\cdot)$ gives us the needed contradiction.

3 Proofs

We start by recalling some notation from [KS99]: $\mathcal{Y}$ is defined as the set of non-negative adapted processes $(Y_t)_{t \in [0,T]}$, $Y_0 = 1$, with the property that $Y(1 + H \cdot S)$ is a super martingale for any 1-admissible strategy H.

Proof of Theorem 2.3: [KS99] assume $\mathcal{M}^e(S)$ - the set of pricing measures - is non-empty and by making appropriate adjustments to their proof on p.925-927 this assumption can be replaced with our weaker Assumption 2.1. An examination of their proof reveals that only the bipolar relation (ii) of Proposition 3.1 in [KS99] needs attention.

Instead of considering the financial market $(1, S)$ we define the market $\tilde{S} \triangleq (\tilde{S}^0, \tilde{S}^1) \triangleq (Z, ZS)$ to obtain $\mathbb{P} \in \mathcal{M}^e(\tilde{S})$. Let $\tilde{\mathbb{P}} \in \mathcal{M}^e(\tilde{S})$ be arbitrary and let $H = (H^0, H^1)$ be an 1-admissible self-financing portfolio in the $(1, S)$-market. From Proposition 2.1 in [GK00] we have the invariance property

$$\begin{aligned}Z_t\big(1 + \int_0^t H_u^1 dS_u\big) &= Z_t(H_t^0 + H_t^1 S_t) \\ &= H_t^0 \tilde{S}_t^0 + H_t^1 \tilde{S}_t^1 = 1 + \int_0^t H_u^0 d\tilde{S}_u^0 + \int_0^t H_u^1 d\tilde{S}_u^1,\end{aligned}$$

and since $\tilde{\mathbb{P}} \in \mathcal{M}^e(\tilde{S})$ it follows that $Z\big(1 + (H^1 \cdot S)\big)$ is a local $\tilde{\mathbb{P}}$-martingale, hence $Z\tilde{Z} \in \mathcal{Y}$ where $\tilde{Z}$ denotes $\tilde{\mathbb{P}}$'s density process with respect to $\mathbb{P}$.

Let $g \geq 0$ satisfy $\mathbb{E}[Y_T g] \leq 1$ for all $Y \in \mathcal{Y}$ and we want to construct a 1-admissible portfolio H such that $g \leq 1 + (H \cdot S)_T$. By the above we have for any $\tilde{\mathbb{P}} \in \mathcal{M}^e(\tilde{S})$

$$\mathbb{E}^{\tilde{\mathbb{P}}}[Z_T g] = \mathbb{E}[\tilde{Z}_T Z_T g] \leq 1.$$

Standard arguments, based on e.g., Corollary 10 of [DS95b], grant a 1-admissible portfolio $H = (H^0, H^1)$ such that $1+(H^0 \cdot \tilde{S}^0)_T + (H^1 \cdot \tilde{S}^1)_T \geq gZ_T$. Using Proposition 2.1 in [GK00] we obtain as before the relation

$$1 + \int_0^t H_u^1 dS_u = \frac{1 + \int_0^t H_u^0 d\tilde{S}_u^0 + \int_0^t H_u^1 d\tilde{S}_u^1}{Z_t},$$

from which it follows that $1 + (H^1 \cdot S)_T \geq g$. By defining the sets

$$\mathcal{C} \triangleq \{g \in \mathbb{L}_+^0 : g \leq W \text{ for some } W \in \mathcal{X}(1)\} \text{ and}$$
$$\mathcal{D} \triangleq \{h \in \mathbb{L}_+^0 : h \leq Y_T \text{ for some } Y \in \mathcal{Y}\},$$

see [KS99] p.912, we can interpret the above as the inclusion

$$\mathcal{C} \supseteq \{g \in \mathbb{L}_0^+ : \mathbb{E}[gY_T] \leq 1 \text{ for all } Y \in \mathcal{Y}\} = \mathcal{D}^0. \tag{3.1}$$

Via the Bipolar Theorem [KS99] proved that $\mathcal{D} = \mathcal{D}^{00}$ and by the definition of $\mathcal{Y}$, we trivially have the inclusion $\mathcal{D} \subseteq \mathcal{C}^0$ and in turn we get $\mathcal{C}^0 = \mathcal{D}$. Taking another polar and using the inclusion (3.1) we obtain

$$\mathcal{C}^{00} = \mathcal{D}^0 \subseteq \mathcal{C},$$

hence, $\mathcal{D}^0 = \mathcal{C}$ which concludes the proof.

