21-301 Combinatorics
Section B

Optional Assignment 6

Due Friday 1st May (at the beginning of the lecture)

Please write up and hand in solutions to all of the starred questions below. Please remember to justify all steps all of your argument, stating exactly what results you are using. To avoid confusion, note that question 1 is not a combinatorial games question!

1.* Consider a graph $G = (V, E)$, where each vertex v has a list $C(v)$ of allowed colours. A list-colouring of G assigns each vertex $v \in V$ a colour from its list $C(v)$. A proper list-colouring is one that ensures that no two vertices with an edge between them are assigned the same colour.

Suppose each vertex has a list of size $10k$. Moreover, for each $v \in V$ and $c \in C(v)$, there are at most k neighbours u of v that contain c in their colour sets $C(u)$. Show that there exists a proper list-colouring of G. (Hint: For each edge $e = \{x, y\}$ and colour $c \in C(v) \cap C(u)$, consider the event $X(e, c)$ to be that both x and y are assigned colour c.)

2.* Consider the game played with n matchsticks where a move consists of removing 1, 3 or 4 matchsticks and the player who takes the last matchstick wins.

(a) For $1 \leq n \leq 10$ determine whether Player 1 or 2 wins.

(b) Do the same for the misère version of the game (i.e. last player to move loses).

3.* Consider the game played on a 3×3 grid. A move consists of choosing a square and removing that square and all squares to the right and above of it. The player who removes the last square (or equivalently, the bottom left square) loses.

Determine the possible board positions for which the current player loses. Justify your results.

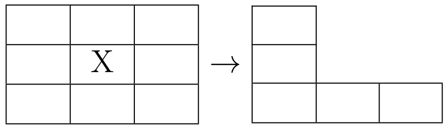

Figure 1: An example 1st move for player 1, choosing the middle square.

4. Consider the game above but played on an $n \times m$ grid. Describe all winning and losing positions for $n = 1$ and $n = 2$.

5. Consider the game played on a graph with n vertices. A move consists of choosing a vertex with even degree and deleting it and all edges incident to that vertex. The last player able to move wins. For what values of n does a winning strategy exist for Player 1?

6.* In class we described the coin pushing game, played on an $n \times m$ grid, with a coin placed at position (n, m) where a move consists of moving the coin to the left or down (but not both). The player who moves the coin to $(0, 0)$ wins the game. Draw the directed graph that represents the board states for the 2×3 version of the game and identify each vertex in this graph that represents a board state that the current player has a winning strategy for.