The Structure of General Mean-Variance Hedging Strategies

Jan Kallsen
TU München

(joint work with **Aleš Černý**, London)

Pittsburgh, February 27, 2006

Quadratic hedging

S (discounted) asset price process
H (discounted) contingent claim

How to hedge the risk from selling the claim?

Hedging error: $v + \varphi \bullet S_T - H$

v (discounted) initial endowment
φ dynamic trading strategy

Quadratic hedging: $\min_{v,\varphi} E\left((v + \varphi \bullet S_T - H)^2\right)$

$v^\star$ variance-optimal initial endowment
$\varphi^\star$ variance-optimal hedging strategy

Quadratic hedging

S (discounted) asset price process
H (discounted) contingent claim

How to hedge the risk from selling the claim?

Hedging error: $v + \varphi \bullet S_T - H$

v (discounted) initial endowment
φ dynamic trading strategy

Quadratic hedging: $\min_{v,\varphi} E\left((v + \varphi \bullet S_T - H)^2\right)$

$v^\star$ variance-optimal initial endowment
$\varphi^\star$ variance-optimal hedging strategy

Quadratic hedging viewed differently

- functional analytic point of view:

  L^2-projection of H on $\{v + \varphi \bullet S_T : v \in \mathbb{R}, \varphi \text{ admissible}\}$

  Is $\{v + \varphi \bullet S_T : v \in \mathbb{R}, \varphi \text{ admissible}\}$ closed?

  (cf. Monat & Stricker 1995, Delbaen et al. 1997, Choulli et al. 1998, Delbaen & Schachermayer 1996)

- or compare to linear regression (= one-period model):

  $\min_{v,\varphi} E\left((v + \varphi S - H)^2\right)$ for random variables H, S

  solution: $\varphi = \dfrac{\mathrm{Cov}(H,S)}{\mathrm{Var}(S)}$

Quadratic hedging viewed differently

- functional analytic point of view:

  L^2-projection of H on $\{v + \varphi \bullet S_T : v \in \mathbb{R}, \varphi \text{ admissible}\}$

  Is $\{v + \varphi \bullet S_T : v \in \mathbb{R}, \varphi \text{ admissible}\}$ closed?

  (cf. Monat & Stricker 1995, Delbaen et al. 1997, Choulli et al. 1998, Delbaen & Schachermayer 1996)

- or compare to linear regression (= one-period model):

  $\min_{v,\varphi} E\left((v + \varphi S - H)^2\right)$ for random variables H, S

  solution: $\varphi = \dfrac{\mathrm{Cov}(H,S)}{\mathrm{Var}(S)}$

Variance-optimal hedging in general

Case 1: S martingale (Föllmer & Sondermann 1986)

$\longrightarrow$ use *Galtchouk-Kunita-Watanabe decomposition*

Case 2: deterministic *mean-variance tradeoff process* of S (Schweizer 1994)

$\longrightarrow$ use *Föllmer-Schweizer decomposition*

Case 3: arbitrary S

(e.g. Schweizer 1996, Rheinländer & Schweizer 1997, Gourieroux et al. 1998, ..., Arai 2005)

Case 1: S martingale

Galtchouk-Kunita-Watanabe decomposition:

$$H = V_0 + \xi \bullet S_T + R_T,$$

where R martingale, orthogonal to S (i.e. RS martingale)

Mean value process of the option: $V_t := E(H|\mathscr{F}_t)$

Variance-optimal hedge: $v^\star = V_0, \quad \varphi_t^\star = \xi_t = \frac{d\langle V, S\rangle_t}{d\langle S, S\rangle_t}$

Hedging error:

$$E\Big(\big(v^\star + \varphi^\star \bullet S_T - H\big)^2\Big) = E\Big(\big\langle V - \varphi^\star \bullet S, V - \varphi^\star \bullet S\big\rangle_T\Big)$$

Case 2: deterministic *mean-variance tradeoff process* of S

Mean-variance tradeoff process: $\widehat{K}_t = \widehat{\lambda} \bullet A^S$, where $\widehat{\lambda}_t = \frac{dA_t^S}{d\langle M^S, M^S \rangle_t}$
and $S = S_0 + M^S + A^S$ Doob-Meyer decomposition of S

