Asymptotic analysis of utility-based hedging strategies for small number of contingent claims

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July 10, 2006

Abstract

In the framework of incomplete financial models the role of hedging strategy is to provide the optimal trade-off between risk (error of replication) and return. We study the linear approximation of utility-based hedging strategies for small number of contingent claims. We show that this approximation is actually a mean-variance hedging strategy under an appropriate choice of a numéraire and a risk-neutral probability.

Keywords: incomplete markets, utility-based hedging, mean-variance hedging, risk-tolerance wealth process, contingent claim, numéraire.

*This research was supported in part by the National Science Foundation under Grant DMS-0505414

†This research was supported in part by the National Science Foundation under Grant DMS-0604643

JEL Classification: G11, G12, C61.

AMS Subject Classification (1991): 90A09, 90A10, 90C26.

1 Introduction

In complete markets, any contingent claim can be replicated by trading. The wealth process of such a *hedging strategy* follows the price process of the claim, which is uniquely defined by no-arbitrage arguments. In *incomplete markets* an agent can not "trade away" (replicate) the *risk* of holding contingent claims in his portfolio but can only attempt to reduce it. The concept of *hedging in incomplete markets* is thus directly related to how *risk* is quantified.

One of the most popular approaches is based on the mean-variance criterion. This line of research was initiated by Föllmer and Sondermann [5] and continued by Föllmer and Schweizer [4], Schweizer [12] and many other authors. We single out here the paper of Gourieroux, Laurent and Pham [6]: through a change of numéraire they convert the problem of mean-variance hedging under historical measure to the hedging under a *martingale measure*, thus reducing it to the Föllmer-Sondermann case. A nice overview of the literature can be found in Schweizer [13].

Mean-variance hedging is tractable, but it has some economic disadvantages (like penalizing equally shortfalls and earnings). Therefore, more recently, a number of authors studied the concept of *utility-based hedging*, where a portfolio's performance is measured by expected utility. We just mention Duffie et al. [2], which uses direct PDE approach in the study of hedging problem for a non-replicable income stream in the case of power utilities, and Delbaen et al. [3], that relies on duality and martingale methods for the case of exponential utility.

Since explicit computations of utility-based hedging strategies are rarely possible, several authors proposed asymptotic techniques. For example, in the framework of financial model with *basic risk* and for power and exponential utilities Davis [1] computes the first order approximation with respect to a small parameter $1-\rho^2$, where ρ is the correlation between traded and non-traded assets, and Henderson and Hobson [8] and Henderson [7] derive the first order expansion with respect to the number of contingent claims.

In this paper we generalize the results of [8] and [7] to the case of general semimartingale financial model and arbitrary utility function defined on positive real line. Our main statement is Theorem 2 which shows that the asymptotic hedging strategy is, in fact, the mean-variance hedging strategy (as in [5]), where the role of the pricing measure is played by $Y'(y)$ (the derivative of the dual minimizer) and the role of the numéraire is played by $X'(x)$ (the derivative of the optimal investment strategy). The paper is a companion to our work [11] and relies heavily on ideas and results there.

2 The Model

We work in the same model as in [11] and refer to this paper for more details. We have $d+1$ assets, one bond and d stocks. The price of the bond is constant and the price process of the stocks $S = (S^i)_{1\leq i\leq d}$ is assumed to be a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$. Here T is a finite time horizon and $\mathcal{F} = \mathcal{F}_T$.

A portfolio is defined as a pair (x, H), where the constant x represents the initial capital and $H = (H^i)_{1\leq i\leq d}$ is a predictable S-integrable process. The wealth process $X = (X_t)_{0\leq t\leq T}$ of the portfolio evolves in time as the stochastic integral of H with respect to S:

$$X_t = x + \int_0^t H_u dS_u, \quad 0 \leq t \leq T. \tag{1}$$

We denote by $\mathcal{X}(x)$ the family of wealth processes with nonnegative capital at any instant and with initial value equal to x:

$$\mathcal{X}(x) \triangleq \{X \geq 0 : \ X \text{ is defined by (1)}\}. \tag{2}$$

A nonnegative wealth process is said to be *maximal* if its terminal value cannot be dominated by that of any other nonnegative wealth process with the same initial value. A (signed) wealth process X is said to be *maximal* if it admits a representation of the form

$$X = X' - X'',$$

where both X' and X'' are nonnegative maximal wealth processes. A wealth process X is said to be *acceptable* if it admits a representation as above, where both X' and X'' are nonnegative wealth processes and, in addition, X'' is maximal.

