

**COHERENT MULTIPERIOD RISK ADJUSTED
VALUES AND BELLMAN'S PRINCIPLE**

PHILIPPE ARTZNER, UNIVERSITÉ LOUIS PASTEUR
FREDDY DELBAEN, EIDGENÖSSISCHE TECHNISCHE HOCHSCHULE
JEAN-MARC EBER, LEXIFI
DAVID HEATH, CARNEGIE MELLON UNIVERSITY
HYEJIN KU*, YORK UNIVERSITY

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Philippe Artzner, Institut de Recherche Mathématique Avancée, 7 rue René Descartes, F 67084
Strasbourg, France, tel. +33 3 90 240 204, artzner@math.u-strasbg.fr

Abstract. Starting with a time-0 coherent risk measure defined for “value processes”, we define, also at intermediate times, a risk measurement process. Two other constructions of such measurement processes are given in terms of sets of test probabilities. These constructions are identical and related to the former construction when the sets fulfill a stability condition also met in multiperiod treatment of ambiguity in decision-making. We finally deduce risk measurements for final value of locked-in positions and repeat a warning concerning Tail-Value-at-risk.

Keywords. Bellman’s principle, capital requirement, coherence, risk-adjusted values, stability by pasting, time consistency.

INTRODUCTION

One-period models of risk measurement considered neither the source of possibly required extra capital at the beginning of an holding period nor the actual consequences of a “bad event” at the end of the same period. These points become relevant for multiperiod risk because:

- first, the availability of intermediate information requires taking into account *intermediate monitoring* by supervisors and/or shareholders of a position even if it is *locked-in*. Intermediate markets may be missing, but release of information over time does matter at date 0, for instance because of intermediate audits, see Bennet (2001), or possible actions of competitors,
- second, the possibility of *intermediate actions*, the availability of extraneous cash flows and of possible capital in- or outflows require handling *sequences* of future, unknown, “values”,
- third, with one period of uncertainty, capital served as a buffer at the initial date and as shareholders’ wealth at the final date. Intermediate dates, whether deterministic or random, raise the question of the nature of capital (valued in a market or accounting way) at such dates.

This paper is organised as follows.

Section 1 briefly reviews the one-period theory with general probability spaces. We define risk-adjusted values which are more convenient in the multiperiod case than their negative, the traditional risk measures.

Section 2 is the core of the paper. Its first part is devoted to the axiomatics of *initial* coherent measurement of the risk of “value” *processes* with our usual emphasis on *acceptability*. It is intended for supervision of behaviour of “values” on *trajectories*, a difference with the current literature, see Cvitanič and Karatzas (1999), Föllmer and Schied (2002b), Nakano (2003), Roorda, Engwerda and Schumacher (2002), as well as Riedel (2002) (see the end of our Section 4.2).

The second part of Section 2 introduces risk measurement at intermediate stopping times, the acceptance at these times and finally risk measurement *processes*.

Section 3 prepares a *particular* construction of *multiperiod* measurement of risk by “pasting” test probabilities, using the likelihood ratio processes. This approach allows for combining different risk attitudes over time. Sets of test probabilities which are stable by pasting appeared in “multiperiod decision theory with ambiguity” as “rectangular sets” in Epstein and Schneider (2003) and generalized Bayes rule in Wang (2003). Our definitions work with general state spaces.

Section 4 also constructs two risk-measurement *processes* of multiperiod-risk by using sets of test probabilities, as in the representation of one-period risk measures. We compare them to each other and to the risk measurement process of the general theory of Section 2. Stability of the set of test probabilities is shown to be equivalent to Bellman’s principle and to time consistency as in Wang (1996, 1999).

Section 5 deals with multiperiod-measurement of a final locked-in position. Again two constructions are given, using either successive conditional expectations with respect to all test probabilities or using a recursive procedure. They are identical if and only if the set of test probabilities is stable. We comment on the case of Tail-Value-at-Risk.

1. REVIEW OF ONE-PERIOD ACCEPTABILITY

The theory of coherent risk measures at date 0 for some date 1 is best approached, see Artzner et al. (1997, p. 69, 1999a, Section 2.2), Delbaen (2000, 2002), Föllmer and Schied (2002a, 2002b), Heath (1998), by taking the primitive object to be an “acceptance set”. It is a set of bounded future random net worths on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$, also simply called “values”, which satisfies some axioms of coherence on the part of a regulator or supervisor, either internal or external.

Remark 1. In a one period model, the value at time “1” does involve an assessment of future payments, like a market value or some form of net worth when markets are missing (as for insurance liabilities). The risk adjusted measurement of a “value” is the maximum amount (of numéraire) to be subtracted from it at the initial date while keeping it acceptable.

Remark 2. It will be convenient in the multiperiod case to work with *risk-adjusted values* defined as the negative of risk measures: this simplifies the presentation of recursive risk measurement (see for example Bellman’s principle in Theorem 4.2, (iii) and the Remark following the statement of Theorem 5.1).

Remark 3. We express everything via a numéraire, for example the money market account or a default-free zero-coupon long bond — doing so in Artzner et al. (1999a) would have made $r = 1$ in Definition 2.2 there. Because of the translation axiom, it is only *after* this choice of numéraire that risk measures can be defined can be qualified as “monetary” (see Föllmer-Schied (2002b)).

The axioms of coherence for acceptance sets or for risk measures (see Artzner et al. (1999a), Sections 2.2 and 2.4, Delbaen (2002), Definition 3, Delbaen (2000), Definition 2.1) translate for coherent risk-adjusted values into:

- monotonicity: for all X and Y , if $X \geq Y$ then $\pi(X) \geq \pi(Y)$,
- translation invariance: if a is a constant then for all X , $\pi(a+X) = a+\pi(X)$,
- positive homogeneity: if $\lambda \geq 0$ then for all X , $\pi(\lambda \cdot X) = \lambda \cdot \pi(X)$,
- superadditivity: for all X and Y , $\pi(X + Y) \geq \pi(X) + \pi(Y)$.

With a general probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$ a continuity property is in order. The Fatou property for a risk-adjusted value π is defined as: for any sequence $(X_n)_{n \geq 1}$ of functions, $\|X_n\|_\infty \leq 1$, and converging to X in probability, $\pi(X) \geq \limsup \pi(X_n)$, see Delbaen (2002), Definition 4 or Delbaen(2000), Definition 3.1.

The representation result (see Proposition 4.1 in Artzner et al. (1999a) for the case of a finite Ω , Theorem 6 in Delbaen (2002) or Theorem 3.2 in Delbaen (2000) for the infinite Ω case) states the following:

One-period result. *For any coherent risk-adjusted value $\pi: L^\infty \rightarrow \mathbb{R}$, having the Fatou property, there exists a convex $L^1(\mathbb{P}_0)$ -closed set \mathcal{P} of \mathbb{P}_0 -absolutely continuous probabilities on (Ω, \mathcal{F}) called generalised scenarios or test probabilities, such that $\pi(X) = \inf_{\mathbb{P} \in \mathcal{P}} \mathbf{E}_{\mathbb{P}}[X]$.*

A future value X is accepted at the initial date if and only if for each test probability $\mathbb{P} \in \mathcal{P}$ we have $\mathbf{E}_{\mathbb{P}}[X] \geq 0$.

