Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics

Section B

Exam 1 - 18th February 2015

Name:

This is a closed book exam, you may not consult your notes, textbooks, other students or electronic equipment during the exam. If you make use of something we proved during lectures, be very explicit in doing so by stating exactly what results or properties you are using and why they apply. You may not cite without proof theorems you proved on homework/review sheet or read in the book/the internet/elsewhere. You must justify your answers and if you are asked for a combinatorial proof, then little to no credit will be provided for non-combinatorial answers.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. Let m and n be positive integers.

(a) How many non-negative integer solutions are there to

$$x_1 + x_2 + \cdots + x_m = n.$$

Solution We can consider this sum by thinking of $n+m-1$ empty space in a sequence. In each space we can place a 1 or a +. If we place $m-1$ pluses and then group adjacent 1's with no + between them into their sum and considering adjacent + signs to have a 0 between them, then we will be left with m non-negative integers that sum to n. This is a unique representation of the required sum and so the number of such solutions is equal to the number of ways of choosing the $m-1$ pluses from the $n+m-1$ spaces. This is equal to $\binom{n+m-1}{m-1}$.

(b) Let $1 \leq p \leq m$ be an integer. How many integer solutions are there to the above that satisfy $x_i \geq 3$ for $1 \leq i \leq p$ and $x_i \geq 0$ otherwise.

Solution If we define $y_i = x_i - 3$ for $1 \leq i \leq p$ and $y_i = x_i$ otherwise, then we have that each y_i can take any non-negative integer value and the following is true,

$$y_1+y_2+\cdots+y_m = x_1-3+x_2-3+\cdots+x_p-3+x_{p+1}+\cdots+x_m = n-3p.$$

We now apply part a) to get that the number of ways of choosing the y_i and hence the x_i is $\binom{n+m-3p-1}{m-1}$.

2. Let $a(x)$ be the ordinary generating function for the sequence a_n with closed form of

$$a(x) = \frac{1}{1-x^2}.$$

(a) Express $a(x)$ as the product of two generating functions and use this to find a_n.

Solution

$$a(x) = \frac{1}{1-x^2} = \frac{1}{(1-x)}\frac{1}{(1+x)}.$$

$$\frac{1}{(1-x)} = \sum_{i=1}^{\infty}(-1)^n x^n.$$

$$\frac{1}{(1+x)} = \sum_{i=1}^{\infty} 1x^n.$$

$$a(x) = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{i} 1(-1)^k\right) x^n.$$

Therefore

$$a_n = \sum_{k=0}^{i} 1(-1)^k = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

(b) Find a closed form for $b(x) = \sum_{n=0}^{\infty} b_n x^n$ if

$$b_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Solution We have that $b_n = a_n + c_n$ where a_n is the solution to part a and $c_n = n - 1$ for n odd and 0 otherwise.

Therefore

$$\begin{aligned} c(x) =& \sum_{n=0}^{\infty} c_n x^n = 0x + 2x^3 + 4x^5 + \dots = 2x(1x^2 + 2x^4 + 3x^6 + \dots) \\ =& 2x(0 + 1x^2 + 2(x^2)^2 + 3(x^2)^3 + \dots) \\ =& 2x \sum_{n=0}^{\infty} n(x^2)^n = 2x \frac{x^2}{(1-x^2)^2} = \frac{2x^3}{(1-x^2)^2}. \end{aligned}$$

This gives us

$$b(x) = a(x) + c(x) = \frac{1}{1-x^2} + \frac{2x^3}{(1-x^2)^2} = \frac{1 - x^2 + 2x^3}{(1-x^2)^2}.$$

3. Let $f(m, n)$ be the number of surjective (onto) mappings from $[m]$ to $[n]$. Prove that

$$f(m, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

Solution Let A_i be the set of mappings from $[m]$ to $[n]$ such that $f(x) \neq i$ for any $x \in [m]$. For $S \subseteq [n]$ we let $A_S = \cap_{i \in S} A_i$. A_S is therefore the mappings that do not map any element of $[m]$ to any element of S. The elements of $[m]$ can be mapped freely to any of the other $n - |S|$ elements of n so there are $(n - |S|)^m$ such mappings in each A_S. The surjective mappings are those that do not lie in any A_i and so by inclusion exclusion

$$f(m, n) = \sum_{S \subseteq [n]} (-1)^{|S|} |A_S| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

The second equality following from grouping the A_S by size, taking $|A_S| = k$, and noting that there are $\binom{n}{k}$ choices for S of size k.

As a side note, you could also use description, involution, exception here. See

https://www.math.hmc.edu/ benjamin/papers/DIE.pdf

for an elegant proof.

4.

(a) Provide a combinatorial proof that

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}.$$

Solution $\binom{n}{m}$ is the number of ways of choosing a set M of size m from $[n]$ and then $\binom{m}{k}$ is the number of ways of choosing a subset K of M of size k. This is equivalent to first choosing a subset of $[n]$ of size k and then choosing the remaining $m-k$ elements of M from $[n] \setminus K$. There are $\binom{n}{k}$ ways to choose K first and $\binom{n-k}{m-k}$ ways to choose the rest of M.

(b) Show that

$$\sum_{k=0}^{m} \binom{n-k}{m-k} = \binom{n+1}{m}.$$

(Hint: How many subsets of $[n+1]$ of size m contain the elements $1, 2, 3, \ldots, k$ but not $k+1$?

Solution There are $\binom{n+1}{m}$ subsets of $[n+1]$ of size m. For each subset $M \subset [n+1]$, consider the smallest element $k+1$ of $[n+1]$ not contained in M. M must contain all the elements $1, 2, \ldots, k$ but not $k+1$. Given that there are $m-k$ remaining elements of M, and these can be chosen freely from the remaining $n-k$ elements of $\{k+2, k+3, \ldots, n+1\}$, there are $\binom{n-k}{m-k}$ such M for each k. Since each sets satisfies this property for exactly one k the total number of M is just the sum of $\binom{n-k}{m-k}$ over all possible values of k as required.

(c) Deduce that

$$\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} = \frac{n+1}{n+1-m}.$$

Solution By part a), we have that

$$\frac{\binom{m}{k}}{\binom{n}{k}} = \frac{\binom{n-k}{m-k}}{\binom{n}{m}}.$$

Therefore, using part b),

$$\begin{aligned}\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} =& \sum_{k=0}^{m} \frac{\binom{n-k}{m-k}}{\binom{n}{m}} \\ =& \frac{1}{\binom{n}{m}} \sum_{k=0}^{m} \binom{n-k}{m-k} \\ =& \frac{1}{\binom{n}{m}} \binom{n+1}{m} \\ =& \frac{m!(n+1)!(n-m)!}{m!(n-m+1)!n!} = \frac{(n+1)n!(n-m)!}{n!(n-m+1)(n-m)!} \\ =& \frac{n+1}{n+1-m},\end{aligned}$$

as required