□

We start by stating a result which we use repeatedly (the proof mimics that of Proposition 3.9 in [LŽ06]).

Proposition 3.1. *Let $\{U^k\}_{k=0,1,2...} \subseteq \mathcal{U}$ and denote by V^n the conjugate of U^k, $k = 0, 1, 2...$, defined by (2.6). If $V^n(y) \to V^0(y)$ for $y \in (0, \infty)$, then $U^n(x^n) \to U^0(x^0)$ for any sequence $\{x^n\}_{n \in \mathbb{N}} \subseteq (0, \infty)$ converging to $x^0 \in (0, \infty)$. The same conclusion holds true if the roles of V^k and U^k are reversed.*

For the rest of this section V will always denote the conjugate of U for a generic utility function $U \in \overline{\mathcal{U}}$ and $\overline{V}$ always denotes the conjugate of $\overline{U}$. From (2.6) we have the inequality $V(\cdot) \leq \overline{V}(\cdot)$ which will be a key ingredient. We define the dual value function by

$$v(y;U) \triangleq \inf_{Y \in \mathcal{Y}} \mathbb{E}[V(yY_T)], \quad y > 0, \quad U \in \overline{\mathcal{U}}.$$

Since V is convex and bounded above by $\overline{V}$, the assumption $\mathbb{E}[\overline{V}(Z_T)] < \infty$ combined with Proposition 6.3(iii) of [KS99] ensure that v is real valued.

Proof of Theorem 2.8 (Existence of $\widehat{W}$): by (2.6) we have for $W \in \mathcal{X}(x)$ the following estimate where Z is given by (2.2)

$$U(W) \leq \overline{U}(W) \leq \overline{V}(Z_T) + WZ_T,$$

$\mathbb{P}$-almost surely. Taking expectations and using that $Z \in \mathcal{Y}$ gives us $\mathbb{E}[U(W)] \leq \mathbb{E}[\overline{V}(Z_T)] + x < \infty$. The existence now follows from Theorem 2.3.

□

We next turn to proof of the continuity of the primal value function given by (2.4) which by means of Proposition 3.1 is equivalent to proving continuity of the dual value function v. This will be done in two steps. We also recall that given the convexity of members of $\overline{\mathcal{U}}$, pointwise convergence is equivalent to uniform convergence on compact subsets of the positive semi-axis. The topology induced by uniform convergence on compact subsets is metrizable and therefore sequences suffice for proving continuity relations.

Proof of Theorem 2.8 (Upper semi-continuity of v, step 1): Let $y^n \to y^0$ and $U^n \to U^0$ pointwise. Proposition 3.1 says that $V^n(\zeta^n) \to V^0(\zeta^0)$ for any sequence $\zeta^n \to \zeta^0$, a property needed below.

To proceed, we first find two constants $0 < y_{min} \leq y_{max} < \infty$ such that $y^n \in [y_{min}, y_{max}]$ for all n and then we denote by $\mathcal{B}$ the set of orthogonal martingales L, $\langle L, M\rangle \equiv 0$, for which there exists a constant $\epsilon > 0$ such that $\mathcal{E}(L)_T \geq \epsilon$ uniformly in $\omega \in \Omega$. Therefore, for any $L \in \mathcal{B}$ we have $\mathbb{P}$-almost surely the upper bound

$$V^n\Big(y^n Z_T \mathcal{E}(L)_T\Big) \leq \overline{V}(y_{min} Z_T \epsilon) \leq C\overline{V}^+(Z_T) + D, \tag{3.2}$$

where the constants C and D are granted by the reasonable asymptotic elasticity, see Proposition 6.3(iii) of [KS99], and by assumption the right-hand-side is integrable. Since $V^n \to V^0$ uniformly on compact sets, we can find a universal affine minorant ϕ such that $V^n(\cdot) \geq \phi(\cdot)$ for all n, in particular we have

$$V^n\Big(y^n Z_T \mathcal{E}(L)_T\Big) \geq \phi\Big(y_{max} Z_T \mathcal{E}(L)_T\Big), \tag{3.3}$$