2. RISK-MEASUREMENT AT INITIAL AND FOLLOWING DATES

This section explores the general way of entering the multi-period world: searching first for an *initial* supervision of the risk resulting from strategies, and then

defining measurements at future random times, providing thereby a risk measurement *process*.

Here we only treat the discrete time case, see Cheredito, Delbaen and Kupper (2002) for the continuous time case.

2.1 Multiperiod information.

Uncertainty over finitely or countably many periods will be described by the probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$ and by the filtration $(\mathcal{F}_n)_{0 \leq n < \infty}$ such that $\bigvee_n \mathcal{F}_n = \mathcal{F}$.

Notation. *In this paper, particularly Sections 2 and 4, the domain where we attempt to define risk measurement is the set, denoted \mathcal{G} , of bounded adapted processes on the filtered space $(\Omega, (\mathcal{F}_n)_{0 \leq n < \infty}, \mathbb{P}_0)$. They will be called “value” processes.*

Once a numéraire has been chosen, *many* interpretations of the meaning of “value” can be made, keeping in mind that for risk supervision purposes we always consider *net* positions:

- market values of equity,
- accounting values of equity i.e. shareholders’ equity or “book value”,
- liquidation values,
- “shareholders’ surplus” as initial capital plus premium received and invested minus claims paid or reserved, in insurance; see 5.3 in Artzner et al. (1999a)
- ...

We thus deal with computations which are both prospective (marking to market, reserving) and retrospective (cumulatively “counting the beans” received or given at each date-event). The generality of the meaning of values will allow us to use the theory for risk management with risk-measurements by various classes of stakeholders: creditors or regulators, managers, and shareholders, see Artzner et al. (1999b) for such a (one-period) study. This generality also allows to consider individual decision theory in the framework of risk measurement, as was initiated by Heath (1998).

The mathematics of the one-period theory is used by looking at processes as special functions on the set of (date-event) pairs as follows: value processes are functions on the product set of states and dates, which are measurable for the optional sigma-field. We use the one-period representation result and interpret the test probabilities in this context. Some of the axioms get a very specific temporal implication (see for example a *different* translation invariance property in the related paper of Riedel (2002) bringing full axiomatics of multiperiod risk measurement, in the case of a finite state space, and the related consequence at the end of our Section 4.2).

Definition 2.1. *(transforming processes into variables) Given a probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$ and a filtration $(\mathcal{F}_n)_{0 \leq n < \infty}$, the probability space $(\Omega', \mathcal{F}', \mathbb{P}'_0)$ is built as follows: Ω' is the product space $\Omega \times \mathbb{N}$, \mathcal{F}' is the σ -field on Ω' whose elements are unions of families $(A_n \times \{n\})_n$, $0 \leq n < \infty$, with $A_n \in \mathcal{F}_n$ for each n . The reference probability \mathbb{P}'_0 is defined on (Ω', \mathcal{F}') by the equality:*

$$\mathbb{P}'_0 \left[\bigcup_{0 \leq n < \infty} \{A_n\} \times \{n\} \right] = \sum_{0 \leq n < \infty} \mu_n \mathbb{P}_0 [A_n],$$

with $(\mu_n)_n$ a given sequence where for each n , $\mu_n > 0$, and $\sum_{0 \leq n < \infty} \mu_n = 1$.

The set \mathcal{G} of bounded adapted processes, becomes the set of bounded measurable functions on the measurable space (Ω', \mathcal{F}') . On \mathcal{G} , we use the topology coming from

the duality between $L^1(\Omega', \mathcal{F}', \mathbb{P}'_0)$ and $L^\infty(\Omega', \mathcal{F}', \mathbb{P}'_0)$. A set is closed in \mathcal{G} if and only if it is closed with respect to the weak* topology coming from $L^1(\Omega', \mathcal{F}', \mathbb{P}'_0)$. The case of unbounded value processes is the topic of on-going research.

2.2 The cone of value processes acceptable at the initial date.

A supervisor, risk manager or regulator, will as in the one-period case decide at date 0 upon a subset of \mathcal{G} denoted \mathcal{A}_{cc} , of acceptable value processes.

Definition 2.2. (*coherent acceptance set*) A coherent acceptance set of value processes is a closed convex cone of \mathcal{G} , with vertex at the origin, containing the positive orthant and intersecting the negative orthant only at the origin.

As in the framework of one-period risk we define

Definition 2.3. (*coherent risk measure*) The risk adjusted value associated with the coherent cone \mathcal{A}_{cc} is defined by $\pi(X) = \sup\{m \mid X - m \in \mathcal{A}_{cc}\}$.

The coherence assumption on \mathcal{A}_{cc} ensures that the associated risk adjusted valuation is “coherent”, i.e. it satisfies the axioms listed at the end of Section 1.

Remark. As in Artzner et al. (1999a), Remark 2.5, acceptability is a numéraire invariant concept while risk-adjusted value is not.

As in Delbaen (2000, 2002) we have the following link between the Fatou property and the closedness of acceptance sets:

Proposition 2.1. For a convex cone $\mathcal{A}_{cc} \subset \mathcal{G}$, containing the cone of nonnegative processes, the following statements are equivalent

- (1) \mathcal{A}_{cc} is $\sigma(\mathcal{G}, L^1(\mathbb{P}'_0))$ closed
- (2) the associated measurement π satisfies the Fatou property: if $X^k \in \mathcal{G}$ is a sequence uniformly bounded by 1 and such that $X^k \rightarrow X$ in \mathbb{P}'_0 probability, then $\pi(X) \geq \overline{\lim} \pi(X^k)$
- (3) the associated measurement π satisfies the property: if X^k is a sequence in \mathcal{G} , $\|X^k_n\|_\infty \leq 1$ and such that for all n , $X^k_n \rightarrow X_n$ in probability, then $\pi(X) \geq \overline{\lim} \pi(X^k)$.

2.3 Translation of the one-period representation result.

Thanks to the identification between value processes and random variables on (Ω', \mathcal{F}') we *directly* deduce from the study of the one-period case (see Section 1) that, given a coherent risk adjusted value measurement π^0 with the Fatou property, there is a closed convex set \mathcal{P}' of probabilities on (Ω', \mathcal{F}') , absolutely continuous w.r.t. \mathbb{P}'_0 , such that:

$$\text{for each } X \in \mathcal{G}, \pi^0(X) = \inf_{\mathbb{Q}' \in \mathcal{P}'} \mathbf{E}_{\mathbb{Q}'} [X].$$

Remark. One could consider more general acceptance sets than convex cones, as was done in Föllmer and Schied (2002a) (see also their use in Heath (1998), to represent the risk measurement *affine* constraints imposed by the shareholders of a firm).

