Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics

Section B

Exam 3 - 10th April 2015

Name:

This is a closed book exam, you may not consult your notes, textbooks, other students or electronic equipment during the exam. You may use known inequalities without proof (unless otherwise stated) as long as you state what you are using. If you make use of something we proved during lectures, be very explicit in doing so by stating exactly what results or properties you are using and why they apply. You may not cite without proof theorems you proved on homework/review sheet or read in the book/the internet/elsewhere. You must justify your answers.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1.

(a) State and prove Markov's inequality.

Solution Given a variable X that takes values in $\{0, 1, 2, \dots\}$ we have from the definition of expectation

$$
\begin{aligned}
\mathbb{E}(X) &= \sum_{i=0}^{\infty} i \times Pr(X = i) \\
&\geq \sum_{i=t}^{\infty} i \times Pr(X = i) \\
&\geq \sum_{i=t}^{\infty} t \times Pr(X = i) = t \sum_{i=t}^{\infty} \times Pr(X = i) \\
&= t Pr(X \geq t).
\end{aligned}
$$

Rearranging this equation gives us

$$Pr(X \geq t) \leq \frac{\mathbb{E}(X)}{t},$$

as required.

(b) Prove Chebyshev's inequality, namely that for a random variable X taking values in $\{0, 1, 2, \dots\}$ with $\mathbb{E}(X) = \mu$ and $Var[X] = \sigma^2$, the following is true,

$$Pr(|X - \mu| \geq t\sigma) \leq \frac{1}{t^2}.$$

Solution Since X takes real values, we have

$$Pr(|X - \mu| \geq t\sigma) = Pr((X - \mu)^2 \geq t^2\sigma^2).$$

Since X is a random variable, so too is $(X-\mu)^2$ and it can only take non-negative values. We apply Markov to $(X-\mu)^2$, giving

$$Pr((X-\mu)^2 \geq t') \leq \frac{\mathbb{E}((X-\mu)^2)}{t'}.$$

Putting $t' = t^2\sigma^2$ and noting that $\mathbb{E}((X-\mu)^2 = Var[X] = \sigma^2$ gives us

$$Pr(|X-\mu| \geq t\sigma) = Pr((X-\mu)^2 \geq t^2\sigma^2) \leq \frac{\sigma^2}{t^2\sigma^2} = \frac{1}{t^2},$$

as required.

(c) Let $\{X_n\}$ be a sequence of random variables, each taking values in $\{0, 1, 2, \dots\}$, such that $\mathbb{E}(X_n) = n$ and $Var[X_n] = n$. Prove that,

$$\lim_{n\to\infty} Pr(X_n = 0) \to 0.$$

Solution Using Chebyshev with $t = \mu/\sigma$ gives

$$Pr(X_n = 0) \leq Pr(|X-\mu| \geq t\sigma) \leq 1/(\mu/\sigma)^2 = \frac{\sigma^2}{\mu^2} = \frac{n}{n^2} = \frac{1}{n},$$

which tends to 0 as $n \to \infty$ as required.

2. Suppose you have fair dice, each with three sides with values $0, 1$ and 2 printed on them. Let X_n be the value obtained by rolling n dice and taking the total of the values obtained on each die.

(a) Calculate the expected value and variance of X_1.

Solution

$$\mathbb{E}(X_1) = (0 + 1 + 2)/3 = 1,$$

and

$$Var[X_1] = \mathbb{E}((X_1 - 1)^2) = ((-1)^2 + 0^2 + 1^2)/3 = 2/3.$$

(b) Calculate the expected value and variance of $(X_1)^2$.

Solution X_1^2 takes values 0, 1 and 4, each with probability 1/3, so

$$\mathbb{E}(X_1^2) = (0^2 + 1^2 + 2^2)/3 = 5/3,$$

and

$$\begin{aligned} Var[X_1^2] =&\mathbb{E}((X_1^2 - 5/3)^2) = ((-5/3)^2 + (1 - 5/3)^2 + (4 - 5/3)^2)/3 \\ =&((25/9 + 4/9 + 49/9)/3 = 26/9. \end{aligned}$$

(c) Calculate the expected value and variance of X_n.