A probability measure $\mathbb{Q} \sim \mathbb{P}$ is called an *equivalent local martingale measure* if any $X \in \mathcal{X}(1)$ is a local martingale under $\mathbb{Q}$. We denote by $\mathcal{Q}$ the set of equivalent martingale measures and assume, as usually, that

$$\mathcal{Q} \neq \emptyset. \tag{3}$$

In addition to the set of traded securities we consider a family of N non-traded European contingent claims with payment functions $f = (f_i)_{1 \leq i \leq N}$, which are $\mathcal{F}$-measurable random variables, and maturity T. We assume that this family is dominated by the terminal value of some nonnegative wealth process X, that is

$$\sum_{i=1}^{N} |f_i| \leq X_T. \tag{4}$$

For $x \in \mathbf{R}$ and $q \in \mathbf{R}^m$ we denote by $\mathcal{X}(x, q)$ the set of *acceptable wealth processes* with initial capital x whose terminal values cover the potential losses from the q contingent claims, that is

$$\mathcal{X}(x, q) \triangleq \{X : X \text{ is acceptable}, X_0 = x \text{ and } X_T + \langle q, f \rangle \geq 0\}.$$

The set of points (x, q) where $\mathcal{X}(x, q)$ is not empty is a closed convex cone in $\mathbf{R}^{m+1}$. We denote by $\mathcal{K}$ the interior of this cone, that is

$$\mathcal{K} \triangleq \operatorname{int}\left\{(x, q) \in \mathbf{R}^{m+1} : \mathcal{X}(x, q) \neq \emptyset\right\}.$$

In this financial model we consider an economic agent whose preferences over terminal wealth are described by a utility function $U : (0, \infty) \to (-\infty, \infty)$. The function U is assumed to be strictly concave and strictly increasing. In addition, motivated by [10] and [11], we make the following assumption on U:

Assumption 1. The utility function U is two-times continuously differentiable on $(0, \infty)$ and its relative risk-aversion coefficient

$$A(x) \triangleq -\frac{xU''(x)}{U'(x)}, \quad x > 0, \tag{5}$$

is uniformly bounded away from zero and infinity, that is, there are constants $c_1 > 0$ and $c_2 < \infty$ such that

$$c_1 < A(x) < c_2, \quad x > 0. \tag{6}$$

Assume that the agent has some initial capital x and quantities $q = (q_i)_{1\leq i\leq N}$ of the contingent claims f such that $(x,q) \in \mathcal{K}$. The quantities q of the contingent claims will be held constant up to maturity. The capital x can be freely invested into the stocks and the bond according to some dynamic strategy. Therefore, the maximal expected utility that the agent can achieve by trading in the financial market is given by

$$u(x,q) \triangleq \sup_{X\in\mathcal{X}(x,q)} \mathbb{E}\left[U\left(X_T + \langle q, f\rangle\right)\right], \quad (x,q) \in \mathcal{K}. \tag{7}$$

Under our conditions there is a unique optimizer $X(x,q)$ in (7), see [9, Theorem 2].

Abusing notation, we denote by $u(x) \triangleq u(x,0)$ the value function for the problem of optimal investment with no random endowment, i.e.

$$u(x) \triangleq u(x,0) = \sup_{X\in\mathcal{X}(x)} \mathbb{E}\left[U\left(X_T\right)\right], \quad x > 0, \tag{8}$$

and by $X(x) \triangleq X(x,0)$ the optimizer in (8). To exclude the trivial case we shall assume that

$$u(x) < \infty \quad \text{for some } x > 0, \tag{9}$$

which together with (4) easily implies that

$$u(x,q) < \infty \quad \text{for all } (x,q) \in \mathcal{K}. \tag{10}$$

The *dual* problem to (8) is given as follows:

$$v(y) \triangleq \inf_{Y\in\mathcal{Y}(y)} \mathbb{E}\left[V\left(Y_T\right)\right], \quad y > 0. \tag{11}$$