$\mathbb{P}$-almost surely. Since $\mathbb{E}[Z_T\mathcal{E}(L)_T] \leq 1$, the right-hand-side is also integrable and therefore by dominated convergence we have for $L \in \mathcal{B}$

$$\mathbb{E}\left[V^n\Big(y^n Z_T \mathcal{E}(L)_T\Big)\right] \to \mathbb{E}\left[V^0\Big(y^0 Z_T \mathcal{E}(L)_T\Big)\right].$$

Corollary 3.4 in [LŽ06] gives the representation

$$v(y;U) = \inf_{L\in\mathcal{B}} \mathbb{E}\Big[V\big(yZ_T\mathcal{E}(L)_T\big)\Big], \quad y > 0,$$

and the upper semi-continuity follows.

□

Before continuing, we recall an equivalent description of the dual value function. Following [Zit05] we introduce the operator $\mathbb{V}$ given by

$$\mathbb{V}(\mathbb{Q};U) \triangleq \sup_{X\in\mathbb{L}^\infty_+} \mathbb{E}[U(X)] - \langle \mathbb{Q}, X\rangle, \quad \mathbb{Q} \in (\mathbb{L}^\infty)^*, \quad U \in \overline{\mathcal{U}},$$

where $(\mathbb{L}^\infty)^*$ - the dual of $\mathbb{L}^\infty$ - is equipped with the weak*-topology.

Lemma 3.2. *When $\overline{\mathcal{U}}$ is equipped with the topology of pointwise convergence and $(\mathbb{L}^\infty)^*$ with the weak*-topology, the functional*

$$(\mathbb{L}^\infty)^* \times \overline{\mathcal{U}} \ni (\mathbb{Q}, U) \to \mathbb{V}(\mathbb{Q}; U) \in (-\infty, \infty]$$

is lower semi-continuous.

PROOF: We start by verifying the relationship

$$\mathbb{V}(\mathbb{Q};U) = \sup_{X\in\mathbb{L}^\infty_{++}} \mathbb{E}[U(X)] - \langle \mathbb{Q}, X\rangle \tag{3.4}$$

where $X \in \mathbb{L}^\infty_{++}$ if we can find a positive constant M such that $X(\omega) \in [\frac{1}{M}, M]$ for all $\omega \in \Omega$. "$\geq$" is obvious and for the other inequality we pick $X \in \mathbb{L}^\infty_+$ and $n \in \mathbb{N}$ and define

$$X^n \triangleq \left(1 - \frac{1}{n}\right) X + \frac{1}{n}$$

which is in $\mathbb{L}^\infty_{++}$. The concavity of U yields the relation

$$\mathbb{E}[U(X^n)] - \langle \mathbb{Q}, X^n \rangle \geq \left(\frac{n-1}{n}\right) \mathbb{E}[U(X)] + \frac{1}{n} U(1) - \left(\frac{n-1}{n}\right) \langle \mathbb{Q}, X \rangle - \frac{1}{n}$$

and passing n to infinity yields the desired inequality.

Given the representation (3.4), the proof can be concluded by showing that for any $X \in \mathbb{L}^\infty_{++}$ the following two mappings are continuous

$$\overline{\mathcal{U}} \ni U \to \mathbb{E}[U(X)], \quad (\mathbb{L}^\infty)^* \ni \mathbb{Q} \to \langle \mathbb{Q}, X \rangle.$$

The first mapping is continuous by means of the dominated convergence theorem: we can find $M > 0$ such that $X \in [\frac{1}{M}, M]$. Since $U^n \to U^0$ pointwise we see $U^n(\frac{1}{M})$ and $U^n(M)$ are convergent and we can therefore find a universal constant $K > 0$ such that $-K \leq U^n(X) \leq K$ for all n, hence $\mathbb{E}[U^n(X)] \to \mathbb{E}[U(X)]$.

