21-301 Combinatorics
Section B
Assignment 2 - Solutions

1.* Find a closed form for

$$\sum_{n=0}^{\infty} \binom{n+k}{k} x^n.$$

Solution

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{n+k}{k} x^n =& 1 + \frac{(k+1)!}{k!1!}x + \frac{(k+2)!}{k!2!}x^2 + \dots \\ =& \frac{1}{k!}\left(k! + \frac{(k+1)!}{1!}x + \frac{(k+2)!}{2!}x^2 + \dots\right)\end{aligned}$$

and we note that this sum in the brackets is the kth derivative of $\sum_{n=0}^{\infty} x^n$. Noting this and using that the closed form of $\sum_{n=0}^{\infty} x^n$ is $1/(1-x)$ tells us that

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{n+k}{k} x^n =& \frac{1}{k!}\left(\frac{d^k}{dn^k}\left(\frac{1}{1-x}\right)\right) \\ =& \frac{1}{k!}\left(\frac{k!}{(1-x)^{k+1}}\right) \\ =& \frac{1}{(1-x)^{k+1}}.\end{aligned}$$

2.* Find a_n if

$$a_n = 2a_{n-1} + 3a_{n-2}$$

and $a_0 = -1$ and $a_1 = -15$.
Solution Let $a(x) = \sum_{i=0}^{\infty} a_n x^n$, then we have,

$$\begin{aligned}\sum_{n=2}^{\infty} a_n =& \sum_{n=2}^{\infty} 2a_{n-1} + 3a_{n-2}, \\ a(x) + 1 + 15x =& 2x(a(x)+1) + 3x^2 a(x), \\ a(x)(1-2x-3x^2) =& -1 - 13x.\end{aligned}$$

and so

$$
\begin{aligned}
a(x) =& \frac{-1-13x}{1-2x-3x^2} \\
=& \frac{-1-13x}{(1-3x)(1+x)} \\
=& \frac{3(1-3x)-4(1+x)}{(1-3x)(1+x)} \\
=& \frac{3}{(1+x)} - \frac{4}{(1-3x)} \\
=& 3\sum_{i=0}^{\infty}(-x)^n - 4\sum_{i=0}^{\infty}(3x)^n \\
\sum_{i=0}^{\infty} a_n x^n =& \sum_{i=0}^{\infty}\left(3(-1)^n - 4(3)^n\right)x^n.
\end{aligned}
$$

From this we see that

$$a_n = 3(-1)^n - 4(3)^n.$$

3.*

(a) How many strings of length n consisting of 0's and 1's have no two consecutive 1's.

Solution Let a_n be the number of strings of length n consisting of 0's and 1's with no two consecutive 1's. The number of such strings that end in 0 is precisely a_{n-1} as we can choose the remaining $n-1$ characters freely, other than ensuring there is no two consecutive 1's in the $n-1$ other characters, this is precisely a_{n-1}. Any such string that ends in a 1 must have a 0 as its $n-1$th character or it would have two consecutive ones, therefore by the same argument as above, there are a_{n-2} strings satisfying our condition that end in a 1. Since these cases are disjoint we have that $a_n = a_{n-1} + a_{n-2}$. There is one string of length 0, namely the empty string and two strings of length 1. This gives us that $a_n = F_{n+1}$ where F_{n+1} is the nth Fibonacci number.

(b) How many strings of length n consisting of 0's and 1's have no three consecutive 1's and no three consecutive 0's?

Solution Let $\{a_n\}$ be a string of length n, satisfying the required condition. Define a new string $\{b_{n-1}\}$ as follows; b_i (the ith character of the string) satisfies $b_i = 1$ if $a_i = a_{i+1}$ and is 0 otherwise. If $\{b_{n-1}\}$

contains two consecutive 1's, that would imply that three consecutive characters of $\{a_n\}$ were the same which would be a contradiction, hence $\{b_{n-1}\}$ contains no two consecutive 1's. For each string $\{b_{n-1}\}$, $\{a_n\}$ is determined entirely by whether the first character of $\{a_n\}$ is 0 or 1 so there are two $\{a_n\}$ for each $\{b_{n-1}\}$.

This tells us that there are $2F_n$ such strings.