Each “test probability” $\mathbb{Q}' \in \mathcal{P}'$ above can be described by its density f' with respect to \mathbb{P}'_0 . This density must first of all be a random variable on (Ω', \mathcal{F}') which we can represent as $f' = (f_n)_{0 \leq n < \infty}$ where each f_n is a non-negative and \mathcal{F}_n -measurable function on Ω such that $\sum_{0 \leq n < \infty} \mu_n \mathbf{E}_{\mathbb{P}'_0} [f_n] = 1$. We then have for

each $X = (X_n)_{0 \leq n < \infty}$, $\mathbf{E}_{\mathbb{Q}'}[X] = \sum_{0 \leq n < \infty} \mu_n \mathbf{E}_{\mathbb{P}_0}[f_n X_n]$. Defining the increasing adapted process A by $A_n = A_{n-1} + \mu_n f_n$, with $A_{-1} = 0$, we get that $\mathbf{E}_{\mathbb{P}_0}[A_\infty] = 1$ and we obtain the representation result by minimizing over the set \mathcal{A} of processes corresponding to the set of test probabilities $\mathbb{P}' \in \mathcal{P}'$:

Proposition 2.2. *For each “initial date” coherent risk-adjusted valuation π^0 of value processes with the Fatou property, there is a closed convex set \mathcal{A} of positive adapted non-decreasing processes A with $\mathbf{E}_{\mathbb{P}_0}[A_\infty] = 1$ such that for each value process X :*

$$\pi^0(X) = \inf_{A \in \mathcal{A}} \mathbf{E}_{\mathbb{P}_0} \left[\sum_{0 \leq n < \infty} X_n \cdot (A_n - A_{n-1}) \right].$$

Remark 1. The set \mathcal{A} can be taken to be a closed set in the normed space of processes with integrable variation, where $\|A\| = \mathbf{E}_{\mathbb{P}_0} \left[|A_0| + \sum_{0 \leq n < \infty} |A_{n+1} - A_n| \right]$.

Remark 2. If, with N periods, the value process X is a \mathbb{P}_0 -martingale, the general expression for $\pi^0(X)$ simplifies to $\inf_{A \in \mathcal{A}} \mathbf{E}_{\mathbb{P}_0}[X_N A_N]$. In words, for our risk measurement method, “only the final value matters” when dealing with \mathbb{P}_0 -martingales. Being defined on the whole of \mathcal{G} , our method handles more general business conditions than “marking to (complete) markets”.

Remark 3. The reader may wonder what kind of multiperiod behaviour is dealt with in our approach. Works like Cvitanić and Karatzas (1999), Nakano (2003), have also several periods for portfolio construction, and measure the risk of *final* wealths which may depend on trajectories. But they do not take into account what happens at intermediate dates. The trajectories have no influence except that they define the final values. Remark 2 above explains a contrario the difference and the first two examples below will illustrate. The Föllmer and Schied (2002a) paper has a very special acceptance cone, deduced from the positive orthant by adding the convex set of securities attainable via the allowed strategies at zero price: this may correspond to an “early warning” function of risk measurement, where the supervisor requires perfect hedging with given instruments. The paper by Jaschke and Küchler (2001) mentions processes as potential objects of risk measurement but does not present either the switch from Ω to Ω' (this Section) nor the use of the representing test probabilities (our Section 4).

2.4 Examples of initial measurement of multiperiod risk.

Example 2.1.

The measurement π^0 defined for each value process X by $\pi^0(X) = \mathbf{E}_{\mathbb{P}_0}[X_\sigma]$, σ a stopping time, is a coherent risk-adjusted measurement, describing “how good or bad processes are at date σ ”.

Define, if $\mathbb{P}_0[\sigma < \infty] > 0$, the non-decreasing process A^σ by $A_n^\sigma = \frac{1}{\mathbb{P}_0[\sigma < \infty]} \mathbf{1}_{\{\sigma \leq n\}}$. Then for each X , $\pi^0(X) = \mathbf{E}_{\mathbb{P}_0} \left[\sum_{0 \leq n < \infty} X_n (A_n^\sigma - A_{n-1}^\sigma) \right]$.

Example 2.2.

The measurement π^0 defined by $\pi^0(X) = \mathbf{E}_{\mathbb{P}_0}[\inf_{0 \leq n \leq N} X_n]$ is a coherent risk-adjusted measurement related to the random times $\bar{\tau}_X$ where

$$\bar{\tau}_X(\omega) = \operatorname{argmin}_{n: n \leq N} (X_n(\omega)).$$

Since dates of extrema are not necessarily *stopping* times, it is a challenge to represent π^0 with *adapted* processes! To do this we define for any *random* time τ the process A^τ where $A_n^\tau = A_{n-1}^\tau + \mathbf{E}_{\mathbb{P}_0}[\mathbf{1}_{C_n} | \mathcal{F}_n]$ with $C_n = \{\tau = n\}$. Denote by \mathcal{A} the closed convex hull of all such A^τ . For each process X we have $\mathbf{E}_{\mathbb{P}_0}[\sum_{0 \leq n \leq N} X_n \cdot \mathbf{1}_{C_n}] = \mathbf{E}_{\mathbb{P}_0}[\sum_{0 \leq n \leq N} X_n \cdot (A_n^\tau - A_{n-1}^\tau)]$. Using the random times $\bar{\tau}_X$ defined above we find that π^0 is given by

$$\pi^0(X) = \inf_{A^\tau \in \mathcal{A}} \mathbf{E}_{\mathbb{P}_0} \left[\sum_{0 \leq n \leq N} X_n \cdot (A_n^\tau - A_{n-1}^\tau) \right].$$

Example 2.3.

The measurement π^0 defined by $\pi^0(X) = \frac{1}{N+1} \mathbf{E}_{\mathbb{P}_0}[\sum_{0 \leq n \leq N} X_n]$ is coherent (it would not fulfill the “translation invariance” property of Riedel (2002)).

2.5 Extension of a risk measurement to future dates.

To keep the presentation simple we suppose that the time interval is finite and we also suppose that the risk adjusted measurements satisfy the Fatou property.

Proposition 2.3. *For each coherent risk adjusted measurement π^0 , each stopping time τ , and each value process X the set of all \mathcal{F}_τ -measurable random variables*

$$\{f \mid f \text{ is } \mathcal{F}_\tau\text{-measurable and for all } A \in \mathcal{F}_\tau : \pi^0(\mathbf{1}_A \mathbf{1}_{[\tau, \infty]}(X - f)) \geq 0\}$$

is closed under taking the maximum of a finite number of elements and it has a maximal element, denoted by $\pi_\tau^0(X)$ and called the risk-adjusted value of the process X at the date τ .

The proof of this statement is left to the reader.