Here V is the convex conjugate to U, that is

$$V(y) \triangleq \sup_{x>0} \left\{U(x) - xy\right\}, \quad y > 0,$$

and $\mathcal{Y}(y)$ is the family of nonnegative supermartingales Y such that $Y_0 = y$ and XY is a supermartingale for all $X \in \mathcal{X}(1)$. The solution to (11) is denoted by $Y(y)$. If $y = u'(x)$ then the ratio $Y(y)/y$ is called *the state price density* corresponding to the cash amount $x > 0$. For such initial position we denote

$$p(x) \triangleq \mathbb{E}\left[\frac{Y_T(u'(x))}{u'(x)} f\right] \tag{12}$$

the vector of marginal utility based prices for the contingent claims f.

In this paper we are interested to know how the above agent "hedges" the q contingent claims he cannot trade, starting from position $(x, q) \in \mathcal{K}$. The formal definition of the hedging strategy is as follows.

Definition 1. Fix $(x, q) \in \mathcal{K}$. The wealth process $G(x, q)$ of *the utility-based hedging strategy* is given by

$$G(x, q) \triangleq X(c(x, q)) - X(x, q), \tag{13}$$

where $c(x, q)$ is the *certainty equivalent value* of the position $(x, q) \in \mathcal{K}$ defined by the equation

$$u(c(x, q)) = u(x, q), \tag{14}$$

$X(c(x, q))$ is the solution of (8) for initial wealth $c(x, q)$ and $X(x, q)$ is the solution of (7).

Remark 1. For $(x, q) \in \mathcal{K}$ we have $u(x, q) \in (u(0), u(\infty))$. Since $u(x) \triangleq u(x, 0)$ is continuous and strictly increasing, equation (14) has a unique solution $c(x, q) > 0$.

Recall that the contingent claim $f = (f_i)_{i \leq i \leq N}$ is *replicable* if for every i there is a maximal wealth process X^i with the terminal value f_i. In his case, it is an easy exercise to show that a utility-based hedging strategy coincides with a replication strategy:

$$G(x, q) = \langle q, X \rangle, \quad (x, q) \in \mathcal{K}.$$

If f is not replicable, then, usually, it is not possible to compute $G(x, q)$ explicitly. The goal of the paper is to study the linear approximation of the hedging strategy in the case of small q.

3 Asymptotic analysis

The main object of the paper is specified in the following

Definition 2. Let $x > 0$. An N-dimensional semimartingale $H(x)$ is called an *asymptotic hedging strategy* for the contingent claims f if each component $H^i(x)$ is a wealth process (that is, a stochastic integral w.r.t. S) and

1. the terminal value $H_T(x)$ satisfies
$$\lim_{\|q\|\to 0} \frac{|G_T(x,q) - \langle q, H_T(x)\rangle\,|}{\|q\|} = 0, \tag{15}$$
where the above limit is in $\mathbb{P}$-probability.
2. for $y = u'(x)$ the product $H(x)Y(y)$ is a martingale, where $Y(y)$ is the solution to (11).

The asymptotic hedging strategy $H(x)$ represents (up to a sign) the *marginal* action the investor needs to take in order to compensate the risk coming from adding to his portfolio a small number of contingent claims. It is easy to see, that $H(x)$ is defined uniquely by Definition (2). Following [11] we specify below precise mathematical conditions for the existence of asymptotic hedging strategy and also describe some methods to compute it.

Since $X(x)Y(y)$ is a uniformly integrable martingale, we can define the probability measure $\mathbb{R}(x)$ by
$$\frac{d\mathbb{R}(x)}{d\mathbb{P}} = \frac{X_T(x)Y_T(y)}{xy}, \quad y = u'(x). \tag{16}$$
Denote by $S^{X(x)}$ the price process of the traded securities discounted by $X(x)/x$, that is,
$$S^{X(x)} = \left(\frac{x}{X(x)}, \frac{xS}{X(x)}\right). \tag{17}$$
Let $\mathbf{H}_0^2(\mathbb{R}(x))$ be the space of square integrable martingales under $\mathbb{R}(x)$ with initial value 0 and
$$\mathcal{M}^2(x) = \left\{ M \in \mathbf{H}_0^2(\mathbb{R}(x)) : \quad M_t = \int_0^t H dS^{X(x)},\ 0 \le t \le T \right\}.$$
Note that if $M \in \mathcal{M}^2(x)$, then $\frac{X(x)}{x} M$ is a wealth process (under the original *numéraire*). We also denote by
$$g_i = x\frac{f_i}{X_T(x)}, \quad 1 \le i \le N, \tag{18}$$
the payoffs of the European options discounted by $X_T(x)/x$. The computation of the asymptotic hedging strategy is based on the solutions of the following auxiliary optimization problems:
$$c_i(x) = \inf_{M\in\mathcal{M}^2(x)} \mathbb{E}_{\mathbb{R}(x)}[A(X_T(x))(p_i(x) + M_T - g_i)^2],\ 1 \le i \le N, \tag{19}$$

