21-301 Combinatorics
Section B

Assignment 4

Due Wednesday 1st April (at the beginning of the lecture)

Please write up and hand in solutions to all of the starred questions below. Please remember to justify all steps all of your argument, stating exactly what results you are using.

1. If a permutation σ on $[n]$ is chosen uniformly at random from all possible permutations on $[n]$, what is the expected number of fixed points of σ. (i.e. the number of $i \in [n]$ such that $\sigma(i) = i$).

2.* Prove that for all integers $n \geq 1$,

$$\left(1 + \frac{1}{n}\right)^{n+1} \geq e.$$

3.* Let X be the value obtained from a standard 6-sided dice roll. Calculate the expected value and variance of X^2.

4.

(a) Let X_1 and X_2 be the values obtained from two independent standard 6-sided dice rolls. Calculate the expected value and variance of X_1+X_2.

(b) Calculate the expected value and variance of $X_1 \times X_2$.

5.*

(a) If the average mark on an exam was 60, prove that the probability that a student chosen at random, scored higher than 80 is less than $\frac{3}{4}$.

(b) If every student scored at least 50 on the exam, prove that the probability that a student chosen at random scored higher than 80 is less than $\frac{1}{3}$.

6. Prove that for any constant k and random variable X

$$Var[k \times X] = k^2 Var[X].$$

7.* A journal receives an average of 10000 submissions a year with a variance of 2000 submissions. Show that the probability that the journal receives between 8000 and 12000 submissions in a given year is at least 0.9995.

8. For a random variable with expected value μ and a variance of 0, what can be said about the values it can take?

9.* Let X be the number of heads that appear from 200 coin flips of a biased coin, such that the expected value of X is 20. Determine the variance of X.