Proposition 2.4. *For each stopping time τ the map π_τ is*

- (1) *The mapping π_τ^0 is positive, super-additive, positively homogeneous,*
- (2) *For each g bounded and \mathcal{F}_τ -measurable we have*

$$\pi_\tau^0(X + g \mathbf{1}_{[\tau, N]}) = \pi_\tau^0(X) + g.$$

- (3) *The mapping π_τ^0 is Fatou, more precisely: for each sequence of processes X^n such that $\|X^n\| \leq 1$ and for all $1 \leq t \leq N$, $X_t^n \rightarrow X_t$ (a.s.), we have $\pi_\tau^0(X) \geq \limsup \pi_\tau^0(X^n)$, where the lim sup is taken in probability.*

3. PASTING OF TEST PROBABILITIES

In contrast with Section 2, we shall keep the test probabilities on the base state space Ω and shall use extensively conditional expectations for “updating” the measurement of risk as time evolves.

We thank J. Hugonnier for pointing out the work of Epstein and Schneider (2003) on the occasion of the presentation of these ideas in the talk Artzner et al. (2001). There is indeed a touch of utility theory in our work since the cone of values acceptable to the supervisor and the translation invariance define a, special, preference order (see also Heath (1998)). We also thank L. Epstein for a discussion on this topic. See also the paper Wang (2003) and the brief discussion of it in Section 3.2 of Epstein and Schneider (2003).

Nevertheless several points singularize our Sections 3 and 4, since in Epstein and Schneider (2003):

- preferences are on lottery-valued processes,
- preferences are not required to be translation-invariant,
- preferences are not required to be monotonic,
- the state space is finite.

Notation. For the remainder of the paper we consider only N periods of uncertainty (see Delbaen (2001) for a continuous time approach) and work with a closed convex set \mathcal{P} of “test” probabilities on (Ω, \mathcal{F}_N) absolutely continuous with respect to \mathbb{P}_0 . For convenience we can assume that the reference probability \mathbb{P}_0 belongs to the chosen set \mathcal{P} .

This section prepares for two specific multiperiod-measurements of risk. We shall combine test probabilities on Ω by identifying them to their densities with respect to the reference probability \mathbb{P}_0 and use the likelihood ratio processes.

To avoid problems with probabilities $\mathbb{Q} \in \mathcal{P}$ which are only absolutely continuous with respect to \mathbb{P}_0 but not equivalent to it, we often use the convex dense set \mathcal{P}^e of elements of \mathcal{P} which are equivalent to \mathbb{P}_0 .

3.1. Stability of the set of test probabilities.

Any closed convex set \mathcal{P} of test probabilities on (Ω, \mathcal{F}_N) absolutely continuous with respect to \mathbb{P}_0 is identified with the closed convex set $\mathcal{Z}_{\mathcal{P}}$ of all the \mathbb{P}_0 -martingales $Z_n^{\mathbb{Q}} = \mathbf{E}_{\mathbb{P}_0} \left[\frac{d\mathbb{Q}}{d\mathbb{P}_0} | \mathcal{F}_n \right]$, $0 \leq n \leq N$, $\mathbb{Q} \in \mathcal{P}$, see Chow, Robbins and Siegmund (1972), Section 2.1, Dothan (1990), Sections 5.4, 6.4.

Definition 3.1. (pasting, stability) We say that the set \mathcal{P} of test probabilities is stable if for elements $\mathbb{Q}^0, \mathbb{Q} \in \mathcal{P}^e$ with associated martingales Z_n^0, Z_n , and for each stopping time τ the martingale L defined as $L_n = Z_n^0$ for $n \leq \tau$ and $L_n = Z_\tau^0 \frac{Z_n}{Z_\tau}$ for $n \geq \tau$ defines an element of \mathcal{P} called the result of pasting \mathbb{Q}^0 and \mathbb{Q} .

The following lemma rephrases the definition of stability:

Lemma 3.1. The stability of the set \mathcal{P} implies that for any three stopping times $\tau \leq \sigma \leq \nu$, the following two sets are equal:

$$\left\{ \left(\frac{Z_\nu}{Z_\sigma}, \frac{Z_\sigma}{Z_\tau} \right) \mid Z \in \mathcal{P}^e \right\} = \left\{ \left(\frac{Z'_\nu}{Z'_\sigma}, \frac{Z_\sigma}{Z_\tau} \right) \mid Z \in \mathcal{P}^e, Z' \in \mathcal{P}^e \right\}.$$

3.2 An example of a stable set of test probabilities.

Let the information structure be given by a binomial tree and a random walk $W_n = U_1 + \dots + U_n$, $0 \leq n \leq N$, the $(U_n)_{0 \leq n \leq N}$ being ± 1 valued independent variables with symmetric distribution. If \mathbb{P}_0 denotes the resulting measure on (Ω, \mathcal{F}_N) , any probability \mathbb{Q} on (Ω, \mathcal{F}_N) has its density with respect to \mathbb{P}_0 written as $Z = \prod_{0 \leq n \leq N} (1 + q_n U_n)$ with the $(q_n)_{0 \leq n \leq N}$ being a predictable process with $|q_n| \leq 1$.

It can then be shown that stability of a closed convex set \mathcal{P} of test probabilities is equivalent to representability of the density processes of elements of \mathcal{P} , meaning that for each n , $0 \leq n \leq N$, there exists a closed convex set \mathcal{Q}_n of $[-1, +1]$, described in an \mathcal{F}_{n-1} -measurable way, such that the density $Z = \prod_{0 \leq n \leq N} (1 + q_n U_n)$ is an

element of $\mathcal{Z}_{\mathcal{P}}$ if and only if each q_n belongs to \mathcal{Q}_n . The continuous time case is treated in Delbaen (2001) with stochastic exponentials.

A special case of the example is obtained by taking for some d , $0 \leq d \leq 1$, all the sets \mathcal{Q}_n to be the interval $[-d, +d]$. The set \mathcal{P} is then the set of all probabilities \mathbb{Q} such that $\frac{d\mathbb{Q}}{d\mathbb{P}_0} = Z_N$ satisfies $Z_n = \mathbf{E}_{\mathbb{P}_0} [Z_N | \mathcal{F}_n] = (1 + q_1 U_1) \dots (1 + q_n U_n)$ where q is a predictable process with $|q| \leq d \leq 1$. This is indeed a stable set.

4. TWO SPECIFIC MULTIPERIOD-MEASUREMENTS OF MULTIPERIOD-RISK

In contrast with the axiomatic approach of Section 2, we build directly *processes* of risk-adjusted values by using a set \mathcal{P} of test probabilities on (Ω, \mathcal{F}_N) . We present two constructions and show that they are identical if and only if \mathcal{P} is stable. Moreover, in this case, the induced initial risk measurement returns, via the construction of Section 2.5, the multiperiod measurement of the present Section.

4.1 A risk-adjusted value process as a generalized Snell envelope.

Given test probabilities on Ω , a value process may well be a martingale under *some* of these but *risk-adjusted* value processes should only be required to be submartingales under each test probability: since information has been provided, uncertainty of the remaining part decreases and risk-adjusted value increases “on (conditional) average”.