where the function A and the vector $p(x)$ were defined in (5) and (12).

To state the result we require two technical assumptions from [11].

Assumption 2. The process $S^{X(x)}$ defined in (17) is sigma-bounded, that is, there is a strictly positive predictable (one-dimensional) process h such that the stochastic integral $\int h dS^{X(x)}$ is well-defined and is locally bounded.

Assumption 3. There are constant $c > 0$ and a process $M \in \mathcal{M}^2(x)$, such that

$$\sum_{i=1}^{N} |g_i| \leq c + M_T. \tag{20}$$

Theorem 1. *Assume* (3) *and* (9) *and also that Assumptions 1, 2 and 3 hold true. Then the asymptotic hedging strategy* $H(x)$ *exists and is given by*

$$H^i(x) = \frac{X(x)}{x}(p_i(x) + M^i(x)), \quad 1 \leq i \leq N, \tag{21}$$

where $p(x)$ *is defined by* (12) *and* $M^i(x)$ *are the solutions of* (19).

The proof of Theorem 1 as well as the proofs of Theorems 2 and 3 below will be given in Section 4. Our next goal is to characterize $H(x)$ in terms of the solution of a mean-variance hedging problem. We denote by $X'(x)$ and $Y'(y)$ the derivatives to $X(x)$ and $Y(y)$ in the sense that $X'(x)Y(y)$ and $Y'(y)X(x)$ are martingales and

$$X'_T(x) = \lim_{\epsilon \to 0} \left(\frac{X_T(x+\epsilon) - X_T(x)}{\epsilon} \right), \tag{22}$$

$$Y'_T(y) = \lim_{\epsilon \to 0} \left(\frac{Y_T(y+\epsilon) - Y_T(y)}{\epsilon} \right), \tag{23}$$

where the convergences take place in probability. Under conditions of Theorem 1 (see [10, Theorem 1]) we have that $X'(x)$ and $Y'(y)$ are well-defined. Hereafter we assume that $X'(x)$ is a strictly positive wealth process, that is,

$$X'_T(x) > 0. \tag{24}$$

(A simple example when this condition is violated can be found in [10].) In this case the product $X'(x)Y'(y)$ is a strictly positive martingale with initial value 1 and, hence, we can define a new probability measure $\widetilde{\mathbb{R}}(x)$ such that

$$\frac{d\widetilde{\mathbb{R}}(x)}{d\mathbb{P}} = X'_T(x)Y'_T(y). \tag{25}$$

We choose the wealth process $X'(x)$ as a numéraire and denote by $S^{X'(x)}$ the price process of the traded securities discounted by $X'(x)$, that is,

$$S^{X'(x)} = \left(\frac{1}{X'(x)}, \frac{S}{X'(x)} \right). \tag{26}$$

Let $\mathbf{H}_0^2(\widetilde{\mathbb{R}}(x))$ be the space of square integrable martingales under $\widetilde{\mathbb{R}}(x)$ with initial value 0 and

$$\widetilde{\mathcal{M}}^2(x) = \left\{ M \in \mathbf{H}_0^2(\widetilde{\mathbb{R}}(x)) : \quad M_t = \int_0^t H dS^{X'(x)},\, 0 \le t \le T \right\}.$$

We denote by $\widetilde{g}$ the payoffs of the contingent claims discounted by $X'(x)$, that is,

$$\widetilde{g}_i = \frac{f_i}{X'_T(x)}, \quad 1 \le i \le N, \tag{27}$$

and define the following N-dimensional martingale under $\widetilde{\mathbb{R}}(x)$:

$$\widetilde{P}_t(x) = \mathbb{E}_{\widetilde{\mathbb{R}}(x)}[\widetilde{g}|\mathcal{F}_t], \quad 0 \le t \le T. \tag{28}$$

