21-301 Combinatorics
Section B

Assignment 5

Not to be handed in. Provided for revision purposes.

These questions are here to give you more practice on Markov and Chebyshev inequalities before the midterm and to help with understanding of random graphs. This sheet does not have to be handed in and will not be assessed.

1. Given a vertex v in $G(n, c/n)$ i.e. $G(n,p)$ with $p = c/n$ for some constant c. Let $d(v)$ be the degree of v.

(a) Prove that

$$Pr(d(v) \geq n/2) < \frac{2c}{n}.$$

(b) Prove that

$$Pr\left(\left|d(v) - \left(1 - \frac{1}{n}\right)c\right| \geq c\sqrt{n}\right) < \frac{1}{n}$$

2. Calculate the expected number of K_4 (complete graph on 4 vertices) in

(i) $G(100, 1/2)$.

(ii) $G(n,p)$ for general n and p.

Show that for $p = o\left(n^{-2/3}\right)$, (i.e. $pn^{2/3} \to 0$),

$$\lim_{n\to\infty} Pr(\text{There exists a } K_4 \text{ in } G(n,p)) = 0.$$

Show that for $p = \frac{\log n}{n^{2/3}}$ and for any constant $d > 1$,

$$\lim_{n\to\infty} Pr(\text{There exists } n/d \text{ distinct } K_4 \text{ in } G(n,p)) = 0.$$

3. In class, we showed that if X is the random variable counting the number of triangles in $G(n,p)$, then the variance of X satisfies

$$Var[X] \leq \binom{n}{3}p^3 + \binom{n}{4}p^5.$$

(a) Show that when $p = \frac{\log n}{n}$, and for n sufficiently large that

$$Var[X] \leq (\log n)^3.$$

(b) Show that (with $p = \frac{\log n}{n}$)

$$Pr\left(\left|X - \binom{n}{3}p^3\right| \geq n\right) \leq \frac{(\log n)^3}{n^2}.$$