Since there is no reason to over-penalize a value process, we want to risk-adjust it by the largest possible process. The theory of the Snell envelope provides the convenient framework (see Chow, Robbins and Siegmund (1972), Section 1.6 and Theorem 4.7, Neveu (1972), Propositions VI.1.1 and VI-1-2, for this as well and for a precise treatment of essential infimum).

Theorem 4.1. *Given the set \mathcal{P} of test probabilities there is for each value process $X \in \mathcal{G}$ a unique process $\bar{\Psi}(X)$ which is the largest of all processes $Y \in \mathcal{G}$ with the following two properties:*

- (i) $Y \leq X$ as processes,
- (ii) for each test probability $\mathbb{Q} \in \mathcal{P}$, Y is a \mathbb{Q} -submartingale.

It is given by the formula

$$\bar{\Psi}_N(X) = X_N; \text{ for } 0 \leq n < N, \bar{\Psi}_n(X) = X_n \wedge \text{ess. inf}_{\mathbb{Q} \in \mathcal{P}^e} \mathbf{E}_{\mathbb{Q}} [\bar{\Psi}_{n+1}(X) | \mathcal{F}_n].$$

Proof. It is clear that $\bar{\Psi}(X)$ satisfies (i) and (ii); for any other process Y with (i) and (ii) satisfied we have as soon as $Y_{n+1} \leq \bar{\Psi}_{n+1}(X)$, for each $\mathbb{Q} \in \mathcal{P}$, $Y_n \leq \mathbf{E}_{\mathbb{Q}} [Y_{n+1} | \mathcal{F}_n] \leq \mathbf{E}_{\mathbb{Q}} [\bar{\Psi}_{n+1}(X) | \mathcal{F}_n]$. These inequalities being valid for each \mathbb{Q} prove, together with $Y_n \leq X_n$, that $Y_n \leq \bar{\Psi}_n(X)$.

4.2 Stability of test probabilities and Bellman’s principle.

We leave the proof of the following lemma to the reader:

Lemma 4.1. *For a fixed stopping time σ and for a stable set \mathcal{P} , the set*

$$\{\mathbf{E}_{\mathbb{Q}}[X_{\tau} | \mathcal{F}_{\sigma}] \mid \tau \geq \sigma \text{ is a stopping time and } \mathbb{Q} \in \mathcal{P}^e\}$$

is closed for taking minima (also maxima).

Corollary. *If \mathcal{P} is stable we also have for every stopping time $0 \leq \sigma$ and every probability measure $\mu \ll \mathbb{P}_0$ (not necessarily in \mathcal{P}) that*

$$\begin{aligned} \mathbf{E}_\mu[\text{ess. inf}_{\mathbb{Q} \in \mathcal{P}^e} \mathbf{E}_\mathbb{Q}[X_\tau \mid \mathcal{F}_\sigma]] \\ = \inf \{ \mathbf{E}_\mu[\mathbf{E}_\mathbb{Q}[X_\tau \mid \mathcal{F}_\sigma]] \mid \mathbb{Q} \in \mathcal{P}^e; \sigma \leq \tau \text{ stopping time} \}. \end{aligned}$$

The corollary follows from Lemma 4.1 by using Proposition VI-1-1 of Neveu (1972).

Theorem 4.2. *Given the set \mathcal{P} of test probabilities, for each value process $X \in \mathcal{G}$, the family $(\Psi_\sigma(X))$, σ a stopping time, where:*

$$\Psi_\sigma(X) = \text{ess. inf} \{ \mathbf{E}_\mathbb{Q}[X_\tau \mid \mathcal{F}_\sigma] \mid \tau \geq \sigma \text{ is a stopping time, } \mathbb{Q} \in \mathcal{P}^e \}.$$

defines a process $\Psi(X)$.

The stability of \mathcal{P} is equivalent to each of the following

- (i) for each $\mathbb{Q} \in \mathcal{P}$ and each value process $X \in \mathcal{G}$, $\Psi(X)$ is a \mathbb{Q} -submartingale,
- (ii) the mappings Ψ and $\bar{\Psi}$ are equal,
- (iii) “Bellman’s principle”: for each $X \in \mathcal{G}$ and stopping times $\sigma, \tau, \sigma \leq \tau$, we have $\Psi_\sigma(X) = \Psi_\sigma(X^{\tau-} + \Psi_\tau(\tau X)\mathbf{1}_{[\tau, N]})$.

Remark. We recall the definitions of $X^{\tau-}$ and τX as $\tau X_n = 0$ for $n < \tau$, $\tau X_n = X_n - X_{\tau-1}$ for $n \geq \tau$, $X_n^{\tau-} = X_n$ for $n < \tau$, $X_n^{\tau-} = X_{\tau-1}$ for $n \geq \tau$. We remark that $\tau(\tau X) = \tau X$, $(\tau X)^{\tau-} = 0$ and that $\Psi_\tau(X) = \Psi_\tau(\tau X) + X_{\tau-1}$.

Proof.

- (1) We first show that $\Psi \geq \bar{\Psi}$, which will ensure that if $\Psi(X)$ is — for each X and each $\mathbb{Q} \in \mathcal{P}$ — a \mathbb{Q} -submartingale, $\Psi(X) = \bar{\Psi}(X)$ (use Theorem 4.1, i.e. if (i) is satisfied so is (ii)). On the other hand (i) follows from (ii) by construction of $\bar{\Psi}$. Since for each $\mathbb{Q} \in \mathcal{P}$, $\bar{\Psi}(X)$ is a \mathbb{Q} -submartingale, for each stopping time τ , $\bar{\Psi}_\sigma(X) \leq \mathbf{E}_\mathbb{Q}[\bar{\Psi}_\tau(X) \mid \mathcal{F}_\sigma] \leq \mathbf{E}_\mathbb{Q}[X_\tau \mid \mathcal{F}_\sigma]$. Taking ess. inf over \mathbb{Q} and τ provides the stated inequality.
- (2) We show now that (i) implies stability of \mathcal{P} . We suppose that Z^1 and Z^2 are two elements in $\mathcal{Z}_{\mathcal{P}^e}$ — coming from the measures $\mathbb{Q}^1, \mathbb{Q}^2$ — and σ is a stopping time. Also suppose that the element $Z_\sigma^1 \frac{Z_N^2}{Z_\sigma^2}$ is not in the closed convex set \mathcal{P} . By the separation theorem for convex sets there is a random variable $f \in L^\infty$, so that

$$\mathbf{E}_\mathbb{P} \left[Z_\sigma^1 \frac{Z_N^2}{Z_\sigma^2} f \right] < \inf_{\mathbb{Q} \in \mathcal{P}} \mathbf{E}_\mathbb{Q} [f].$$

Defining the process X by $X_N = f, X_n = \|f\|$ for $n < N$, we find that $\Psi_0(X) = \inf_{\mathbb{Q} \in \mathcal{P}} \mathbf{E}_\mathbb{Q} [f]$. The left hand side of the displayed inequality can be written as $\mathbf{E}_{\mathbb{Q}^1} [\mathbf{E}_{\mathbb{Q}^2} [f \mid \mathcal{F}_\sigma]]$.

