21-301 Combinatorics
Section B

Assignment 4 - Solutions

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1. If a permutation σ on $[n]$ is chosen uniformly at random from all possible permutations on $[n]$, what is the expected number of fixed points of σ. (i.e. the number of $i \in [n]$ such that $\sigma(i) = i$).

Solution Let X be the number of fixed points of a permutation chosen uniformly at random from all possible permutations on $[n]$. Then we have that $X = \sum_{i=1}^{n} X_i$ where X_i is the indicator variable for i being a fixed point in the permutation, i.e. if our permutation is π then $X_i = 1$ if $\pi(i) = i$ and 0 otherwise.

For each i, $\pi(i)$ can be any value in $[n]$ with equal probability and so the probability that $\pi(i) = i$ is just $1/n$. This gives us that the expectation of $X_i = 1/n$ and so by linearity of expectation, the expected value of X which is the expected number of fixed points is $n/n = 1$.

As an interesting aside, note that this does not depend on n.

2.* Prove that for all integers $n \geq 1$,

$$\left(1 + \frac{1}{n}\right)^{n+1} \geq e.$$

Solution In lectures we proved that

$$(1 - x) \leq e^{-x}.$$

This tells us that,

$$\frac{1}{(1-x)} \geq \frac{1}{e^{-x}} = e^{x}.$$

We observe that

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{n+1} &= \left(\frac{n+1}{n}\right)^{n+1} \\ &= \frac{1}{\left(\frac{n}{n+1}\right)^{n+1}} \\ &= \frac{1}{\left(1-\frac{1}{n+1}\right)^{n+1}} \\ &\geq \left(e^{\frac{1}{n+1}}\right)^{n+1} = e.\end{aligned}$$

3.* Let X be the value obtained from a standard 6-sided dice roll. Calculate the expected value and variance of X^2.

Solution Since each outcome is equally likely, we have

$$\mathbb{E}(X^2) = (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2)/6 = 91/6,$$

and

$$\mathbb{E}(X^4) = (1^4 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4)/6 = 2275/6.$$

Using these, we have,

$$Var[X^2] = \mathbb{E}(X^4) - \mathbb{E}(X^2)^2 = 2275/6 - (91/6)^2 = 5369/36$$

4.

(a) Let X_1 and X_2 be the values obtained from two independent standard 6-sided dice rolls. Calculate the expected value and variance of X_1+X_2.

Solution We always have $\mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2)$ and we saw in lectures that if X_1 and X_2 are independent, as they are in this question, then we also have $Var[X_1 + X_2] = Var[X_1] + Var[X_2]$. We have

$$\mathbb{E}(X_1) = \mathbb{E}(X_2) = (1 + 2 + 3 + 4 + 5 + 6)/6 = 7/2,$$

and

$$Var[X_1] = Var[X_2] = \mathbb{E}(X_1^2) - \mathbb{E}(X_1)^2 = 91/6 - (7/2)^2 = 35/12.$$

This tells us that

$$\mathbb{E}(X_1 + X_2) = 7,$$

and

$$Var[X_1] + Var[X_2] = 35/6.$$

(b) Calculate the expected value and variance of $X_1 \times X_2$.

Solution Using the fact that for independent variables $\mathbb{E}(X_1 \times X_2) = \mathbb{E}(X_1) \times \mathbb{E}(X_2)$ we have

$$\mathbb{E}(X_1 \times X_2) = (7/2)^2 = 49/4,$$

and

$$\begin{aligned} Var[X_1 \times X_2] =& \mathbb{E}\left((X_1 \times X_2)^2\right) - \mathbb{E}\left(X_1 \times X_2\right)^2 \\ =& \mathbb{E}\left(X_1^2 \times X_2^2\right) - \mathbb{E}\left(X_1 \times X_2\right)^2 \\ =& \mathbb{E}\left(X_1^2\right) \times \mathbb{E}\left(X_2^2\right) - \mathbb{E}\left(X_1 \times X_2\right)^2. \end{aligned}$$

By the results of question 3, we can calculate $\mathbb{E}\left(X_1^2\right) = \mathbb{E}\left(X_2^2\right) = 91/6$ and so

$$Var[X_1 \times X_2] = (91/6)^2 - (49/4)^2 = 11515/144$$

5.*

(a) If the average mark on an exam was 60, prove that the probability that a student chosen at random, scored higher than 80 is less than $\frac{3}{4}$.

Solution Let X be the score of the randomly chosen student, by the Markov inequality

$$Pr[X > 80] \leq Pr[X \geq 80] \leq \frac{\mathbb{E}(X)}{80} = \frac{60}{80} = \frac{3}{4}.$$

(b) If every student scored at least 50 on the exam, prove that the probability that a student chosen at random scored higher than 80 is less than $\frac{1}{3}$.

Solution Let X be the score of the randomly chosen student. Let $Y = X - 50$. We know that Y is a random variable taking values between 0 and 50 and the expected value of Y is $\mathbb{E}(X - 50) = 60 - 50 = 10$. By Markov

$$Pr[X > 80] \leq Pr[X \geq 80] = Pr[Y \geq 30] \leq \frac{\mathbb{E}(Y)}{30} = \frac{10}{30} = \frac{1}{3}.$$

6. Prove that for any constant k and random variable X

$$Var[k \times X] = k^2 Var[X].$$

Solution Starting from the definition of variance and using that $\mathbb{E}(kX) = k\mathbb{E}(X)$, we have

$$\begin{aligned}
Var[kX] =& \mathbb{E}((kX)^2) - \mathbb{E}(kX)^2 \\
=& \mathbb{E}(k^2 X^2) - (k\mathbb{E}(X))^2 \\
=& k^2 \mathbb{E}(X^2) - k^2 \mathbb{E}(X)^2 \\
=& k^2 \left(\mathbb{E}(X^2) - \mathbb{E}(X)^2\right) \\
=& k^2 Var[X],
\end{aligned}$$

as required.

7.* A journal receives an average of 10000 submissions a year with a variance of 2000 submissions. Show that the probability that the journal receives between 8000 and 12000 submissions in a given year is at least 0.9995.

Solution Letting X be the number of submissions in a given year, we have

$$Pr(8000 < X < 12000) = Pr(|X - 10000| < 2000) = 1 - Pr(|X - 10000| \geq 2000).$$

Using Chebychev, we have that

$$Pr(|X - 10000| \geq 2000) \leq \frac{2000}{2000^2} = \frac{1}{2000} = 0.0005.$$

Putting these two results together gives the required result that

$$Pr(8000 < X < 12000) \geq 1 - 0.0005 = 0.9995$$

8. For a random variable with expected value μ and a variance of 0, what can be said about the values it can take?
Solution This tells us that the variable must have constant value i.e. $X = \mu$ with probability 1. This follows from noting that $(X - \mu)^2 \geq 0$ and $\mathbb{E}((X - \mu)^2) = Var[X] = 0$ and therefore $(X - \mu)^2 = 0 \rightarrow X = \mu$.

9.* Let X be the number of heads that appear from 200 coin flips of a biased coin, such that the expected value of X is 20. Determine the variance of X.
Solution This is a binomial distribution with $n = 200$ and mean $np = 20$ and hence $p = 1/10$. This tells us that the variance is $np(1-p) = 200 \times \frac{1}{10} \times \frac{9}{10} = 18$.