The latter mapping is also continuous: by the very definition of the weak*-topology we have $\mathbb{Q}^\alpha \to \mathbb{Q}^0$ if $\langle \mathbb{Q}^\alpha, X \rangle \to \langle \mathbb{Q}^0, X \rangle$, $X \in \mathbb{L}^\infty_+$, where $\{\mathbb{Q}^\alpha\}_{\alpha \in A}$ is a net in $(\mathbb{L}^\infty)^*$ (A is some directed set).

All in all, we have showed that $\mathbb{V}$ is given as a supremum of continuous functions and hence is itself lower semi-continuous.

□

We define the none-empty weak*-compact set

$$\mathcal{D} \triangleq \{\mathbb{Q} \in (\mathbb{L}^\infty)^*_+ : \langle \mathbb{Q}, 1 \rangle \leq 1 \text{ and } \langle \mathbb{Q}, W \rangle \leq x \; \forall W \in (\mathcal{X}(x) - \mathbb{L}^0_+) \cap \mathbb{L}^\infty\}.$$

From [Zit05] we have the relation

$$v(y; U) = \inf_{\mathbb{Q} \in \mathcal{D}} \mathbb{V}(y\mathbb{Q}; U), \quad y > 0, \quad U \in \overline{\mathcal{U}}. \tag{3.5}$$

Furthermore, [Zit05] shows by means of the Banach-Alaoglu theorem that we can find $\widehat{\mathbb{Q}} = \widehat{\mathbb{Q}}(y; U) \in \mathcal{D}$ attaining the infimum of (3.5) for $y > 0$ and $U \in \overline{\mathcal{U}}$.

PROOF OF THEOREM 2.8 (Continuity of v, step 2): Let $y^n \to y^0$ in $(0, \infty)$ and let $U^n \to U^0$ pointwise in $\overline{\mathcal{U}}$. For each $n \in \mathbb{N}$, we let $\widehat{\mathbb{Q}}^n \in \mathcal{D}$ be a minimizer. Given the weak*-compactness of $\mathcal{D}$, we can extract a weak*-convergent subnet, $\{\widehat{\mathbb{Q}}^\alpha\}_{\alpha \in A}$ for a directed set A, of these optimizers

$$\mathcal{D} \ni \widehat{\mathbb{Q}}^0 = \lim_\alpha \widehat{\mathbb{Q}}^\alpha,$$

where the limit is weak*. We then have

$$\begin{aligned}
v(y^0; U^0) &\geq \limsup_n v(y^n; U^n) \\
&\geq \limsup_\alpha v(y^\alpha; U^\alpha) \\
&= \limsup_\alpha \mathbb{V}(y^\alpha \widehat{\mathbb{Q}}^\alpha; U^\alpha) \\
&\geq \liminf_\alpha \mathbb{V}(y^\alpha \widehat{\mathbb{Q}}^\alpha; U^\alpha) \\
&\geq \mathbb{V}(y^0 \widehat{\mathbb{Q}}^0; U^0) \geq v(y^0; U^0),
\end{aligned}$$

where the first inequality follows from the already proven upper semi-continuity, the second inequality follows by the subnet property, the forth inequality follows from the previous lemma and the last inequality follows since $\widehat{\mathbb{Q}}^0 \in \mathcal{D}$. This shows that

$$v(y^0; U^0) = \limsup_n v(y^n; U^n) \tag{3.6}$$

for any sequence $\{(y^n, U^n)\}_{n \in \mathbb{N}}$ converging to (y^0, U^0). To see that this implies $v(y^0; U^0) = \liminf_n v(y^n; U^n)$ we argue by contradiction: suppose that $v(y^0; U^0) > \liminf_n v(y^n; U^n)$. We can then find a subsequence $\{(y^{n_k}, U^{n_k})\}_{k \in \mathbb{N}}$ such that $\lim_k v(y^{n_k}; U^{n_k}) = \liminf_n v(y^n; U^n) < v(y^0; U^0)$ which contradicts (3.6) since $(y^{n_k}, U^{n_k}) \to (y^0, U^0)$.

□

Finally, we can conclude the proof of our main theorem by proving the continuity of the optimal terminal wealth.