This is clearly at least equal to $\mathbf{E}_{\mathbb{Q}^1} [\Phi_\sigma(X)]$, a quantity which by the \mathbb{Q}^1 -submartingale property is at least equal to $\Psi_0(X)$. This is a contradiction since the right hand side of the displayed inequality is precisely $\Psi_0(X)$.

- (3) Let us prove that stability implies Bellman's principle (iii). To avoid complicated notation, the measures $\mathbb{Q}, \mathbb{Q}', \mathbb{Q}''$ are taken in \mathcal{P}^e . By definition

$$\begin{aligned}
\Psi_\sigma(X) &= \text{ess. inf}_{\mathbb{Q}, \nu \geq \sigma} \mathbf{E}_{\mathbb{Q}}[X_\nu | \mathcal{F}_\sigma] \\
&= \text{ess. inf}_{\mathbb{Q}, \nu \geq \sigma} \mathbf{E}_{\mathbb{Q}}[\mathbf{E}_{\mathbb{Q}}[X_\nu | \mathcal{F}_\tau] | \mathcal{F}_\sigma] \\
&= \text{ess. inf}_{\mathbb{Q}, \nu \geq \sigma} \mathbf{E}_{\mathbb{Q}}[X_\nu \mathbf{1}_{\nu < \tau} + \mathbf{E}_{\mathbb{Q}}[X_\nu \mathbf{1}_{\nu \geq \tau} | \mathcal{F}_\tau] | \mathcal{F}_\sigma] \\
&= \text{ess. inf}_{\mathbb{Q}, \nu \geq \sigma} \mathbf{E}_{\mathbb{Q}}[X_\nu \mathbf{1}_{\nu < \tau} + \mathbf{1}_{\nu \geq \tau}(X_{\tau-} + \mathbf{E}_{\mathbb{Q}}[X_\nu - X_{\tau-} | \mathcal{F}_\tau]) | \mathcal{F}_\sigma]
\end{aligned}$$

The Corollary of Lemma 4.1 together with Lemma 3.1 allow us to rewrite the result of the simple ess. inf as a compounded expression:

$$\begin{aligned}
\Psi_\sigma(X) &= \text{ess. inf}_{\mathbb{Q}', \nu \geq \sigma} \mathbf{E}_{\mathbb{Q}'}[X_\nu \mathbf{1}_{\nu < \tau} \\
&\quad + \mathbf{1}_{\nu \geq \tau} \text{ess. inf}_{\mathbb{Q}'', \nu' \geq \tau} (X_{\tau-} + \mathbf{E}_{\mathbb{Q}''}[X_{\nu'} - X_{\tau-} | \mathcal{F}_\tau]) | \mathcal{F}_\sigma] \\
&= \text{ess. inf}_{\mathbb{Q}', \nu \geq \sigma} \mathbf{E}_{\mathbb{Q}'}[X_\nu \mathbf{1}_{\nu < \tau} + \mathbf{1}_{\nu \geq \tau}(X_{\tau-} + \Psi_\tau({}^\tau X) | \mathcal{F}_\sigma] \\
&= \text{ess. inf}_{\mathbb{Q}', \nu \geq \sigma} \mathbf{E}_{\mathbb{Q}'}[(X^{\tau-} + \Psi_\tau({}^\tau X))_\nu | \mathcal{F}_\sigma] \\
&= \Psi_\sigma(X^{\tau-} + \Psi_\tau({}^\tau X) \mathbf{1}_{[\tau, N]})
\end{aligned}$$

- (4) Bellman's principle (iii) ensures that

$$\begin{aligned}
\Psi_\sigma(X) &\leq \Psi_\sigma(\|X\| + \Psi_\tau({}^\tau X) \mathbf{1}_{[\tau, N]}) \\
&\leq \text{ess. inf}_{\mathbb{Q} \in \mathcal{P}^e} \{\mathbf{E}_{\mathbb{Q}}[\Psi_\tau({}^\tau X) | \mathcal{F}_\sigma] | \mathbb{Q} \in \mathcal{P}^e\}
\end{aligned}$$

hence for each $\mathbb{Q} \in \mathcal{P}$, $\Psi_\sigma(X) \leq \mathbf{E}_{\mathbb{Q}}[\Psi_\tau(X) | \mathcal{F}_\sigma]$, so (i) is a consequence of (iii) and the proof of Theorem 4.2 is complete.

Remark 1. The risk measurement of Example 2.3, Section 2, satisfies Bellman's principle but it is not of the form Ψ_0 for any set \mathcal{P} of test probabilities.

Remark 2. With our sign convention the risk-adjustment measurement of *cash-flows* streams in Riedel (2002) would appear as

$$\lambda_t(D) = \text{ess. inf}_{\mathbb{Q} \in \mathcal{P}} \mathbf{E}_{\mathbb{Q}} \left[\sum_{t \leq k \leq N} D_k | \mathcal{F}_t \right],$$

that is in terms of cumulative cash-flows

$$\mu_t(Y) = \text{ess. inf}_{\mathbb{Q} \in \mathcal{P}} \mathbf{E}_{\mathbb{Q}} [Y_N - Y_{t-1} | \mathcal{F}_t].$$

This is in contrast with our theory which, dealing with

$$\Psi_t(X) = \text{ess. inf}_{\mathbb{Q} \in \mathcal{P}, \sigma \geq t} \mathbf{E}_{\mathbb{Q}} [X_\sigma | \mathcal{F}_t],$$

does not only look at final values but also at the whole trajectories.

4.3 Time consistent set of test probability sets for value processes.

Definition 4.1. A set \mathcal{P} of test probabilities is time consistent for value processes if for each pair of value processes $X \in \mathcal{G}$ and $Y \in \mathcal{G}$ and each pair of stopping times σ and τ , $\sigma \leq \tau$ the conditions

- (i) $X^{\tau-} = Y^{\tau-}$,
- (ii) $\Psi_\tau(X) \leq \Psi_\tau(Y)$

provide the inequality $\Psi_\sigma(X) \leq \Psi_\sigma(Y)$.

The notion goes back to Wang (1996, 1999), and we present it more generally, in terms of random (stopping) times and test probabilities.

The following theorem identifies stability with time consistency for value processes. It is related to decision theory with multipriors, see again Epstein and Schneider (2003). A corresponding version for risk measurement of final values, related in the finite Ω case to Riedel (2002) and Roorda, Engwerda and Schumacher (2002), will be established in Theorem 5.1.

Theorem 4.3. A set \mathcal{P} of test probabilities is stable if and only if it is time consistent for value processes.

Proof. If \mathcal{P} is time consistent we define for each $X \in \mathcal{G}$ and each stopping time τ , the process $Y = X^{\tau-} + \Psi_\tau({}^\tau X)\mathbf{1}_{[\tau, N]}$ and we find that $X^{\tau-} = Y^{\tau-}$, $\Psi_\tau(X) = \Psi_\tau(Y)$. By time consistency we find for each stopping time σ , $\sigma \leq \tau$ that $\Psi_\sigma(X) = \Psi_\sigma(Y)$, which proves Bellman's principle.

