21-301 Combinatorics
Section B

Assignment 3

Due Monday 2nd March (at the beginning of the lecture)

Please write up and hand in solutions to all of the questions below. Please remember to justify all steps all of your argument, stating exactly what results you are using.

1.*

(a) Prove the linearity of expectation, namely that for a finite collection of discrete random variables $X_1, X_2, X_3, \dots, X_n$ on a single probability space, we have that

$$\mathbb{E}\left(\sum X_i\right) = \sum \mathbb{E}(X_i).$$

(b) Prove the first moment method, namely that if X is a random variable taking values in $\{0, 1, 2, 3, \dots\}$, then

$$Pr[X \geq 1] \leq \mathbb{E}\left(X\right).$$

2.* Prove that for a graph with $2n$ vertices and $m > 0$ edges, it is possible to partition the vertices of the graph into two sets of size n such that more than $m/2$ edges go between these two sets.

3.* Prove that if $\binom{n}{k} 3^{1-\binom{k}{2}} < 1$ then it is possible to colour the edges of K_n, the complete graph on n vertices, with three colours such that there is no monochromatic K_k.

4. * In an $n \times n$ array, each of the numbers $1, 2, \dots, n$ appears exactly n times (not randomly). Let X be the random variable determined by counting the number of distinct numbers in a randomly chosen row or column. Use X to prove that there must exist a row or column containing at least $\sqrt{n}$ distinct numbers.