Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics

Section B

Exam 2 - 3rd March 2015

Name:

This is a closed book exam, you may not consult your notes, textbooks, other students or electronic equipment during the exam. You may use known series expansions for functions without proof as long as you state what you are using. If you make use of something we proved during lectures, be very explicit in doing so by stating exactly what results or properties you are using and why they apply. You may not cite without proof theorems you proved on homework/review sheet or read in the book/the internet/elsewhere. You must justify your answers.

Problem	Points	Score
1	30	
2	35	
3	35	
Total:	100	

1.

(a) Prove that (for $1 < k < n$),

$$\binom{n}{k} > \left(\frac{n}{k}\right)^k.$$

Solution

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k(k-1)\ldots 2 \times 1} = \frac{n}{k}\left(\frac{n-1}{k-1}\right)\left(\frac{n-2}{k-2}\right)\cdots\left(\frac{n-k+1}{k-k+1}\right).$$

For $1 \leq i \leq k-1$ we have that $\frac{n-i}{k-i} > \frac{n}{k}$. This follows since the following are equivalent statements,

$$\begin{aligned}
&\frac{n-i}{k-i} > \frac{n}{k} \\
&nk - ik > nk - in \\
&in > ik \\
&n > k,
\end{aligned}$$

and we know the final statement is true. This gives us that

$$\binom{n}{k} > \left(\frac{n}{k}\right)\left(\frac{n}{k}\right)\cdots\left(\frac{n}{k}\right) = \left(\frac{n}{k}\right)^k.$$

(b) Prove that

$$\frac{2^n}{n+1} \leq \binom{n}{\lfloor n/2 \rfloor} \leq 2^n.$$

Solution $\binom{n}{k}$ is the number of ways of choosing a subset of size k from a set of size n. We know that the number of ways of choosing a subset of any size from n elements is 2^n since each element of $[n]$ can be either in or out of the subset. Therefore for all k, $\binom{n}{k} < 2^n$.

Equally, the above tells us that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

We also know that $\binom{n}{\lfloor n/2 \rfloor}$ is the largest of the possible $\binom{n}{k}$ and hence must be at least as large as the average over the values of k. The values of k range between 0 and n and hence there are $n+1$ of them, so the average value of $\binom{n}{k}$ is equal to $\frac{2^n}{n+1}$ and as such we have

$$\frac{2^n}{n+1} \leq \binom{n}{\lfloor n/2 \rfloor},$$

as required.

(b) Prove that

$$\left(1 - \frac{1}{n}\right)^n \leq \frac{1}{e}.$$

Solution As shown in class, we have that for all $x \in \mathbb{R}$ we have $(1-x) \leq e^{-x}$. Putting $x = \frac{1}{n}$ gives us

$$\left(1 - \frac{1}{n}\right)^n \leq \left(e^{-\frac{1}{n}}\right)^n = e^{-\frac{n}{n}} = e^{-1} = \frac{1}{e}.$$

2. You are organising a school sports day. The school will be split into two teams who will compete in many different sporting events. The students will decide what sports will be played and have formed 30 committees of 6 students each to discuss organisation and rules for each sport.

To ensure the competition is fair, you must choose the two teams, and must make sure that each committee has at least one student from each team. Show that this is possible.

Solution Call the teams A and B. Assign each student to A or B with probability half each, uniformly and independently at random. The probability that a committee is entirely on team A is therefore $\frac{1}{2^6}$. Equally the probability that it is entirely on team B is the same. Therefore, since these events are distinct events on the same probability space, the probability that a committee is entirely on the same team is

$$2 \times \frac{1}{2^6} = 2^{-5} = \frac{1}{32}.$$

The expected number of committees with all members of one team is therefore $30 \times \frac{1}{32}$. Since this is clearly less than 1, and the number of committees with all members on one team can only take integer values, there must be a greater than 0 probability that this value is 0. This tells us that there must exist at least one assignment of students to teams such that the number of committees with all members on one team is 0 and this is the assignment we require.

3. Let $n^2 < 2^{k-1}$. Show that there exists an (edge) colouring of the complete graph on n vertices with two colours such that there exists no mono-coloured clique of size k (i.e. no k vertices where all the edges between those k vertices are one colour.)

Solution Randomly colour the edges of the graph either red or blue independently of the colour of any other edge, with probability half each. There are $\binom{n}{k}$ cliques of size k and each clique contains $\binom{k}{2}$ edges.

The probability that a clique is mono-coloured is $2 \times (1/2)^{\binom{k}{2}}$, since there are two choices of colour for the mono-coloured clique to be and the probability that all of the edges are that colour is $(1/2)^{\binom{k}{2}}$.

The expected number of mono-coloured cliques is therefore

$$\begin{aligned}\binom{n}{k} 2^{1-\binom{k}{2}} =& \binom{n}{k} 2^{1-\frac{k(k-1)}{2}} \\ =& 2\binom{n}{k} 2^{-\frac{k(k-1)}{2}} \\ \leq& n^k 2^{-\frac{k(k-1)}{2}} \\ =& \left(n 2^{-\frac{(k-1)}{2}}\right)^k \\ =& \left(n^2 2^{-(k-1)}\right)^{\frac{k}{2}}.\end{aligned}$$

The third line follows from noting that,

$$\begin{aligned}2\binom{n}{k} =& 2\frac{n(n-1)\dots(n-k+1)}{k(k-1)\dots 3 \times 2 \times 1} \\ =& \frac{n(n-1)\dots(n-k+1)}{k(k-1)\dots 3 \times 1} \leq n^k.\end{aligned}$$

Returning to the expected number of mono-coloured cliques, we have that it is equal to

$$\left(\frac{n^2}{2^{k-1}}\right)^{\frac{k}{2}}.$$

We are given that $n^2 < 2^{k-1}$ and so this value is less than 1 and hence the probability that there exists less than 1 (and hence 0) mono-coloured cliques is greater than 0. This tells us that such a colouring must exist as required.