4.* Find a closed form for

$$\sum_{n=0}^{\infty} n^2 x^n.$$

Solution We firstly note that the derivative of x^n is nx^{n-1} and so we have that

$$x\frac{d}{dx}x^n = nx^n.$$

Equally we can use this operation twice to get

$$x\frac{d}{dx}x\frac{d}{dx}x^n = x\frac{d}{dx}(nx^n) = xn^2x^{n-1} = n^2x^n.$$

Therefore

$$\begin{aligned}
\sum_{n=0}^{\infty} n^2 x^n =& \sum_{n=0}^{\infty} x n^2 x^{n-1} \\
=& \sum_{n=0}^{\infty} x \frac{d}{dx}(n x^n) \\
=& \sum_{n=0}^{\infty} x \frac{d}{dx} x \frac{d}{dx}(x^n) \\
=& x \frac{d}{dx} x \frac{d}{dx} \sum_{n=0}^{\infty} (x^n) \\
=& x \frac{d}{dx} x \frac{d}{dx} \left(\frac{1}{(1-x)} \right) \\
=& x \frac{d}{dx} x \left(\frac{1}{(1-x)^2} \right) \\
=& x \left(\frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3} \right) \\
=& \frac{x(1-x)}{(1-x)^3} + \frac{2x}{(1-x)^3} \\
=& \frac{x^2+x}{(1-x)^3} = \frac{x(x+1)}{(1-x)^3}.
\end{aligned}$$

5. Find a closed form for

$$\sum_{n=0}^{\infty} n^3 x^n.$$

Solution By the same argument as above, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} n^3 x^n =& x \frac{d}{dx} \left(\frac{x(x+1)}{(1-x)^3} \right) \\
=& \frac{x^3+4x^2+x}{(1-x)^4}.
\end{aligned}$$

6. The number of ways of partitioning n into k parts, i.e. finding a set of positive integers $\{a_1, a_2, \dots a_k\}$ (repetitions allowed) such that

$$a_1 + a_2 + \cdots + a_k = n$$

is equal to the number of ways of partitioning n into any number of parts, the largest of which is k. For example, if $n = 4$ and $k = 2$ then

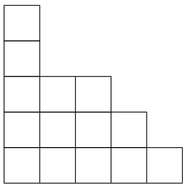
$$2 + 2 = 3 + 1 = 4$$

and also

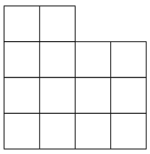
$$2 + 2 = 2 + 1 + 1 = 4.$$

Find a combinatorial proof of this for general n and k.

Solution One can interpret the ways of partitioning n into k blocks as the ways of building k towers using n blocks, with the towers ordered by their height. This can be seen visually using young diagrams.



is a partition of 14 into 5 blocks, corresponding to $14 = 5+3+3+2+1$.



is a partition of 14 into 4 blocks, corresponding to $14 = 4 + 4 + 3 + 3$.

It is clear that any partition can uniquely be defined by such a diagram and each diagram must correspond to a partition of n into k parts by simply counting the number of blocks in each of the k columns (which must add up to n.

Equally however, rather than counting the values of the columns we could count the values in each row. This will correspond to a partition into some number of parts, the largest of which will be at most k, as the bottom row(s) will have exactly k elements, one for each of the columns. Since any such partition can be uniquely represented in this way we have a bijection and so the number of each type of partition is equal.

7.* A composition of n into k parts is the same as a partition of n (as above) but we care about the order of the sum. For example the compositions of 4 are 4, $1 + 3$, $3 + 1$, $2 + 2$, $1 + 1 + 2$, $1 + 2 + 1$, $2 + 1 + 1$, $1 + 1 + 1 + 1$.

(a) How many compositions of n (a positive integer) are there?

Solution Consider the following

$$n = 1 + 1 + 1 + \cdots + 1,$$

where there are n ones. Each composition can be uniquely defined by choosing some number of the pluses present and keeping them, and contracting the ones with no pluses between them into their sum. For example

$$10 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1.$$

And choosing 2 of the pluses (the 3rd and 7th pluses, marked below in red),

$$10 = 1 + 1 + 1 {\color{red}+} 1 + 1 + 1 + 1 {\color{red}+} 1 + 1 + 1$$

corresponds to

$$10 = 3 + 4 + 3.$$

Since each $+$ can be chosen or not and there are $n - 1$ of them, the number of ways of choosing and hence the number of compositions is 2^{n-1}.

(b) How many compositions of n into k parts are there for general positive integers $k \leq n$?

Solution By the same argument as above the number of compositions into k parts is the number of ways of choosing $k - 1$ of the $+$ signs and so is equal to

$$\binom{n-1}{k-1}.$$