21-301 Combinatorics
Section B

Optional Assignment 6 - Solutions

1.* Consider a graph $G = (V, E)$, where each vertex v has a list $C(v)$ of allowed colours. A list-colouring of G assigns each vertex $v \in V$ a colour from its list $C(v)$. A proper list-colouring is one that ensures that no two vertices with an edge between them are assigned the same colour.

Suppose each vertex has a list of size $10k$. Moreover, for each $v \in V$ and $c \in C(v)$, there are at most k neighbours u of v that contain c in their colour sets $C(u)$. Show that there exists a proper list-colouring of G. (Hint: For each edge $e = \{u, v\}$ and colour $c \in C(u) \cap C(v)$, consider the event $X(e, c)$ to be that both u and v are assigned colour c.)

Solution Assign each vertex a colour chosen uniformly at random from the list assigned to it. The colouring is a proper list colouring if and only if no two adjacent vertices that have a colour in common in their lists choose that common colour.

Given this, we see that the event the colouring is proper is equivalent to the intersection of the events $\neg X(e, c)$, where for each edge $e = \{u, v\}$ and colour $c \in C(v) \cap C(u)$, let $X(e, c)$ to be that both u and v are assigned colour c. For any edge, if the colour c is not present in the lists of both endpoints then the probability of $X(e, c)$ occurring is 0. Therefore we only need to consider $e = (u, v)$ and c such that $c \in C(v) \cap C(u)$. The probability that any one $X(e, c)$ happens is $p = 1/(10k)^2 = 1/(100k^2)$. Each $X(e, c)$ depends only on $X(e', c')$ for e' adjacent to either u or v. For each colour in $C(u)$ there are at most k possible y that also have that colour and so only $10k^2$ possible $X(e', c')$ can exist adjacent to u. The same applies to v and so each $X(e, c)$ depends on at most $d = 20k^2 - 1$ (the minus one coming from not counting itself) other variables.

Given these facts, we can apply the Local Lemma with $p = 1/(10k)^2 = 1/(100k^2)$, $d + 1 = 20k^2$ and so $ep(d + 1) = e/5 < 1$ and so the probability that none of the $X(e, c)$ occur is strictly greater than 0. This tells us that there must exist a list-colouring such that no $X(e, c)$ occurs and hence a proper list-colouring exists, as required.

2.* Consider the game played with n matchsticks where a move consists of removing 1, 3 or 4 matchsticks and the player who takes the last matchstick wins.

(a) For $1 \leq n \leq 10$ determine whether Player 1 or 2 wins.

Solution 0 is a sink and hence a P-position and can be reached from $n = 1, 3$ and 4 and hence these are N-positions. From 2 the only valid move is to remove 1 matchstick which leads to an N-position and hence 2 is a P-position. From 5 and 6, one can remove 3 or 4 matchsticks to reach 2 and so 5 and 6 are both N-positions. From 7 you can only reach $3, 4$ or 6 which are all N-positions and so 7 is a P-position. From 8 and 10, you can reach 7 and so these are both N-positions. Lastly from 9 you can reach $5, 6$ and 8 which are all N-positions and so 9 is a P-position.

1:N 2:P 3:N 4:N 5:N 6:N 7:P 8:N 9:P 10:N

(b) Do the same for the misère version of the game (i.e. last player to move loses).

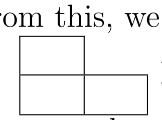
Solution 0 is an N-position, since the player who moved to 0 lost. This tells us that 1 must be a P-position since the only valid move from 1 is to 0. This tells us that 2,4 and 5 are all N-positions. Valid moves from 3 are to 0 or 2, both of which are N-positions and so 3 is a P-position. Therefore 6 and 7 are both N-positions. From 8 the only valid moves are to 4,5 and 7 which are all N-positions so 8 is a P-position. This tells us 9 must be an N-position and lastly 10's valid moves are to 6,7 and 9 and so is a P-position.

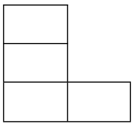
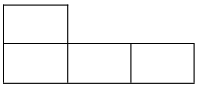
1:P 2:N 3:P 4:N 5:N 6:N 7:N 8:P 9:N 10:P

3.* Consider the game played on a 3×3 grid. A move consists of choosing a square and removing that square and all squares to the right and above of it. The player who removes the last square (or equivalently, the bottom left square) loses.

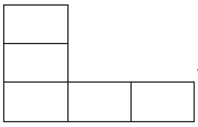
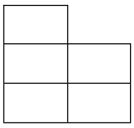
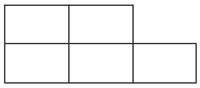
Determine the possible board positions for which the current player loses. Justify your results.

Solution A single square is clearly a P-position, since the next player must remove it and so loses. From this, any 1×2, 1×3 2×1 or 3×1 board must be an N-position, since it can be reduced in one move to a single square. From this, we see that

 Must be a P-position, since any move either removes the last square or leaves one of the N-positions above. this tells us that all three of

 ,  and 

are all N-positions. From here, we see that all moves from

 ,  and 

lead either to one of the above or a $1 \times X$ or $X \times 1$ block which are all N-positions, making these P-positions. The remaining positions are all reducible to one of the above and hence are all N-positions.

4. Consider the game above but played on an $n \times m$ grid. Describe all winning and losing positions for $n = 1$ and $n = 2$.

Solution We have seen above that any $1 \times m$ board for $m > 1$ is an N-position since it can be reduced to a single square in one move. We saw that the board with the three squares in the lower left-hand corner present was a P-position. We claim that all P-positions with $n = 2$ with are of this form, i.e. there is one less square in the second row than the first.

We prove this by induction. Clearly if there is one or two squares in the first row, we are done, since this corresponds to the case above or a single square. Assume as the inductive step that the only P-position with m squares in the first row is the board with $m - 1$ squares in the second.

Suppose our board has $m + 1$ squares in the first row. If the board has m squares in the top row then any move in the top row reduces to a board position where the number of squares in the top row is $k < m$. The next player can therefore play in the first row to reduce it to $k + 1 \leq m$ blocks, which by induction is a P-position and hence this move was an N-position. If the player moved in the first row then this reduces to an $2 \times k$ block with $k \leq m$ and again by induction this is an N-position. Since this describes all moves from this board and they are all N-positions, it too must be a P-position.

Any board with $m + 1$ squares in the first row and $k < m$ in the top can be reduced to having $k + 1$ squares in the first row, which by induction is a P-position and hence all boards that differ by other than one square in the two rows are N-positions as required.

5. Consider the game played on a graph with n vertices. A move consists of choosing a vertex with even degree and deleting it and all edges incident to

that vertex. The last player able to move wins. For what values of n does a winning strategy exist for Player 1?
Solution The game ends when all vertices have odd degree. Any graph has an even number of odd degree vertices (by summing all degrees you count each edges twice and so must be an even number) and hence the number of vertices at the end of the game must be even. Since a move reduced the number of vertices by 1 and hence changes it from an odd to even value or vice versa, player 1 can only win if n is initially odd and any valid move will lead to victory.

6.* In class we described the coin pushing game, played on an $n \times m$ grid, with a coin placed at position (n, m) where a move consists of moving the coin to the left or down (but not both). The player who moves the coin to $(0, 0)$ wins the game. Draw the directed graph that represents the board states for the 2×3 version of the game and identify each vertex in this graph that represents a board state that the current player has a winning strategy for.
Solution

The graph below represents the valid moves from each position in the 2×3 board and is labelled as an N or P position.