In Lemma 1 we shall show that $\widetilde{P}(x)$ is a square integrable martingale under $\widetilde{\mathbb{R}}(x)$. Hence, it admits the the following Kunita-Watanabe decomposition:

$$\widetilde{P}(x) = \widetilde{p}(x) + \widetilde{M}(x) + \widetilde{N}(x), \tag{29}$$

where

$$\widetilde{p}(x) = \mathbb{E}_{\widetilde{\mathbb{R}}(x)}[\widetilde{g}] = \mathbb{E}[Y'_T(y) f], \tag{30}$$

$\widetilde{M}^i(x)$ belongs to $\widetilde{\mathcal{M}}^2(x)$ and $\widetilde{N}^i(x)$ is an element of $\mathbf{H}_0^2(\widetilde{\mathbb{R}}(x))$ orthogonal to $\widetilde{\mathcal{M}}^2(x)$.

Theorem 2. *Assume conditions of Theorem 1 and that $X'(x)$ is a strictly positive wealth process. Then the asymptotic hedging strategy $H(x)$ admits the representation:*

$$H^i(x) = X'(x)(p_i(x) + \widetilde{M}^i(x)), \quad 1 \le i \le N, \tag{31}$$

where $p(x)$ is defined by (12) *and $\widetilde{M}^i(x)$ are given by the Kunita-Watanabe decomposition* (29).

Theorems 1 and 2 provide characterizations of the asymptotic hedging strategy in terms of the numéraires $X(x)$ and $X'(x)$ and the corresponding risk-neutral probabilities $\mathbb{R}(x)$ and $\widetilde{\mathbb{R}}(x)$. In our final Theorem 3 we give more explicit description of $H(x)$ under the original numéraire (bank account) and the risk-neutral probability measure $\mathbb{Q}(y)$ defined by

$$\frac{d\mathbb{Q}(y)}{d\mathbb{P}} \triangleq \frac{Y_T(y)}{y}.$$

Of course, for $\mathbb{Q}(y)$ to be a probability measure we need the following

Assumption 4. $Y(y)$ is a uniformly integrable martingale, i.e. $\mathbb{E}[Y_T(y)] = y$.

Recall from [11] that a semimartingale $R(x)$ is called a *risk-tolerance wealth process* if it is a maximal positive wealth process and

$$R_T(x) = -\frac{U'(X_T(x))}{U''(X_T(x))}. \tag{32}$$

Assumption 5. The risk-tolerance wealth process $R(x)$ exists.

It was shown in [11, Theorem 4] that the existence of $R(x)$ is equivalent to the fact that

$$Y'(y) = \frac{Y(y)}{y}, \tag{33}$$

and that in this case

$$X'(x) = \frac{R(x)}{R_0(x)}. \tag{34}$$

Moreover, Assumption 5 is a necessary and sufficient condition for some "nice" qualitative properties of marginal utility-based prices to hold true when $q \approx 0$, see [11, Theorem 9]. Hence, one can argue that this assumption should be valid for any "practical" incomplete financial model.

To state the result we also have to impose the following

Assumption 6. The stock price process S is continuous.

We would like to point out that Assumption 6 is stronger than Assumption 2 and, as simple examples show, is needed for the validity of Theorem 3 below.

Consider now the $\mathbb{Q}(y)$-martingale $P(x)$ (the marginal utility-based price process)

$$P_t(x) = \mathbb{E}_{\mathbb{Q}(y)}[f|\mathcal{F}_t], \quad 0 \le t \le T, \tag{35}$$

and let

$$P^i(x) = p_i(x) + \int K^i dS + L^i, \quad 1 \le i \le N, \tag{36}$$

be its Kunita-Watanabe decomposition, where $L = (L^i)_{1 \le i \le N}$ is a local martingale under $\mathbb{Q}(y)$ orthogonal to S such that $L_0 = 0$ and we used the fact that $P_0(x) = p(x)$.