For a stable \mathcal{P} , Bellman's principle together with conditions (i), (ii), and monotonicity of Ψ_σ ensures that

$$\begin{aligned} \Psi_\sigma(X) &= \Psi_\sigma(X^{\tau-} + \Psi_\tau({}^\tau X)\mathbf{1}_{[\tau, N]}) \\ &= \Psi_\sigma(Y^{\tau-} + \Psi_\tau(X)\mathbf{1}_{[\tau, N]}) \\ &\leq \Psi_\sigma(Y^{\tau-} + \Psi_\tau(Y)\mathbf{1}_{[\tau, N]}) \\ &= \Psi_\sigma(Y^{\tau-} + \Psi_\tau({}^\tau Y)\mathbf{1}_{[\tau, N]}) \\ &= \Psi_\sigma(Y), \end{aligned}$$

which proves time consistency.

4.4 Relation with the construction in Section 2.

Proposition 4.1. For any set \mathcal{P} of test probabilities, the deduced initial risk-adjusted measurement Ψ_0 again generates Ψ via the construction defined in Propositions 2.3, 2.4.

Proof. We indeed have for each value process X and each stopping time ρ ,

$$\begin{aligned} \psi_\rho(X) &= \text{ess. inf}_{\mathbb{Q} \in \mathcal{P}, \tau \geq \rho} \mathbf{E}_{\mathbb{Q}}[X_\tau \mid \mathcal{F}_\rho] \\ &= \text{ess. sup} \{g \in \mathcal{F}_\rho \mid \text{for all } \mathbb{Q} \in \mathcal{P} \text{ and all } \tau \geq \rho : g \leq \mathbf{E}_{\mathbb{Q}}[X_\tau \mid \mathcal{F}_\rho]\}, \end{aligned}$$

The last condition on g can also be written as: for each $A \in \mathcal{F}_\rho$, $\mathbb{Q} \in \mathcal{P}^e$, $\tau \geq \rho$, $\mathbf{E}_{\mathbb{Q}}[\mathbf{1}_A(X - g\mathbf{1}_{[\rho, N]})_\tau] \geq 0$. This means that $\psi_0(\mathbf{1}_A(X - g\mathbf{1}_{[\rho, N]})) \geq 0$.

5. MULTIPERIOD-MEASUREMENT OF FINAL VALUES

This section deals with *multiperiod* risk-measurement of a “final value” (bounded \mathcal{F}_N -measurable random variable) for which a whole *process* of risk-adjusted values will be built. One may consult Wilkie, Waters and Yang (2003), Sections 3.4.2 and 3.4.3, for an example of early discussion (ca. 1980) of multiperiod “reserving” i.e. capital requiring for such a “locked-in” position.

5.1 Direct and derived constructions of a risk-adjusted value process.

In earlier versions of this paper we gave in detail, out of a given set \mathcal{P} of test probabilities, two constructions of processes of risk-adjusted values. Here we shall rather connect such constructions to the definitions of Ψ and $\bar{\Psi}$ of Sections 4.1 and 4.2. For this we use the following

Definition 5.1. *A bounded \mathcal{F}_N -random variable f is called a “final value” and we denote by X_f the element of \mathcal{G} such that $(X_f)_N = f, (X_f)_n = \|f\|$ for $n < N$. For each stopping time σ we define*

$$\Phi_\sigma(f) = \Psi_\sigma(X_f), \bar{\Phi}_\sigma(f) = \bar{\Psi}_\sigma(X_f).$$

Proposition 5.1. *For each “final value” f we have*

$$\begin{aligned} \Phi_\tau(f) &= \text{ess. inf}_{\mathbb{Q} \in \mathcal{P}^e} \mathbf{E}_{\mathbb{Q}}[f | \mathcal{F}_\tau] \text{ for each stopping time } \tau. \\ \bar{\Phi}_n(f) &= \text{ess. inf}_{\mathbb{Q} \in \mathcal{P}^e} \mathbf{E}_{\mathbb{Q}}[\bar{\Phi}_{n+1}(f) | \mathcal{F}_n], 0 \leq n < N, \bar{\Phi}_N(f) = f. \end{aligned}$$

We leave the proof, based on the following equalities, to the reader

$$\begin{aligned} &\text{ess. inf}_{\mathbb{Q} \in \mathcal{P}^e} \mathbf{E}_{\mathbb{Q}}[f | \mathcal{F}_n] \\ &= \text{ess. sup} \{g \mid \text{for all } \mathbb{Q} \in \mathcal{P} : g \leq \mathbf{E}_{\mathbb{Q}}[f | \mathcal{F}_n], \mathbb{Q} \text{ a.s.}\} \\ &= \text{ess. sup} \{g \mid \text{for all } \mathbb{Q} \in \mathcal{P}^e : g \leq \mathbf{E}_{\mathbb{Q}}[f | \mathcal{F}_n], \mathbb{Q} \text{ a.s.}\}. \end{aligned}$$

5.2 Time consistency of test probability sets for final values.

We want to single out test probability sets for which “acceptability tomorrow in any event” ensures “acceptability today”.

Definition 5.2. *A set \mathcal{P} of test probabilities is called time consistent for final values if for each pair of stopping times $\sigma \leq \tau$ and each pair of final values f, g , we have that $\Phi_\tau(f) \leq \Phi_\tau(g)$ implies that $\Phi_\sigma(f) \leq \Phi_\sigma(g)$.*

Remark. Notice that without any assumption on \mathcal{P} the inequality $\Phi_\tau(g) \geq 0$ implies that for $\sigma \leq \tau, \Phi_\sigma(g) \geq 0$, a weaker form of time consistency. It will be shown in Section 5.3 that the tail value-at-risk measurement does *not* satisfy this weak form of time consistency.

The equivalence between Definition 4.1 and Definition 5.2 will follow from the Theorem 5.1. The following theorem is clearly similar to Theorem 4.2 and identifies stability with time consistency.

Theorem 5.1. *The properties (1) to(5) below are equivalent.*

- (1) *stability of the set \mathcal{P} ,*
- (2) *recursivity: For each final value f , the family of all the $\Phi_\sigma(f)$ satisfies: for every two stopping times $\sigma \leq \tau$ we have $\Phi_\sigma(f) = \Phi_\sigma(\Phi_\tau(f))$,*
- (3) *for each final value f , for every stopping time σ we have $\Phi_0(f) \leq \Phi_0(\Phi_\sigma(f))$,*
- (4) *time consistency of the set \mathcal{P} for final values,*
- (5) *for each final value f the family of the $\Phi_\sigma(f)$ satisfies the submartingale property: for each $\mathbb{Q} \in \mathcal{P}$ and each pair of stopping times $\sigma \leq \tau$ we have that $\Phi_\sigma(f) \leq \mathbf{E}_{\mathbb{Q}}[\Phi_\tau(f) | \mathcal{F}_\sigma]$.*

Remark. Property (2) would be written in terms of risk measures as $\rho_\sigma(f) = \rho_\sigma(-\rho_\tau(f))$. With three periods we would get formulas like $\rho_0(f) = \rho_0(-\rho_1(-\rho_2(f)))$ and it again clear that it is better to work with the “risk-adjusted value” $\pi = -\rho$.