Föllmer-Schweizer decomposition: $H = V_0 + \xi \bullet S_T + R_T$,
where R martingale, orthogonal to the martingale part M^S of S

Mean value process of the option: $V_t := E_Q(H|\mathscr{F}_t)$,
where Q *minimal (signed) martingale measure* with density process $\mathscr{E}(-\widehat{\lambda} \bullet M^S)$

Variance-optimal hedge: $v^\star = V_0$, $\xi_t = \frac{d\langle V, S \rangle_t}{d\langle S, S \rangle_t}$,

$\varphi_t^\star = \xi_t + \tilde{\lambda}(V_{t-} - v^\star - \varphi^\star \bullet S_{t-})$, where $\tilde{\lambda}_t = \frac{dA_t^S}{d\langle S, S \rangle_t}$

Hedging error:

$$E\Big(\big(v^\star + \varphi^\star \bullet S_T - H\big)^2\Big) = E\Big(\mathscr{E}(\widehat{K}) \bullet \big\langle V - \xi \bullet S, V - \xi \bullet S \big\rangle_T\Big) \frac{1}{\mathscr{E}(\widehat{K})_T}$$

Case 3: arbitrary S

- (Schweizer 1996)
  Variance-optimal hedge: $v^\star := E_{Q^\star}(H)$, $\varphi_t^\star = \varrho_t - \tilde{a}(v^\star + \varphi^\star \bullet S_{t-})$,
  where $Q^\star$ *variance-optimal (signed) martingale measure (VOMM)* with density
  $$\frac{dQ^\star}{dP} = \frac{\mathscr{E}(-\tilde{a} \bullet S)_T}{E(\mathscr{E}(-\tilde{a} \bullet S)_T)}$$
  backward stochastic differential equations for *adjustment process* $\tilde{a}$ and ϱ

- (Rheinländer & Schweizer 1997) for continuous S
  Mean value process of the option: $V_t := E_{Q^\star}(H|\mathscr{F}_t)$
  where $Q^\star$ *variance-optimal martingale measure (VOMM)*

  Variance-optimal hedge: $v^\star = V_0$, $\xi_t = \frac{d\langle V, S\rangle_t^{Q^\star}}{d\langle S, S\rangle_t^{Q^\star}}$,

  $\varphi_t^\star = \xi_t + \tilde{a}(V_t - v^\star - \varphi^\star \bullet S_t)$

  How to obtain the adjustment process $\tilde{a}$?

Case 3: arbitrary S

- (Schweizer 1996)
  Variance-optimal hedge: $v^\star := E_{Q^\star}(H)$, $\varphi^\star_t = \varrho_t - \tilde{a}(v^\star + \varphi^\star \bullet S_{t-})$,
  where $Q^\star$ *variance-optimal (signed) martingale measure (VOMM)* with density
  $$\frac{dQ^\star}{dP} = \frac{\mathscr{E}(-\tilde{a} \bullet S)_T}{E(\mathscr{E}(-\tilde{a} \bullet S)_T)}$$
  backward stochastic differential equations for *adjustment process* $\tilde{a}$ and ϱ

- (Rheinländer & Schweizer 1997) for continuous S
  Mean value process of the option: $V_t := E_{Q^\star}(H|\mathscr{F}_t)$,
  where $Q^\star$ *variance-optimal martingale measure (VOMM)*

  Variance-optimal hedge: $v^\star = V_0$, $\xi_t = \frac{d\langle V, S\rangle_t^{Q^\star}}{d\langle S, S\rangle_t^{Q^\star}}$,
  $$\varphi^\star_t = \xi_t + \tilde{a}(V_t - v^\star - \varphi^\star \bullet S_t)$$
  How to obtain the adjustment process $\tilde{a}$?