Theorem 3. *Let the conditions of Theorem 1 and Assumptions 4, 5 and 6 hold true. Then the asymptotic hedging strategy satisfies the equation*

$$H^i_t(x) = p_i(x) + \int_0^t K^i_u dS_u + \int_0^t (H^i_t(x) - P^i_{t-}(x)) \frac{dR_t(x)}{R_t(x)}, \tag{37}$$

where K^i is defined by (36)*, $P(x)$ is given by* (35) *and $R(x)$ is the risk-tolerance wealth process.*

Remark 2. The message of Theorem 3 is that asymptotic hedging is performed the following way: start with $p(x)$ cash and buy at any moment the quantities of stocks S one would buy to hedge quadratically the payoff f under the martingale measure $\mathbb{Q}(y)$. The missing dollar amount up to the marginal price $P_{t-}(x)$ is invested in the money market. Since perfect replication may not be possible, this strategy is not self-financing. The miss-match $(H_t(x) - P_{t-}(x))$ should be financed by investing in (borrowing from) the risk-tolerance wealth process.

4 Proofs

Proof of Theorem 1. Let $g_0 = 1$ and consider the optimization problems

$$a_i(x) = \inf_{N \in \mathcal{M}^2(x)} \mathbb{E}_{\mathbb{R}(x)}[A(X_T(x))(g_i + N_T)^2], \quad 0 \le i \le N. \tag{38}$$

For $0 \le i \le N$ we denote by $N^i(x)$ the solution to (38). From [11, Theorem 1] we know that

$$X'(x) = \frac{X(x)}{x}(1 + N^0(x)) \tag{39}$$

($X'(x)$ is defined by (22) and the martingale property of $X'(x)Y(y)$). Also, denoting

$$Z^i(x) \triangleq \frac{X(x)}{x} N^i(x), \quad 1 \leq i \leq N, \tag{40}$$

we have by the same [11, Theorem 1] that

$$\lim_{|\Delta x|+|q|\to 0} \left(\frac{X_T(x+\Delta x, q) - X_T(x) - X'_T(x)\Delta x - \langle Z_T(x), q\rangle}{|\Delta x| + |q|} \right) = 0, \tag{41}$$

and the process $Z(x)Y(y)$ is a uniformly integrable martingale. Taking into account the linearity of the solutions of the quadratic optimization problems (38) with respect to g_i, we conclude that the solution $M(x)$ of (19) can be written as

$$M^i(x) = p_i(x)N^0(x) - N^i(x).$$

Using (39) and (40), we obtain

$$\frac{X(x)}{x}(p_i(x) + M^i(x)) = p_i(x)X'(x) - Z^i(x).$$

We know from [11, Theorem 10] that

$$\frac{\partial c(x,q)}{\partial q^i}|_{q=0} = p_i(x), \quad 1 \leq i \leq N,$$

so we can use Definition 2, relation (41) and a simple chain rule argument to finish the proof. $\square$

For the proof of Theorem 2 we denote by $\mathcal{N}^2(y)$ the orthogonal complement of $\mathcal{M}^2(x)$ in $\mathbf{H}_0^2(\mathbb{R}(x))$ and by $\widetilde{\mathcal{N}}^2(y)$ the orthogonal complement of $\widetilde{\mathcal{M}}^2(x)$ in $\mathbf{H}_0^2(\widetilde{\mathbb{R}}(x))$.

Lemma 1. *Assume the hypotheses of Theorem 2. Then:*

1. *For a random vector h, we have*

$$\frac{xh}{X_T(x)} \in \mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{R}(x)) \text{ if and only if } \frac{h}{X'_T(x)} \in \mathbf{L}^2(\Omega, \mathcal{F}, \widetilde{\mathbb{R}}(x)).$$

2. *For a semimartingale Z and a fixed number a we have*

$$\frac{xZ}{X(x)} \in a + \mathcal{M}^2(x) \text{ if and only if } \frac{Z}{X'(x)} \in a + \widetilde{\mathcal{M}}^2(x).$$

3. *For a semimartingale W and a fixed number b we have*

$$\frac{yW}{Y(y)} \in b + \mathcal{N}^2(y) \text{ if and only if } \frac{W}{Y'(y)} \in b + \widetilde{\mathcal{N}}^2(y).$$