Solution Since X_n is the sum of n independent variables, each equivalent to X_1, we have that $\mathbb{E}(X_n) = n\mathbb{E}(X_1)$ and equally, $Var[X_n] = nVar[X_1]$. This tells us that $\mathbb{E}(X_n) = n$ and $Var[X_n] = 2n/3$.

(d) Show that

$$Pr(X_n \leq \sqrt{n}) \leq \frac{n}{2n - \sqrt{n}}.$$

Solution Note that X_n is symmetrical with a range between 0 and $2n$ and hence

$$Pr(X_n \leq \sqrt{n}) = Pr(X_n \geq 2n - \sqrt{n}).$$

This can be shown by letting $Y_n = 2n - X_n$ where Y_n is the sum of random variables $2 - X_1$. Each of these variables takes values $2, 1, 0$ each with probability $1/3$ and hence Y_n has the same expectation as X_n. Using Markov we get

$$Pr(X_n \leq \sqrt{n}) = \Pr(Y_n \geq 2n - \sqrt{n}) \leq E(Y_n)/(2n - \sqrt{n})$$
$$= \frac{n}{2n - \sqrt{n}},$$

as required.

3. Suppose the second hand of a clock is pointing at 6. The clock is broken and so after each second, the hand either moves clockwise or anti-clockwise by one second (1/60th of the circle), each with probability 1/2.

Show that after 40 seconds have passed, the probability that the second hand is less than or equal to 10 seconds (a 6th of the circle) away from 12 is less than $2e^{-5}$.

Solution To end up within 10 seconds of 12, the hand must have moved a minimum of 20 seconds either clockwise or anti-clockwise. After 40 seconds the furthest the hand could have travelled from 6 would be 40 seconds in one direction which would place it still within 10 seconds of 12. This tells us that the probability that the hand lies within 10 seconds of 12 is the same as it having ended up at a distance of at least 20 seconds either clockwise or anti-clockwise from 6.

Define the random variable X_i to be equal to 1 if the hand moved clockwise, and -1 if it moved anti-clockwise at second i. Each occurs with probability 1/2 and the distance from 6 that the hand ends up at is equal to $X = \sum_{i=1}^{40} X_i$, with positive distances counting the distance clockwise and negative, the distance from 6 in the anti-clockwise direction.

Hoeffding's inequality with $t = 20$ and $n = 40$ tells us that

$$Pr(|X| \geq 20) \leq 2e^{-20^2/(2\times 40)} = 2e^{-5},$$

as required.

4. Let H_n be the number of heads obtained from n flips of a biased coin such that $\mathbb{E}(H_n) = 0.2n$.

Show that

$$Pr\left(|H_n - 0.2n| \geq \frac{2n}{5}\right) \leq \frac{1}{n}.$$

Solution Note that H_n is the number of successes in n independent random events each happening with the same probability p. This makes H_n a binomial distribution with mean

$$\mathbb{E}(H_n) = np = 0.2n$$

and variance

$$Var[H_n] = np(1-p).$$

The first equation tells us that $p = 0.2$ or $1/5$ which tells us that

$$Var[H_n] = n\left(\frac{1}{5}\right)\left(\frac{4}{5}\right) = \frac{4n}{25}.$$

Noting that,

$$\sigma = \sqrt{Var[H_n]} = \frac{2\sqrt{n}}{5},$$

and using Chebyshev with $t = \sqrt{n}$, and hence $t\sigma = \frac{2n}{5}$, we get

$$Pr\left(|H_n - 0.2n| \geq \frac{2n}{5}\right) \leq \frac{1}{t^2} = \frac{1}{n},$$

as required.