PROOF OF THEOREM 2.8 (Continuity of $\widehat{W}$): Given the above proved continuity of v, we get via convexity that also the derivatives of $u(x; U)$ converge implying that the Lagrange multipliers converge, $y^n \to y^0$. Therefore, by the relation

$$(U^n)'(\widehat{W}^n) = y^n \widehat{Y}^n, \quad \widehat{Y}^n \triangleq Z_T \mathcal{E}(\widehat{L}^n)_T,$$

the fact that $(V^n)'$ converges uniformly on compact sets and since the inverse of $(U^n)'(\cdot)$ is given by $-(V^n)'(\cdot)$, it suffices to prove that $\widehat{Y}^n \to \widehat{Y}^0$ in probability. The analysis on p.23 of [LŽ06] gives for any $\epsilon > 0$, $\varphi \in (0,1)$ and $N > \epsilon$ the following estimate

$$\mathbb{P}(y^n|\widehat{Y}^n - \widehat{Y}^0| > \epsilon) \leq \frac{2y^n\varphi}{\epsilon} + \frac{2y^n}{N} + \frac{1}{2\beta^n}\Big(\mathbb{E}[V^n(y^n Z_T H^\varphi)] - v(y^n; U^n)\Big) \tag{3.7}$$

where $H^\varphi \triangleq (1-\varphi)\mathcal{E}(\widehat{L}^0)_T + \varphi \geq \varphi > 0$ and the constant β^n is given by the convexity of V^n on the interval $[\epsilon, N]$. Specifically, we can take

$$\beta^n \triangleq \frac{V^n(N-\epsilon) + V^n(N)}{2} - V^n\left(N - \frac{\epsilon}{2}\right) > 0$$

and we see that $\beta^n \to \beta^0 > 0$. Since H^φ is bounded away from zero we can obtain estimates similar to (3.2) and (3.3) which allow limit operation and expectation to be interchanged and we get

$$\Big(\mathbb{E}[V^n(y^n Z_T H^\varphi)] - v(y^n; U^n)\Big) \to \mathbb{E}[V^0(y^0 Z_T H^\varphi)] - v(y^0; U^0),$$

using the continuity of v. The convexity of V^0 gives us the estimate

$$\mathbb{E}[V^0(y^0 Z_T H^\varphi)] - v(y^0; U^0) \leq \varphi\Big(\mathbb{E}[V^0(y^0 Z_T)] - \mathbb{E}[V^0(y^0\widehat{Y}^0)]\Big).$$

Applying the lim sup-operator on both sides of (3.7), passing φ to zero and finally passing N to $+\infty$ show that $y^n(\widehat{Y}^n - \widehat{Y}^0) \to 0$ in probability, however, as already used $y^n \to y^0$ and the result follows.

□

The remaining proofs are based on the same inequality that provides an upper bound for $\mathbb{E}[\overline{V}(Z_T)]$. Recall for any continuous local martingale X' the stochastic exponential $\mathcal{E}(X')$ is a super martingale and hence $\mathbb{E}[\mathcal{E}(X')_T] \leq 1$. In particular, for any continuous local martingale X we have

$$\mathbb{E}[\exp(-X_T - \langle X\rangle_T)^2] = \mathbb{E}[\exp(-2X_T - 0.5\langle 2X\rangle_T)] = \mathbb{E}[\mathcal{E}(-2X)_T] \leq 1.$$

We will apply this with the $X_t \triangleq -\alpha \int_0^t \lambda_u dM_u$ for some model of the market price of risk process λ and α being a constant. X's quadratic variation reads $\langle X \rangle_t = \alpha^2 \int_0^t \lambda_u^2 d\langle M \rangle_u$. This combined with Hölder's inequality allow us to compute the following upper bound