Proof. From [10, Theorem 1] we know that the function u is two-times differentiable at x, and

$$U''(X_T(x))X'_T(x) = u''(x)Y'_T(y), \quad y = u'(x). \tag{42}$$

Relation (42) together with $U'(X_T(x)) = Y_T(y)$ imply

$$A(X_T(x))\frac{xX'_T(x)}{X_T(x)} = -\frac{xu''(x)}{u'(x)}\frac{yY'_T(y)}{Y_T(y)}.$$

By Assumption 1 we have that $c_1 \leq A(X_T(x)) \leq c_2$ and by [10, Theorem 1] we know that

$$0 < c_1 \leq a(x) \triangleq -\frac{xu''(x)}{u'(x)} \leq c_2 < \infty.$$

Since $\frac{xX'(x)}{X(x)}$ and $\frac{yY'(y)}{Y(y)}$ are uniformly integrable martingales under $\mathbb{R}(x)$ we conclude that

$$\frac{c_1}{c_2}\frac{xX'(x)}{X(x)} \leq \frac{yY'(y)}{Y(y)} \leq \frac{c_2}{c_1}\frac{xX'(x)}{X(x)}. \tag{43}$$

Note that $\frac{xh}{X_T(x)} \in \mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{R}(x))$ if and only if

$$\mathbb{E}\left[\|h\|^2\frac{Y_T(y)}{X_T(x)}\right] < \infty,$$

and, similarly, $\frac{h}{X'_T(x)} \in \mathbf{L}^2(\Omega, \mathcal{F}, \widetilde{\mathbb{R}}(x))$ if and only if

$$\mathbb{E}\left[\|h\|^2\frac{Y'_T(y)}{X'_T(x)}\right] < \infty.$$

Taking into account relation (43) (at time T) we complete the proof of the first item.

If $\frac{xZ}{X(x)} \in a + \mathcal{M}^2(x)$ then $\frac{Z}{X'(x)}$ is a wealth process starting at a (under $X'(x)$ *numéraire*) and, according to item 1, we also know that $\frac{Z_T}{X'_T(x)} \in \mathbf{L}^2(\Omega, \mathcal{F}, \widetilde{\mathbb{R}}(x))$. Furthermore, since

$$\frac{d\widetilde{\mathbb{R}}(x)}{d\mathbb{R}(x)} = \frac{xyX'_T(x)Y'_T(y)}{X_T(x)Y_T(y)},$$

and

$$\frac{Z}{X'(x)}\frac{X'(x)Y'(y)}{X(x)Y(y)} = \frac{Z}{X(x)}\frac{Y'(y)}{Y(y)}$$

is a uniformly integrable martingale under $\mathbb{R}(x)$ (because $\frac{yY'(y)}{Y(y)} \in 1+\mathcal{N}^2(y)$, see [10, Theorem 1]), we conclude that $\frac{Z}{X'(x)}$ is a uniformly integrable martingale under $\widetilde{\mathbb{R}}(x)$. We obtain

$$\frac{Z}{X'(x)} \in a + \widetilde{\mathcal{M}}^2(x). \tag{44}$$

Assume now (44). Then $\frac{xZ}{X(x)}$ is a wealth process starting at a, under the *numéraire* $X(x)/x$. Because of Assumption 2, the process $S^{X(x)}$ is a *sigma-martingale* under the measure $\mathbb{R}(x)$. Therefore, in order to prove that

$$\frac{xZ}{X(x)} \in a + \mathcal{M}^2(x)$$

it is enough to show that

$$\sup_{0\leq\tau\leq T} \mathbb{E}_{\mathbb{R}(x)}\left[\frac{Z_\tau^2}{X_\tau^2(x)}\right] < \infty, \tag{45}$$

where the supremum above is taken with respect to all stopping times τ. In view of relation (43) this amounts to

$$\sup_{0\leq\tau\leq T} \mathbb{E}_{\widetilde{\mathbb{R}}(x)}\left[\frac{Z_\tau^2}{(X'_\tau(x))^2}\right] < \infty,$$

which is true because of assumption (44). The proof of item 2 of the lemma is also complete.