Case 3: arbitrary S (Černý & K 2005)

Key idea: change of measure $P \to P^\star$ (determined by characteristic equation)

Föllmer-Schweizer decomposition relative to $P^\star$: $H = V_0 + \xi \bullet S_T + R_T$,
where R $P^\star$-martingale, orthogonal to the $P^\star$-martingale part of S

Mean value process of the option: $V_t := E_{Q^\star}(H|\mathscr{F}_t)$,
where $Q^\star$ *variance-optimal (signed) martingale measure (VOMM)*
= minimal martingale measure relative to $P^\star$

Variance-optimal hedge: $\quad v^\star = V_0, \qquad \xi_t = \dfrac{d\langle V, S\rangle_t^{P^\star}}{d\langle S, S\rangle_t^{P^\star}},$

$\varphi_t^\star = \xi_t + \tilde{a}(V_{t-} - v^\star - \varphi^\star \bullet S_{t-})$, where $\tilde{a}_t = \dfrac{dA_t^{S\star}}{d\langle S, S\rangle_t^{P^\star}}$ *adjustment process*
and $S = S_0 + M^{S\star} + A^{S\star}$ Doob-Meyer decomposition of S relative to $P^\star$

Hedging error:

$$E\Big(\big(v^\star + \varphi^\star \bullet S_T - H\big)^2\Big) = E\Big(L \bullet \big\langle V - \xi \bullet S, V - \xi \bullet S\big\rangle_T^{P^\star}\Big)$$

Case 3: arbitrary S (Černý & K 2005)

Key idea: change of measure $P \to P^\star$ (determined by characteristic equation)

Föllmer-Schweizer decomposition relative to $P^\star$: $H = V_0 + \xi \bullet S_T + R_T$,
where R $P^\star$-martingale, orthogonal to the $P^\star$-martingale part of S

Mean value process of the option: $V_t := E_{Q^\star}(H|\mathscr{F}_t)$,
where $Q^\star$ *variance-optimal (signed) martingale measure (VOMM)*
= minimal martingale measure relative to $P^\star$

Variance-optimal hedge: $v^\star = V_0$, $\xi_t = \frac{d\langle V, S\rangle_t^{P^\star}}{d\langle S, S\rangle_t^{P^\star}}$,

$\varphi_t^\star = \xi_t + \tilde{a}(V_{t-} - v^\star - \varphi^\star \bullet S_{t-})$, where $\tilde{a}_t = \frac{dA_t^{S\star}}{d\langle S, S\rangle_t^{P^\star}}$ *adjustment process*

and $S = S_0 + M^{S\star} + A^{S\star}$ Doob-Meyer decomposition of S relative to $P^\star$

Hedging error:

$$E\Big(\big(v^\star + \varphi^\star \bullet S_T - H\big)^2\Big) = E\Big(L \bullet \big\langle V - \xi \bullet S, V - \xi \bullet S\big\rangle_T^{P^\star}\Big)$$

Case 3: arbitrary S (Černý & K 2005)

Key idea: change of measure $P \to P^\star$ (determined by characteristic equation)

Föllmer-Schweizer decomposition relative to $P^\star$: $H = V_0 + \xi \bullet S_T + R_T$,
where R $P^\star$-martingale, orthogonal to the $P^\star$-martingale part of S

Mean value process of the option: $V_t := E_{Q^\star}(H|\mathscr{F}_t)$,
where $Q^\star$ *variance-optimal (signed) martingale measure (VOMM)*
= minimal martingale measure relative to $P^\star$

Variance-optimal hedge: $\quad v^\star = V_0, \qquad \xi_t = \dfrac{d\langle V, S\rangle_t^{P^\star}}{d\langle S, S\rangle_t^{P^\star}},$

$\varphi_t^\star = \xi_t + \tilde{a}(V_{t-} - v^\star - \varphi^\star \bullet S_{t-})$, where $\tilde{a}_t = \dfrac{dA_t^{S\star}}{d\langle S, S\rangle_t^{P^\star}}$ *adjustment process*
and $S = S_0 + M^{S\star} + A^{S\star}$ Doob-Meyer decomposition of S relative to $P^\star$

Hedging error:

$$E\Big(\big(v^\star + \varphi^\star \bullet S_T - H\big)^2\Big) = E\Big(L \bullet \big\langle V - \xi \bullet S, V - \xi \bullet S\big\rangle_T^{P^\star}\Big)$$

The equations for the *opportunity-neutral measure* $P^\star$

$$L_t = \inf_\vartheta E\left((1 - (1_{]\!]t,T]\!]}\vartheta) \bullet S_T)^2 \middle| \mathscr{F}_t \right)$$

is called *opportunity process*.

