21-301 Combinatorics
Section B

Assignment 5 - Solutions

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1. Given a vertex v in $G(n, c/n)$ i.e. $G(n, p)$ with $p = c/n$ for some constant c. Let $d(v)$ be the degree of v.

(a) Prove that

$$Pr(d(v) \geq n/2) < \frac{2c}{n}.$$

Solution Since each vertex is adjacent to $n-1$ possible edges, each present with probability c/n, we have that

$$\mathbb{E}(d(v)) = \frac{c(n-1)}{n} = \left(1 - \frac{1}{n}\right) c < c.$$

By Markov

$$Pr(d(v) \geq n/2) < \frac{\mathbb{E}(d(v))}{n/2} < \frac{c}{n/2} = \frac{2c}{n},$$

as required.

(b) Prove that

$$Pr\left(\left|d(v) - \left(1 - \frac{1}{n}\right) c\right| \geq c\sqrt{n}\right) < \frac{1}{n}$$

Solution The variance of $d(v)$ can be calculated by observing that $d(v) = \sum_{i=1}^{n-1} X_i$ where X_i is the indicator variable for each of the possible $n-1$ edges being present or not. Since each of these X_i are independent, this is a binomial distribution with $n-1$ trials each with probability of success c/n. This tells us that we have

$$\sigma^2 = Var[d(v)] = (n-1)\frac{c}{n}\left(1 - \frac{c}{n}\right) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{c}{n}\right) c < c.$$

We observe that this tells us that $\sigma < \sqrt{c}$ and so (actually, we require that $\sqrt{c} < c$ which is only true for $c \geq 1$ which I forgot to state. Sorry!)

$$Pr\left(\left|d(v)-\left(1-\frac{1}{n}\right)c\right|\geq\sqrt{n}c\right)\leq Pr\left(\left|d(v)-\left(1-\frac{1}{n}\right)c\right|\geq\sqrt{n}\sigma\right), \tag{1}$$

since the event on the left is contained within the event on the right. Lastly Chebyshev's inequality tells us that

$$Pr\left(|d(v)-\mathbb{E}(d(v))|\geq t\sigma\right)\leq 1/t^2,$$

and we know $\mathbb{E}(d(v))$ from part a and putting $t=\sqrt{n}$ gives us

$$Pr\left(\left|d(v)-\left(1-\frac{1}{n}\right)c\right|\geq\sqrt{n}\sigma\right)\leq\frac{1}{(\sqrt{n})^2}=\frac{1}{n}. \tag{2}$$

Putting these results together gives the required result.

2. Calculate the expected number of K_4 (complete graph on 4 vertices) in
   (i) $G(100,1/2)$.
   (ii) $G(n,p)$ for general n and p.

Solution There are $\binom{n}{4}$ sets of 4 vertices that could possibly be a K_4. Each of these is a K_4 with probability p^6, so in the first case when $n=100$ and $p=1/2$ this is

$$\binom{100}{4}1/2^6=\frac{100\times 99\times 98\times 97}{2^6 4!}=3921225/64,$$

while in general it is equal to

$$\binom{n}{4}p^6.$$

Show that for $p=o\left(n^{-2/3}\right)$, (i.e. $pn^{2/3}\to 0$),

$$\lim_{n\to\infty}Pr(\text{There exists a } K_4 \text{ in } G(n,p))=0.$$

Solution Let X be the number of K_4 in $G(n,p)$. We have

$$\mathbb{E}(X) = \binom{n}{4} p^6 \leq n^4 p^6 = o\left(n^4 \left(n^{-2/3}\right)^6\right) = o\left(n^4 n^{-4}\right) = o(1).$$

This tells us that $\mathbb{E}(X) \to 0$ as $n \to \infty$. Using Markov

$$Pr(\text{There exists a } K_4 \text{ in } G(n,p)) = Pr(X \geq 1) \leq \mathbb{E}(X)/1 \to 0.$$

Show that for $p = \frac{\log n}{n^{2/3}}$ and for any constant $d > 1$,

$$\lim_{n\to\infty} Pr(\text{There exists } n/d \text{ distinct } K_4 \text{ in } G(n,p)) = 0.$$

Solution We again use Markov on X, the number of K_4 in $G(n,p)$. Now we have

$$\mathbb{E}(X) = \binom{n}{4} \left(\frac{\log n}{n^{2/3}}\right)^6 \leq n^4 \frac{(\log n)^6}{n^4} = (\log n)^6.$$

Markov gives us

$$\begin{aligned} Pr(\text{There exists } n/d \text{ distinct } K_4 \text{ in } G(n,p)) \leq & Pr(X \geq n/d) \\ \leq & \frac{\mathbb{E}(X)}{n/d} \leq \frac{d(\log n)^6}{n}. \end{aligned}$$

this value clearly tends to 0 as required.

3. In class, we showed that if X is the random variable counting the number of triangles in $G(n,p)$, then the variance of X satisfies

$$Var[X] \leq \binom{n}{3} p^3 + \binom{n}{4} p^5.$$

(a) Show that when $p = \frac{\log n}{n}$, and for n sufficiently large that

$$Var[X] \leq (\log n)^3.$$

Solution Substituting our value of p, we have

$$\begin{aligned} Var[X] \leq \binom{n}{3} p^3 + \binom{n}{4} p^5 \leq & \frac{n^3}{3!} \left(\frac{\log n}{n}\right)^3 + n^4 \left(\frac{\log n}{n}\right)^5 \\ = & \frac{(\log n)^3}{6} + \frac{(\log n)^5}{n} \\ = & (\log n)^3 - \left(\frac{5\,(\log n)^3}{6} - \frac{(\log n)^5}{n}\right). \end{aligned}$$

For n sufficiently large, the term in the bracket on the right must be greater than 0 since $(\log n)^3 \to \infty$ and $\frac{(\log n)^5}{n} \to 0$, and hence we have the desired result.

(b) Show that (with $p = \frac{\log n}{n}$)

$$Pr\left(\left|X - \binom{n}{3}p^3\right| \geq n\right) \leq \frac{(\log n)^3}{n^2}.$$

Solution We apply Chebyshev with $t\sigma = n$ and hence $t = \frac{n}{\sigma}$ and recall that the expected number of triangles has expectation $\binom{n}{3}p^3$.

$$Pr\left(\left|X - \binom{n}{3}p^3\right| \geq n\right) \leq \frac{1}{t^2} = \frac{\sigma^2}{n^2} \leq \frac{(\log n)^3}{n^2},$$

as required.