Choose now two arbitrary semimartingales Z and W. We observe that the process

$$\frac{xZ}{X(x)}\frac{yW}{Y(y)}$$

is a uniformly integrable martingale under $\mathbb{R}(x)$ if and only if

$$\frac{Z}{X'(x)}\frac{W}{Y'(y)}$$

is a uniformly integrable martingale under $\widetilde{\mathbb{R}}(x)$. The above observation, applied for fixed W and any Z, together with the assertion of the second item, finishes the proof of item 3. $\square$

Proof of Theorem 2. From Lemma 1 (item 1) and Assumption 3 we have $\widetilde{g} \in \mathbf{L}^2(\Omega, \mathcal{F}, \widetilde{\mathbb{R}}(x))$. This implies that the process $(\widetilde{P}_t(x))_{0\leq t\leq T}$ defined in (28) is a square integrable martingale under $\widetilde{\mathbb{R}}(x)$ and, hence, admits the unique Kunita-Watanabe decomposition (29).

A standard argument in constraint optimization applied to problem (19) leads to

$$\frac{-U''(X_T(x))X_T(x)}{Y_T(y)}\left(p_i(x) + M_T^i(x) - \frac{xf_i}{X_T(x)}\right) = L_T^i, \tag{46}$$

where

$$L^i \in b_i + \mathcal{N}^2(y)$$

for some real number b_i. Using (42) we obtain

$$\frac{f_i}{X_T'(x)} = \frac{X_T(x)}{xX_T'(x)}(p_i(x) + M_T^i(x)) + \frac{y}{xu''(x)}\frac{Y_T(y)L_T^i}{yY_T'(y)}.$$

According to Lemma 1 we have

$$\frac{X(x)}{xX'(x)}(p_i(x) + M^i(x)) \in p_i(x) + \widetilde{\mathcal{M}}^2(x)$$

and

$$\frac{y}{xu''(x)}\frac{Y(y)L^i}{yY'(y)} \in \frac{y}{xu''(x)}b_i + \widetilde{\mathcal{N}}^2(y).$$

Since the Kunita-Watanabe decomposition (29) is unique, we obtain

$$\widetilde{M}^i(x) = \frac{X(x)}{xX'(x)}(p_i(x) + M^i(x)) - p_i(x).$$

Taking into account Theorem 1 we finally conclude that

$$H^i(x) = X'(x)(p_i(x) + \widetilde{M}^i(x)).$$

□

Proof of Theorem 3. We remind the reader that under the assumptions of Theorem 3 we have

$$Y'(y) = \frac{Y(y)}{y}, \quad X'(x) = \frac{R(x)}{R_0(x)}.$$

Consider decomposition (29). Since $\widetilde{p}(x) = p(x)$ and $P(x) = R(x)\widetilde{P}(x)/R_0(x)$, we know from Theorem 2 that

$$P(x) = H(x) + \frac{R(x)}{R_0(x)}\widetilde{N}(x). \tag{47}$$

Under the measure $\widetilde{\mathbb{R}}(x)$, the process $\widetilde{N}(x)$ is a martingale orthogonal to the continuous local martingale $S^{X'(x)}$. This implies easily that $\widetilde{N}(x)$ and S are orthogonal local martingales under $\mathbb{Q}(y)$. The process $R(x)$ is a stochastic integral with respect to S, so $[\widetilde{N}(x), R(x)] = 0$. We can now apply the Itô formula to the product $\widetilde{N}(x)R(x)$ in (47) to obtain

$$P_t(x) = H_t(x) + \int_0^t \frac{\widetilde{N}_{u-}(x)}{R_0(x)} dR_u(x) + \int_0^t \frac{R_u(x)}{R_0(x)} d\widetilde{N}_u(x).$$

Using again the fact that $\widetilde{N}(x)$ and S are orthogonal local martingales under $\mathbb{Q}(y)$, we can identify the terms in the Kunita-Watanabe decomposition (36) as

$$p(x) + \int_0^t K_u dS_u = H_t(x) + \int_0^t \frac{\widetilde{N}_{u-}(x)}{R_0(x)} dR_u(x). \tag{48}$$

Using (47) we have

$$\widetilde{N}(x) = \frac{R_0(x)(P(x) - H(x))}{R(x)}$$

so (48) can be rewritten as

$$H_t(x) = p(x) + \int_0^t K_u dS_u - \int_0^t (P_{u-}(x) - H_u(x)) \frac{dR_u(x)}{R_u(x)},$$

which ends the proof. □

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