Proof. Let us show that (4) follows from (2). Suppose that for two bounded random variables f, g and two stopping times $\sigma \leq \tau$, we have $\Phi_\tau(f) \leq \Phi_\tau(g)$. Then we have by monotonicity of Φ_σ , if (2) holds, $\Phi_\sigma(f) = \Phi_\sigma(\Phi_\tau(f)) \leq \Phi_\sigma(\Phi_\tau(g)) = \Phi_\sigma(g)$.

Conversely to prove (2) out of (4), we take for the two random variables, the functions f and $g = \Phi_\tau(f)$. We have equality $\Phi_\tau(\Phi_\tau(f)) = \Phi_\tau(f)$ and therefore (applying item(2) twice) that $\Phi_\sigma(\Phi_\tau(f)) = \Phi_\sigma(f)$.

Obviously (2) implies (3).

We show now that (3) implies (1); this is the same as in Theorem 4.2. We suppose that Z^1 and Z^2 are two elements in \mathcal{P}^e – coming from the measures $\mathbb{Q}^1, \mathbb{Q}^2$ – and σ is a stopping time. Also suppose that the element $Z_\sigma^1 \frac{Z_N^2}{Z_\sigma^2}$ is not in the closed convex set \mathcal{P} . By the separation theorem for convex sets there is a random variable $f \in L^\infty$, so that

$$\mathbf{E}_{\mathbb{P}} \left[Z_\sigma^1 \frac{Z_N^2}{Z_\sigma^2} f \right] < \inf_{\mathbb{Q} \in \mathcal{P}} \mathbf{E}_{\mathbb{Q}} [f].$$

We can write the left hand side as $\mathbf{E}_{\mathbb{Q}^1} [\mathbf{E}_{\mathbb{Q}^2} [f | \mathcal{F}_\sigma]]$. This is clearly at least equal to $\mathbf{E}_{\mathbb{Q}^1} [\Phi_\sigma(f)]$, a quantity at least equal to $\Phi_0[\Phi_\sigma(f)]$, hence by property (3), at least equal to $\Phi_0(f)$. This is a contradiction since the right hand side is precisely $\Phi_0(f)$.

If (2) holds we take $g = \Phi_\tau(f)$ in the definition of $\Phi_\sigma(g)$ to find that for each $\mathbb{Q} \in \mathcal{P}$, $\Phi_\sigma(f) \leq \mathbf{E}_{\mathbb{Q}}[\Phi_\tau(f) | \mathcal{F}_\sigma]$ which proves (5).

Suppose now that (5) holds and let us prove (3). For each $f \in L^\infty$ and each pair $\sigma \leq \tau$ of stopping times we have $\Phi_\sigma(f) \leq \mathbf{E}_{\mathbb{Q}}[\Phi_\tau(f) | \mathcal{F}_\sigma]$, hence $\Phi_\sigma(f) \leq \Phi_\sigma(\Phi_\tau(f))$. Since always $\Phi_\sigma(f) \geq \Phi_\sigma(\Phi_\tau(f))$, we have equality.

Since by Theorem 4.2, (1) implies Bellman’s principle, the proof of the theorem shall be completed by showing that Bellman’s principle implies time consistency (2) for *values*. This results from the relation

$$\begin{aligned} \Phi_\sigma(f) &= \Psi_\sigma(X_f) = \Psi_\sigma((X_f)^{\tau-} + \Psi_\tau(\tau(X_f))\mathbf{1}_{[\tau, N]}) \\ &= \Psi_\sigma(\|f\|, \dots, \|f\|, \Psi_\tau(X_f), \dots, \Psi_\tau(X_f)) \\ &= \Phi_\sigma(\Psi_\tau(X_f)) = \Phi_\sigma(\Phi_\tau(f)). \end{aligned}$$

5.3 Discussing the Tail Value-at-risk measurement.

Tail Value-at-risk, see Embrechts (1995), Artzner et al. (1997, 1999), Delbaen (2000, 2002), has been well received, see Wilkie, Waters and Yang (2003), as it gives an idea of “how bad is bad”. We explain here why care must be used in the multiperiod setting, as we had warned in Artzner (2002).

The good definition of Tail Value-at-risk of level α for the one-period case reads as follows:

$$TVaR^\alpha(X) = \inf_{\mathbb{Q}} \{ \mathbf{E}_{\mathbb{Q}}[X] \mid \frac{d\mathbb{Q}}{d\mathbb{P}_0} \leq \alpha^{-1} \}$$

that is also:

$$TVaR^\alpha(X) = \inf \{ \mathbf{E}_{\mathbb{P}_0}[Z \cdot X] \mid 0 \leq Z \leq \alpha^{-1}, \mathbf{E}_{\mathbb{P}_0}[Z] = 1 \}.$$

Definition 5.5. Given a final value f and dates $n, n+h, 0 \leq n \leq n+h \leq N$, Tail Value-at-risk for the time interval $(n, n+h)$ at level α is given by $TVaR_{n,n+h}^\alpha(f) = \text{ess. inf}\{\mathbf{E}_{\mathbb{P}_0}[Z \cdot f \mid \mathcal{F}_n] \mid 0 \leq Z \leq \alpha^{-1}, \mathbf{E}_{\mathbb{P}_0}[Z \mid \mathcal{F}_n] = 1, Z \text{ is } \mathcal{F}_{n+h}\text{-measurable}\}$.

Remark. Notice that for $n+h = N$ fixed and n variable, we use *different* sets of test probabilities.

We first notice that the set of test probabilities providing the coherent one-period risk-adjusted measurement $TVaR_{0,N}^\alpha$ is not stable, see Artzner (2002). We also show that the system of the $(TVaR_{n,N}^\alpha)_n$ does not come from a mapping like Φ or $\bar{\Phi}$ of Section 5.1, since the weak form of time consistency is not even satisfied. Here is an example of a future value g at date $N = 2$ with a “TailVaR” positive as random variable at date 1 and negative at date 0:

$\Omega = \{[uu], [um], [ud], [du], [dm], [dd]\}$, \mathbb{P}_0 uniform on Ω , \mathcal{F}_1 generated by $[u]$ and $[d]$,

$g([uu]) = -10, g([um]) = 12, g([ud]) = 14, g([du]) = -20, g([dm]) = 22, g([dd]) = 22$.

We find for g the TVaR values, at the level $\alpha = \frac{2}{3}$, at date 0 and at date 1:

$$TVaR_{1,2}^\alpha(g)([u]) = TVaR_{1,2}^\alpha(g)([d]) = 1, TVaR_{0,2}^\alpha(g) = -1.$$

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