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\int_0^T \lambda_t dM_t - \frac{1}{2}\int_0^T \lambda_t^2 d\langle M \rangle_t\right)^{\alpha}\right] \\
&= \mathbb{E}\left[\exp\Big(-X_T - \langle X \rangle_T\Big)\exp\left(\left(\alpha^2 - \frac{1}{2}\alpha\right)\int_0^T \lambda_t^2 d\langle M \rangle_t\right)\right] \\
&\le \mathbb{E}\left[\exp\Big(-X_T - \langle X \rangle_T\Big)^2\right]^{\frac{1}{2}} \mathbb{E}\left[\exp\left(\alpha\left(\alpha - \frac{1}{2}\right)\int_0^T \lambda_t^2 d\langle M \rangle_t\right)^2\right]^{\frac{1}{2}} \\
&\le \mathbb{E}\left[\exp\left(\alpha\,(2\alpha - 1)\int_0^T \lambda_t^2 d\langle M \rangle_t\right)\right]^{\frac{1}{2}}
\end{aligned}$$

For Z_T defined by (2.2), $\overline{V}^{\delta}$ and δ' given by (2.7) we therefore obtain the following estimate

$$\delta' \mathbb{E}\left[\overline{V}^{\delta}(Z_T)\right] \le \mathbb{E}\left[\exp\left(\frac{\delta(\delta+1)}{(\delta-1)^2}\int_0^T \lambda_t^2 d\langle M \rangle_t\right)\right]^{\frac{1}{2}}. \tag{3.8}$$

Proof of Corollary 2.9: Let $\overline{V}$ denote the conjugate of $k_1 + k_2 x^{\delta}$ with k_1, k_2 positive constants and $\delta \in (0,1)$. A computation shows that for $y > 0$ we have $\overline{V}(y) = k_1 + \overline{V}^{\delta}(y/k_2)$ where $\overline{V}^{\delta}$ is given by (2.7). Therefore for λ bounded, (3.8) shows $\mathbb{E}[\overline{V}(Z_T)] < \infty$ and the result follows. In the Brownian setting (2.3) with $\tilde{\lambda} \triangleq \mu/\sigma$ being a bounded process, we have

$$\int_0^T \lambda_t^2 d\langle M \rangle_t = \int_0^T \tilde{\lambda}_t^2 dt$$

is uniformly bounded in $\omega \in \Omega$ and the result follows as before.

□

The following proof is a generalization of Lemma 4.3 in [BK05].

PROOF OF LEMMA 2.11: To ease the notation we define the constant $K \triangleq \frac{\delta(\delta+1)}{(\delta-1)^2}$. The estimate (3.8) and Jensen's inequality give us

$$\delta' \mathbb{E}\left[\overline{V}^{\delta}(Z_T)\right] \leq \mathbb{E}\left[\exp\left(K\int_0^T \tilde{\lambda}_u^2 du\right)\right] \leq \frac{1}{T}\int_0^T \mathbb{E}\left[\exp\left(TK\tilde{\lambda}_u^2\right)\right] du,$$

where $\overline{V}^{\delta}$ is given by (2.7). For a normally distributed random variable $X \sim \mathcal{N}(m, v)$ we have for any constant ξ satisfying $1 > 2\xi v$ the moment generating function

$$\mathbb{E}\left[\exp\left(\xi X^2\right)\right] = \frac{\exp\left(\frac{\xi m^2}{1-2\xi v}\right)}{\sqrt{1-2\xi v}}.$$

The regularity condition (2.13) allows us to apply this computation with $\xi \triangleq TK$ and we see $u \to \mathbb{E}\left[\exp\left(TK\tilde{\lambda}_u^2\right)\right]$ is a continuous function on the interval $[0, T]$ and the result follows.

□

PROOF OF LEMMA 2.14: Condition (2.17) ensures that $\mu^2 K \leq \frac{\kappa^2}{2\beta^2}$, $K \triangleq \frac{\delta(\delta+1)}{(\delta-1)^2}$. Therefore Proposition 5.1 in [Kra05] together with (3.8) give

$$\delta' \mathbb{E}[\overline{V}^{\delta}(Z_T)] \leq \mathbb{E}\left[\exp\left(K\int_0^T \tilde{\lambda}_u^2 du\right)\right] = \mathbb{E}\left[\exp\left(\mu^2 K\int_0^T \omega_u du\right)\right] < \infty,$$

where ω is the solution of (2.14) and $\overline{V}^{\delta}$ is given by (2.7).

□

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