It is the unique semimartingale such that

1. L, L_- are $(0,1]$-valued,
2. $L_T = 1$,
3. the joint characteristics $(b^{S,L}, c^{S,L}, F^{S,L}, A)$ of (S, L) solve

$$b_t^L = L_{t-}\frac{\overline{b}_t^2}{\overline{c}_t},$$

where

$$\overline{b}_t := b_t^S + c_t^{SL}\frac{1}{L_{t-}} + \int x\frac{y}{L_{t-}}F_t^{S,L}(d(x,y))$$

and

$$\overline{c}_t := c_t^S + \int x^2\left(1 + \frac{y}{L_{t-}}\right)F_t^{S,L}(d(x,y)),$$

4. some (unpleasant) integrability conditions hold.

The equations for the *opportunity-neutral measure* $P^\star$

$$L_t = \inf_{\vartheta} E\left((1 - (1_{]\!]t,T]\!]}\vartheta) \bullet S_T)^2 \middle| \mathscr{F}_t \right)$$

is called *opportunity process*.

It is the unique semimartingale such that

1. L, L_- are $(0, 1]$-valued,
2. $L_T = 1$,
3. the joint characteristics $(b^{S,L}, c^{S,L}, F^{S,L}, A)$ of (S, L) solve

$$b^L_t = L_{t-} \frac{\bar{b}^2_t}{\bar{c}_t},$$

where

$$\bar{b}_t := b^S_t + c^{SL}_t \frac{1}{L_{t-}} + \int x \frac{y}{L_{t-}} F^{S,L}_t(d(x, y))$$

and

$$\bar{c}_t := c^S_t + \int x^2 \left(1 + \frac{y}{L_{t-}}\right) F^{S,L}_t(d(x, y)),$$

4. some (unpleasant) integrability conditions hold.

In this case we define

1. the *adjustment process*

$$\tilde{a}_t := \frac{\overline{b}_t}{\overline{c}_t},$$

2. the density process of $P^\star$ relative to P:

$$Z^{P^\star} := \frac{L}{E(L_0)\mathscr{E}\left(\frac{\overline{b}_t^2}{\overline{c}_t} \bullet A\right)},$$

3. and the density process of $Q^\star$ relative to P:

$$Z^{Q^\star} := \frac{L\mathscr{E}(-\tilde{a} \bullet S)}{E(L_0)}.$$

Opportunity process L in specific situations

- discrete time: backward recursion

$$L_T = 1, \quad L_{t-1} = E(L_t|\mathscr{F}_{t-1}) - \frac{(E(\Delta S_t L_t|\mathscr{F}_{t-1}))^2}{E((\Delta S_t)^2 L_t|\mathscr{F}_{t-1})},$$

adjustment process $\tilde{a}_t = \dfrac{E(\Delta S_t L_t|\mathscr{F}_{t-1})}{E((\Delta S_t)^2 L_t|\mathscr{F}_{t-1})}$

- affine stochastic volatility models (S, v)

Try $L_t = \exp(\alpha(t) + v_t\beta(t))$ with α, β deterministic

$\longrightarrow$ ordinary differential equations for α, β

Opportunity process L in specific situations

- discrete time: backward recursion

  $$L_T = 1, \quad L_{t-1} = E(L_t|\mathscr{F}_{t-1}) - \frac{(E(\Delta S_t L_t|\mathscr{F}_{t-1}))^2}{E((\Delta S_t)^2 L_t|\mathscr{F}_{t-1})},$$

  adjustment process $\tilde{a}_t = \dfrac{E(\Delta S_t L_t|\mathscr{F}_{t-1})}{E((\Delta S_t)^2 L_t|\mathscr{F}_{t-1})}$

- affine stochastic volatility models (S, v)

  Try $L_t = \exp(\alpha(t) + v_t\beta(t))$ with α, β deterministic

  $\longrightarrow$ ordinary differential equations for α, β